

4^o Math. P. 228 c / 2

SCRIPTORES LOGARITHMICI

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OR,

A COLLECTION

OF

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

LOGARITHMS,

MENTIONED IN

DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW EDITION OF
SHERWIN'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS CON-
NECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME II.

LONDON:

PRINTED BY J. DAVIS,

AND SOLD BY B. WHITE AND SON, FLEET-STREET.

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BIBLIOTHECA
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A
T A B L E
OF THE
C O N T E N T S
OF THE
S E C O N D V O L U M E.

I.

A TRACT of Mr. James Gregory, of Aberdeen in Scotland, written in Latin, and intitled, *Nicolai Mercatoris Quadratura Hyperboles Geometricè demonstrata*; which was first published at London in the year 1668, with some other small tracts, under the title of *Exercitationes Geometricæ*; containing a geometrical demonstration of the method of squaring the Hyperbola, by means of an infinite series of decreasing quantities, then lately published by *Mr. Nicholas Mercator*. In pages 2—5.

II.

Another Tract of the same author, intitled, *Analogia inter Lineam Meridianam Planisphærii Nautici et Tangentes Artificiales, Geometricè demonstrata: seu, Quòd Secantium Naturalium Additio efficiat Tangentes Artificiales: Item, Quòd Tangentium Naturalium Additio efficiat Secantes Artificiales: Item, Quadratura Conchoëidis, et Quadratura Cissoëidis*: taken from the same collection of tracts, intitled, *Exercitationes Geometricæ*. In pages 6—15.

III.

A third Tract of the same author, intitled, *Methodus componendi Tabulas Tangentium et Secantium Artificialium ex Tabulis Tangentium et Secantium Naturalium, exactissimè et minimo cum Labore*: taken from the same collection of tracts, intitled, *Exercitationes Geometricæ*. In pages 16—18.

IV. An

IV.

An Extract from a Letter of the same author to Mr. John Collins, formerly Secretary to the Royal Society of London: dated on the 15th day of February in the year 1670-1, and first published in the *Commercium Epistolicum Domini Johannis Collins et aliorum de Analyfi promotâ*, in the year 1712; containing some infinite serieses relating to the tangents and secants of circular arcs, and to the Logarithms of the ratios of such tangents and secants to the Radius.

In pages 18 and 19.

V.

An Extract from Mr. Isaac Newton's first Epistle to Mr. Henry Oldenburgh, Secretary to the Royal Society of London; with a direction to communicate the contents of it to Mr. Godfrey William Leibnitz: dated on the 13th of June, in the year 1676, from Cambridge, where Mr. Newton (who was afterwards Sir Isaac Newton, knight) was at that time Professor of the Mathematicks upon Dr. Lucas's foundation: containing a discovery relating to Logarithms.

In page 20.

VI.

An Extract from an Epistle of Mr. Godfrey William Leibnitz, of Hanover, to Mr. Henry Oldenburgh, Secretary of the Royal Society of London; with a direction to communicate the contents of it to Mr. Isaac Newton: dated the 27th day of August, in the year 1676: containing a passage relating to Logarithms.

In pages 21, 22.

VII.

An Extract from Mr. Isaac Newton's Second Epistle to Mr. Henry Oldenburgh, Secretary of the Royal Society of London; with a direction to communicate the contents of it to Mr. Godfrey William Leibnitz: dated on the 24th day of October, in the year 1676: containing some discoveries relating to Logarithms.

In pages 22—26.

VIII.

The twelfth Chapter of Dr. John Wallis's Treatise of Algebra, intituled, *Of Logarithms, their Invention and Use*. Published in the year 1685.

In pages 27—34.

IX. A

IX.

A Letter from the Reverend Dr. Wallis, Profefſor of Geometry in the University of Oxford, and Fellow of the Royal Society of London, to Mr. Richard Norris; concerning the Collection of Secants, and the true Division of the Meridians in the Sea Chart. Published in the Philosophical Transactions of the year 1686, Number 176. In pages 35—41.

X.

Logarithmotechnia: or the making of the numbers called Logarithms to twenty-five places of figures, from a geometrical figure, with speed, ease, and certainty. By Euclid Speidell, Philomath. Published at London in the year 1688. In pages 44—75.

XI.

An easy Demonstration of the Analogy of the Logarithmic Tangents to the Meridian Line, or Sum of the Secants; with various methods for computing the same to the utmost exactness. By Dr. Edmund Halley. Published in the Philosophical Transactions for the year 169 $\frac{5}{6}$; Number 219. In pages 76—84.

XII.

A most compendious and facile Method of constructing the Logarithms, exemplified and demonstrated from the nature of Numbers, without any regard to the Hyperbola: with a speedy method for finding the number from the Logarithm given. In pages 84—91.

By Dr. Edmund Halley. Published in the Philosophical Transactions for the year 1695, Number 215.

XIII.

Notes on some of the more difficult Passages of the foregoing Discourse of Dr. Edmund Halley. In pages 92—122.

By Francis Maſeres, Eſq. F. R. S. Curſitor Baron of the Court of Exchequer.

XIV. An

XIV.

An Appendix to the foregoing Tract of Dr. Edmund Halley upon Logarithms, being a direct method of computing the logarithms of ratios, either in Briggs's system, or any other that may be proposed, by the help of Sir Isaac Newton's Binomial Theorem, without the intervention of the Hyperbola, or the Logarithmick Curve, or any other geometrical figure, and likewise without having recourse to the method of Indivisibles or the Arithmetick of Infinites.

In pages 123—152.

By the same.

XV.

A Demonstration of Sir Isaac Newton's Binomial Theorem in the Case of Integral Powers, or Powers of which the Indexes are whole Numbers; together with an extension of the said demonstration to Sir Isaac Newton's Residual Theorem, relating to the powers of a residual quantity as $a - b$, in the case of the integral powers of such quantity, or when the indexes of the said powers are whole numbers.

In pages 153—169.

By the same.

XVI.

A Demonstration of Sir Isaac Newton's Binomial Theorem, in the cases of roots and the powers of roots, as well as in the case of integral powers; published by Mr. John Landen in the year 1758.

In pages 170—175.

XVII.

An Explanation of the foregoing Demonstration of the Binomial Theorem, in the case of the fractional index $\frac{m}{n}$, invented by Mr. John Landen.

In pages 176—193.

By Francis Maferes, Esq. F. R. S. Curfitor Baron of the Court of Exchequer.

XVIII.

A Discourse concerning the Binomial Theorem, in the case of fractional powers, or powers of which the indexes are fractions; containing a demonstration of the said theorem in that case of it.

In pages 194—344.

By the same.

XIX.

A Discourse concerning Sir Isaac Newton's Residual Theorem, or Theorem for raising the powers of the Residual Quantity $1 - x$, in the case of fractional powers, or powers of which the indexes are fractions; containing a demonstration of the said theorem in that case of it. In pages 345—378.
By the same.

XX.

A Method of extending Cardan's Rule for resolving the Cubick Equation $y^3 + qy = r$, or $qy + y^3 = r$, to the resolution of the Cubick Equation $qy - y^3 = r$, when $\frac{rr}{4}$ is of any magnitude less than $\frac{q^3}{54}$, or than $\frac{1}{2} \times \frac{q^3}{27}$, or when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$; by the help of Sir Isaac Newton's binomial and residual theorems, which have been demonstrated in the two preceding discourses. In pages 381—440.
By the same.

XXI.

A Method of extending Cardan's Rule for resolving the Cubick Equation $y^3 - qy = r$, in the first case of it, or when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$, to the other case of the same equation, in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which the said rule is not naturally fitted to resolve; provided that the absolute term r (though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) be greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{rr}{4}$ (though less than $\frac{q^3}{27}$) be greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. In pages 443—575.
By the same.

N. B. This Tract was first published in the Philosophical Transactions, for the year 1778; but is here very much enlarged.

XXII.

A Conjecture concerning the Method by which Cardan's Rules for resolving the Cubick Equation $x^3 + qx = r$ in all cases (or in all magnitudes of the known quantities q and r) and the Cubick Equation $x^3 - qx = r$ in the first case of it (or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$)
Vol. II. b were

were probably discovered by Scipio Ferreus of Bononia, and Nicholas Tartalea, or whoever else were the first inventors of them. In pages 579—586.

By the same.

N. B. This Tract was first published in the Philosophical Transactions for the year 1780.

XXIII.

An Appendix to the Tract contained in the foregoing part of this second volume of Mathematical Tracts, in pages 153, 154, 155, &c, to page 169, intituled "A Demonstration of Sir Isaac Newton's Binomial Theorem in the case of Integral Powers, or powers of which the Indexes are whole Numbers:" containing an Investigation of the Law by which the coefficients of the third and fourth and other following terms of the series which is equal to any integral power of a binomial quantity, are derived from the co-efficient of the second term of the said series; grounded on a probable induction from particular examples. In pages 587—591.

By the same.

E R R A T A.

- ✓ In the figure in page 3 the line XZ ought to pass through the point Y.
 ✓ In page 4, line 6, instead of "NCDH," read "NODH."
 ✓ In the same page 4, line 9, instead of "*infinitas*," read "*infinitis*."
 ✓ In the same page 4, line 15, instead of "*æqualis*," read "*æquales*."
 ✓ In page 55, lines 9, 13, 14, 15, 16, instead of "*trapezia*," read "*trapezium*."
 ✓ In page 67, line 17, instead of "*to his doctrine*," read "*according to his doctrine*."
 ✓ In page 84, line 17 from the bottom, instead of "*conuntryman*," read "*countryman*."
 ✓ In page 85, the last line, instead of "*infinite infinite*," read "*infinîtè infinite*."
 In page 88, line 10 from the bottom, instead of "*demonstration*," read "*demonstration*."
 In page 100, line 11, instead of "*the ratio of 22×4* ," read "*the ratio of 22×24* ."
 In page 105, line 8 from the bottom, instead of "*Suppose b to be*," read "*Suppose k to be*."
 In page 110, line 6, instead of " $\frac{1l}{1 - \frac{1}{2}l}$," read " $\frac{al}{1 - \frac{1}{2}l}$."

And in the same page 110, in the last line but one and the last line, instead of "*the second $1 - \frac{bl^2}{2}$*
etc.," read "*the series $bl - \frac{bl^2}{2}$ etc.*"

In page 111, note ix, line 3, instead of " $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$," read " $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$."

In page 112, line 4, instead of " $\frac{ad}{\frac{1}{m} - \frac{1}{2}d}$," read " $\frac{ad}{\frac{1}{m} - \frac{1}{2}d}$."

And in the same page 112, line 15, instead of " $\frac{bd}{\frac{1}{m} + \frac{1}{2}d}$," read " $\frac{bd}{\frac{1}{m} + \frac{1}{2}d}$."

And in the same page 112, line 10 from the bottom, instead of " $\frac{l}{2}$," read " $\frac{l}{2}$."

And in the same page 112, line 8 from the bottom, dele the mark) of a parenthesis after
 $\frac{ad}{\frac{1}{m} - \frac{d}{2}}$, and substitute a semicolon ; in its stead.

In page 154, line 2, instead of " $\text{or to } \frac{1}{m} \times \frac{m-1}{2} \times \frac{m-2}{3}$," read " $\text{or to } \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$."

ERRATA.

- In the same page 154, line 3, instead of "*several*," read "*several*."
- In the same page, line 14, instead of " $1 \times am$," read " $1 \times a^m$."
- In page 156, line 11 from the bottom, instead of "or am " read "or a^m ."
- In page 157, line 5, instead of " $\overline{a + b}m$," read " $\overline{a + b}^m$."
- In the same page 157, line 7, instead of " $\overline{1 + 1}m$," read " $\overline{1 + 1}^m$."
- In page 158, line 5, after the words "to do which" place a comma.
- In page 164, line 7 from the bottom, instead of " $\frac{m-3}{4}$," read " $\frac{n-3}{4}$."
- In page 190, line 7th from the bottom, after " $G, H, I, K, L, \&c$ " insert the word "*equal*."
- And in the same page 190, line 6th from the bottom, after the word "respectively" insert the mark of a parenthesis,).
- In page 197, line 8th from the bottom, after the word "respectively," insert the mark of a parenthesis,).
- In page 202, article 11, line 1st, instead of " $1x$ " read " $1 + x$."
- In page 226, line 8, instead of " $\frac{1}{n}y$ " read " $\frac{1}{n}y$."
- In page 322, line 4th from the bottom, in the numerator of the last fraction in the said line, instead of "48,000" read "48,000 x ".
- In page 341, line 6, instead of " $1 \frac{120y}{720}$," read " $-\frac{120y}{720}$."
- In page 363, line 10th from the bottom, instead of " $\overline{1 - x}^{\frac{1}{n}}$ " read " $\overline{1 + x}^{\frac{1}{n}}$."
- In page 388, article 9, line 6, instead of "other following terms," read "other following even terms."
- In page 396, line 3d from the bottom, after $\frac{e^{13}}{5^{12}}$, &c, dele the word *which*.
- In page 418, the last line, instead of "the sign" read "the sign +."
- In page 439, the last line, instead of " $\sqrt[3]{a - \sqrt{bb}}$ " read " $\sqrt[3]{a - \sqrt{-bb}}$."
- In page 449, in the running title, instead of "THE RESOLUTION OF CUBICK EQUATIONS," read "RESOLVING THE CUBICK EQUATION."
- In page 584, line 11, instead of " $\frac{2\sqrt{q}}{\sqrt{q}}$," read " $\frac{2\sqrt{q}}{\sqrt{3}}$."
- And in the same page 584, line 20, instead of, "if $x^3 - qx = r$, or r ," read "if $x^3 - qx$, or r ."

E X T R A C T S

FROM

M R. J A M E S G R E G O R Y ' S

E X E R C I T A T I O N E S G E O M E T R I C Æ,

R E L A T I N G T O

L O G A R I T H M S.

P R I N T E D A T L O N D O N I N 1668.

VOL. II.

B

N. M E R C A T O R I S

QUADRATURA HYPERBOLES

GEOMETRICE DEMONSTRATA.

P R O P. I.

SI fuerint quantitates continuè proportionales $A, B, C, D, E, F, \&c$, numero infinitæ, quarum prima & maxima A ; erit $A - B$ ad A ut A ad summam omnium; hoc enim passim demonstratur apud geometras.

P R O P. II.

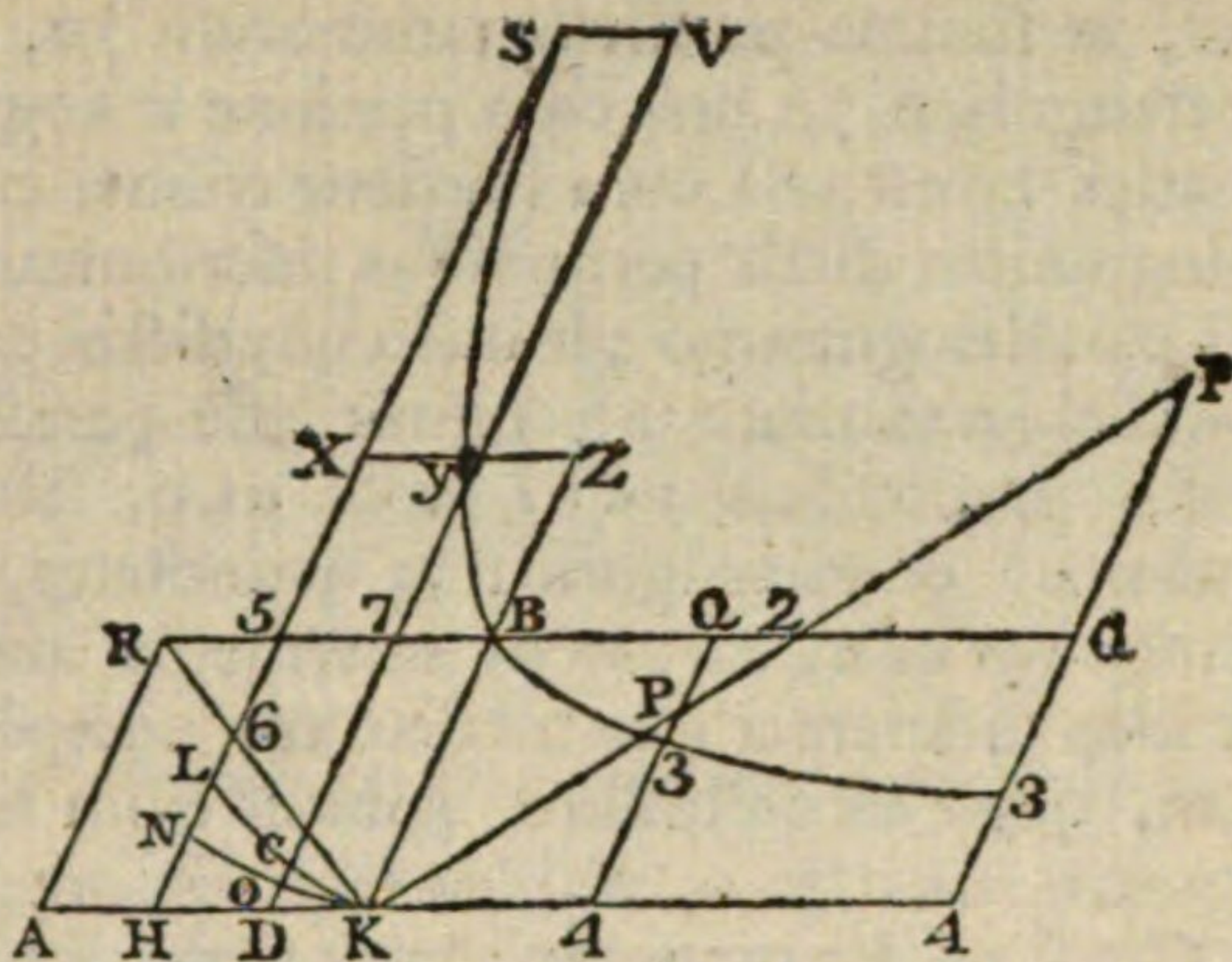
Eisdem positis quæ antecedente; dico $A + B$ esse ad A ut A est ad excessum omnium $A, C, E, G, \&c$, in locis imparibus, supra omnes $B, D, F, \&c$, in locis paribus: est enim dictus excessus summa seriæ infinitæ continuè proportionalium in ratione A ad C , nempe $A - B, C - D, E - F, \&c$, & ideo (ex præcedente) ut $A - C$ ad A vel $A^2 - B^2$ ad A^2 , ita $A - B$ ad summam dicte seriæ, quam vocamus z ; & priores analogiæ terminos applicando ad $A - B$, $A + B$ est ad $\frac{A^2}{A - B}$ ut $A - B$ ad z , & ideo $Az + Bz = A^2$, & proinde $A + B$ est ad A ut A ad z excessum omnium $A, C, E, G, \&c$, supra omnes $B, D, F, \&c$. Quod demonstrare oportuit.

P R O P.

PROP. III.

Sit hyperbola $SB3$, cujus vertex B , asymptotæ AR , $A4$, asymptotæ RA ducatur parallela BK , & altera ad libitum inter rectas BK , RA , utrique parallela YD ; dico YD esse summam infinitæ seriæ continuè proportionalium, cujus primus terminus $BK = KA$ & secundus KD ; est enim $BK - KD = AD$ ad BK ut BK ad DY ; & ideo ex hujus prima patet propositum.

Eisdem rectis RA , BK , ultra punctum K fiat parallela 34 ; dico rectam 34 æqualem esse excessui omnium terminorum imparium supra omnes terminos pares infinitæ seriæ cujus primus terminus KB , & secundus $K4$: est enim $KB + K4 = A4$ ad BK ut BK ad 34 ; & ideo ex hujus 2^a patet propositum.



PROP. IV.

Sit $SBKH$ spatium hyperbolicum, contentum à curva hyperbolica SB , asymptotæ portione HK , & rectis SH , BK , alteri asymptotæ parallelis, posito B hyperbolæ vertice: sit parallelogrammum $BKHS$, et producat $B5$ in R , jungaturque KR quæ $5H$ secet in 6 : deinde continuetur series infinita continuè proportionalium nempe $5H$, $6H$, LH , NH , & sic deinceps; sitque $5BKH$ parallelogrammum, $K6H$ triangulum KLH trilineum quadraticum, KNH trilineum cubicum, & ita deinceps in infinitum. Dico spatium hyperbolicum, $BKHS$ æquale esse dicto parallelogrammo, dicto triangulo, unà cum infinitis illis trilineis, quorum omnium summam vocamus ω . Si figura $BKHS$ & ω non sunt æquales, sit inter illas differentia α ; & dividatur recta HK in tot partes æquales à rectis asymptotæ RA parallelis, ut rectangula (ab illis & portionibus rectæ KH contenta) figuræ $BKHS$ circumscripta, nempe VH , ZD , differant à rectangulis figuræ $BKHS$ inscriptis, nempe YH , BD , minore intervallo quàm α ; hoc enim fieri potest ab indefinita divisione rectæ KH . Quoniam B est hyperbolæ vertex, parallelogrammum $BKAR$ est æquilaterum; & proinde recta $6H$ ad libitum est æqualis rectæ HK , cumque $5H$, $6H$, LH , NH , &c, sint continuè proportionales in infinitum, ex hujus 3 erit recta SH æqualis summæ omnium, & parallelogrammum SD æquale summæ omnium parallelogrammorum $5D$, $6D$, LD , ND , &c, in infinitum; atque summa omnium parallelogrammorum $5D$, $6D$, LD , ND , &c, in infinitum, major est parallelogrammo $5D$ unà cum portione trianguli $6FDH$ unà cum portione trilinei quadratici $LCDH$ unà cum portione trilinei cubici $NC DH$, &c, in infinitum, quoniam predictæ portiones dictis parallelogrammis inscribuntur, & ideo parallelogrammum SD majus est parallelogrammo $5D$ unà cum dictis portionibus; eodem modo demonstratur parallelogrammum YK majus esse rectangulo $7K$ unà cum infinitis numero portionibus FKD , CKD , OKD , &c, & proinde rectilineum $SVYZKH$ majus est quam ω . De-

indè recta FD est æqualis rectæ DK ; atque $7D$, FD , CD , OD , &c sunt rectæ continuè proportionales in infinitum, & igitur recta YD est æqualis ipsarum summæ, & parallelogrammum XD æquale parallelogrammis $7H$, FDH , CDH , ODH , &c, at summa parallelogrammorum $7H$, FDH , CDH , ODH , &c minor est quàm rectangulum $7H$ unà cum portione trianguli $6FDH$ unà cum portione trilinei quadratici $LCDH$ unà cum portione trilinei cubici $NOBH$, &c, quoniam dicta parallelogramma dictis portionibus inscribuntur, & ideo parallelogrammum YH minus est parallelogrammo $7H$ unà cum dictis portionibus; eodem modo demonstratur parallelogrammum BD minus esse parallelogrammo BD unà cum infinitis numero portionibus FKD , CKD , OKD , & ideo rectilineum $XY7BKH$ minus est quàm ω : quatuor igitur sunt quantitates, quarum maxima & minima sunt rectilinea $SVYZKH$, $XY7BKH$, intermediæ autem ω & spatium hyperbolicum $SBKH$; & ideo differentia intermediarum, nempe ω minor est quàm differentia extremarum, quod est absurdum, ponitur enim major; nulla igitur est differentia inter figuram $SBKH$ & ω , & ideo æqualis sunt, quod demonstrandum erat.

Conf. 1. Et proinde si fuerit series infinita quantitatum continuè proportionalium in ratione KB ad $KH = 6H$, cujus primus terminus est parallelogrammum BH ; erit primus terminus $+$ $\frac{1}{2}$ secundi $+$ $\frac{1}{3}$ tertii $+$ $\frac{1}{4}$ quarti $+$ $\frac{1}{5}$ quinti $+$ &c in infinitum $=$ spatio hyperbolico $SBKH$, hoc enim evidenter sequitur ex quadratura trilineorum.

Conf. 2. Si autem ultra K sumatur spatium $B34K$, positâ 34 parallela rectæ BK , & sit series infinita quantitatum in continua ratione BK ad $K4$, cujus primus terminus est parallelogrammum KQ ; erit primus terminus $-$ $\frac{1}{2}$ secundi $+$ $\frac{1}{3}$ tertii $-$ $\frac{1}{4}$ quarti $+$ &c in infinitum $=$ spatio hyperbolico $B34K$: poterit hoc confectarium eodem ferè modo demonstrari geometricè ex secunda conclusione hujus tertie, quo antecedens ex ejusdem conclusione priore; utrumque autem ex methodo indivisibilium Cavallerianâ nullo negotio demonstratur; sed quoniam magni sunt momenti, placuit methodum rigorosam adhibere.

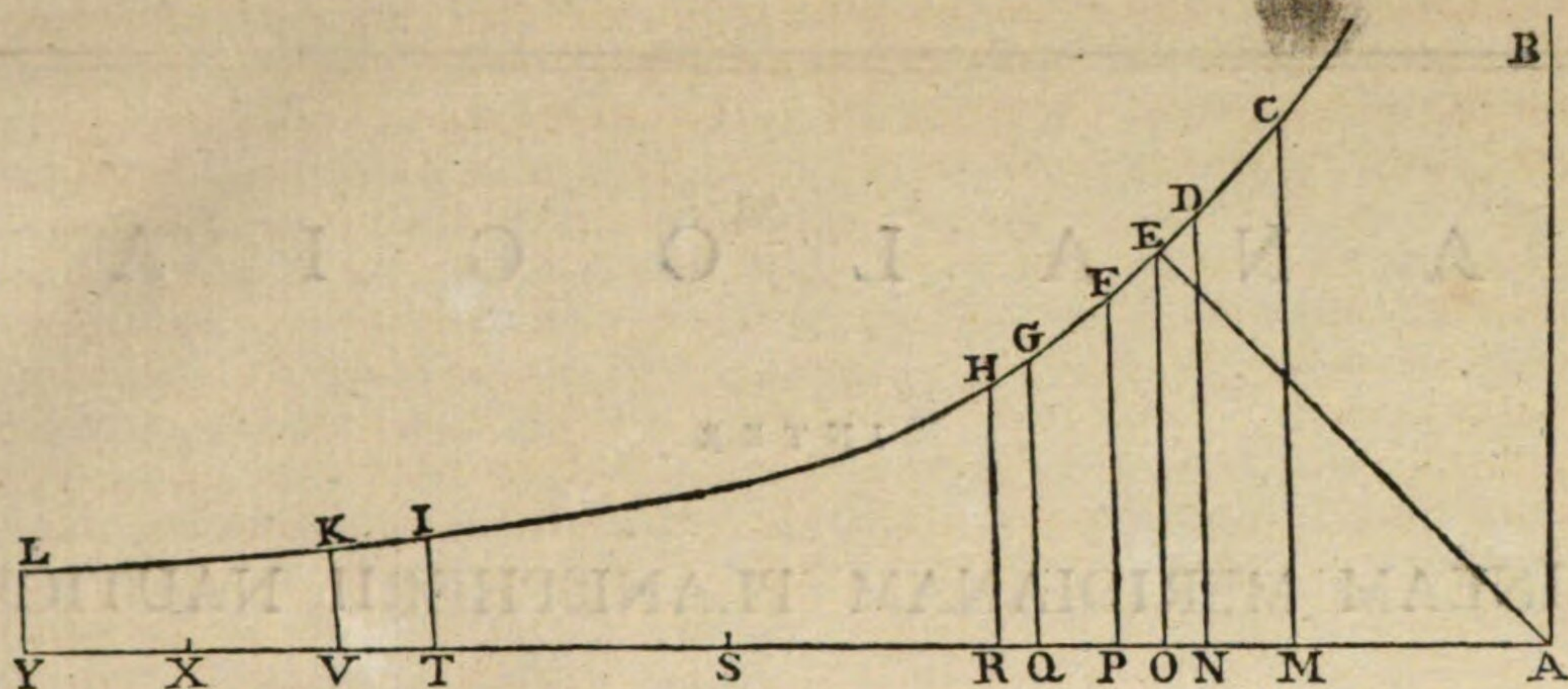
Conf. 3. Quod si $K4 = KH$, & fuerit series infinita quantitatum in continua ratione BK ad $K4 = KH$, cujus primus terminus est $BH = B4$; erit excessus spatii $SBKH$ supra spatium $B34K =$ toti secundo termino $+$ $\frac{1}{2}$ quarti $+$ $\frac{1}{3}$ sexti $+$ $\frac{1}{4}$ octavi $+$ &c in infinitum: nam ex primo confectario, spatium $SBKH =$ primo termino $+$ $\frac{1}{2}$ secundi $+$ $\frac{1}{3}$ tertii $+$ $\frac{1}{4}$ quarti $+$ &c in infinitum: & ex secundo confectario spatium $B34K =$ primo termino $-$ $\frac{1}{2}$ secundi $+$ $\frac{1}{3}$ tertii $-$ $\frac{1}{4}$ quarti $+$ &c; at manifestum est horum differentiam $=$ toti secundo $+$ $\frac{1}{2}$ quarti $+$ $\frac{1}{3}$ sexti $+$ $\frac{1}{4}$ octavi $+$ &c; & ideo patet propositum.

Conf. 4. Eisdem positis quæ in antecedente confectario, manifestum est spatium hyperbolicum $SH43 =$ duplo primi termini $+$ $\frac{2}{3}$ tertii $+$ $\frac{2}{5}$ quinti $+$ $\frac{2}{7}$ septimi $+$ $\frac{2}{9}$ noni $+$ &c in infinitum.

P R O P. V.

Sit hyperbola CEL , cujus vertex E , & asymptotæ AB , AY ; in qua sumantur duo spatia hyperbolica ad libitum $HITR$, $KLYV$, contenta à curvâ hyperbolicâ, unâ asymptotâ & rectis alteri asymptotæ parallelis: dividantur rectæ RT , VY , bifariam in s & x punctis. Dico spatium $HITR$ esse ad spatium $KLYV$, ut

$$\frac{RS \times AO}{AS} + \frac{RS^3 \times AO}{3AS^3} + \frac{RS^5 \times AO}{5AS^5} + \frac{RS^7 \times AO}{7AS^7} + \&c \text{ in infinitum ad } \frac{VX \times AO}{AX} +$$



$\frac{vx^3 \times AO}{3AX^3} + \frac{vx^5 \times AO}{5AX^5} + \frac{vx^7 \times AO}{7AX^7} + \&c$ in infinitum. Asymptotæ AB fiat recta parallela EO; sitque ut AS ad AR ita AO = EO ad AM; & ut AS ad AT ita AO ad AQ; fimiliter, sit ut AX ad AV ita AO ad AN, & AO ad AP ut AX ad AY; manifestum est MO = OQ & NO = OP. Ductis rectis MC, ND, PF, QG, evidens est (ex hyperbolæ proprietatibus) spatium CGQM esse æquale spatio HCTR & spatium DFPN spatio KLYV; at patet (ex conscripto 4 antecedentis) spatium CGQM esse ad spatium DFPN, ut MO + $\frac{MO^3}{3AO^2} + \frac{MO^5}{5AO^4} + \frac{MO^7}{7AO^6} + \&c$ ad NO + $\frac{NO^3}{3AO^2} + \frac{NO^5}{5AO^4} + \frac{NO^7}{7AO^6} + \&c$, quæ analogia eadem est cum proposita, quod demonstrandum erat.

Conf. Hinc manifestum est (ob analogiam inter spatia hyperbolica & logarithmos) differentiam inter logarithmos numerorum A, B, esse ad differentiam inter logarithmos numerorum D, E, (posito c medio arithmetico inter A, B, & F medio arithmetico inter D, E, item N differentiâ inter c & A, & o differentiâ inter F & D) ut $\frac{N}{c} + \frac{N^3}{3c^3} + \frac{N^5}{5c^5} + \frac{N^7}{7c^7} + \&c$ ad $\frac{o}{F} + \frac{o^3}{3F^3} + \frac{o^5}{5F^5} + \frac{o^7}{7F^7} + \&c$, & ideo si ponatur A = D = I, hinc patet methodus inveniendi logarithmum quemcunque ex uno dato, absque ulla hyperbolæ consideratione, sed calculo plerumque nimis laborioso. Quod si ponatur A = 999, B = 1001, cum datis logarithmis, item D, E, numeri dati majores, unitate vel parvo aliquo intervallo differentes, nullo negotio invenietur differentia logarithmorum numeris D, E, debitorum; hâc methodo non difficulter computatur integra logarithmorum tabula ad quotvis notas.

Facilè quoque deducitur (ex 3 conscript. antecedentis prop.) differentiam secundam logarithmorum numerorum in ratione arithmetica G, H, I, esse ad differentiam secundam logarithmorum numerorum in ratione arithmetica K, L, M, (posito P differentia inter G, H, & Q differentia inter K, L) ut $\frac{P^2}{H^2} + \frac{P^4}{2H^4} + \frac{P^6}{3H^6} + \&c$ ad $\frac{Q^2}{L^2} + \frac{Q^4}{2L^4} + \frac{Q^6}{3L^6} + \&c$, non dissimili ferè methodo determinantur differentiæ logarithmorum tertiæ, quartæ, quintæ, & sic deinceps; sed non est operæ pretium illas prosequi: primæ enim differentiæ compositioni logarithmorum abundè sufficiunt.

ANALOGIA

A N A L O G I A

INTER

LINEAM MERIDIANAM PLANISPHERII NAUTICI,

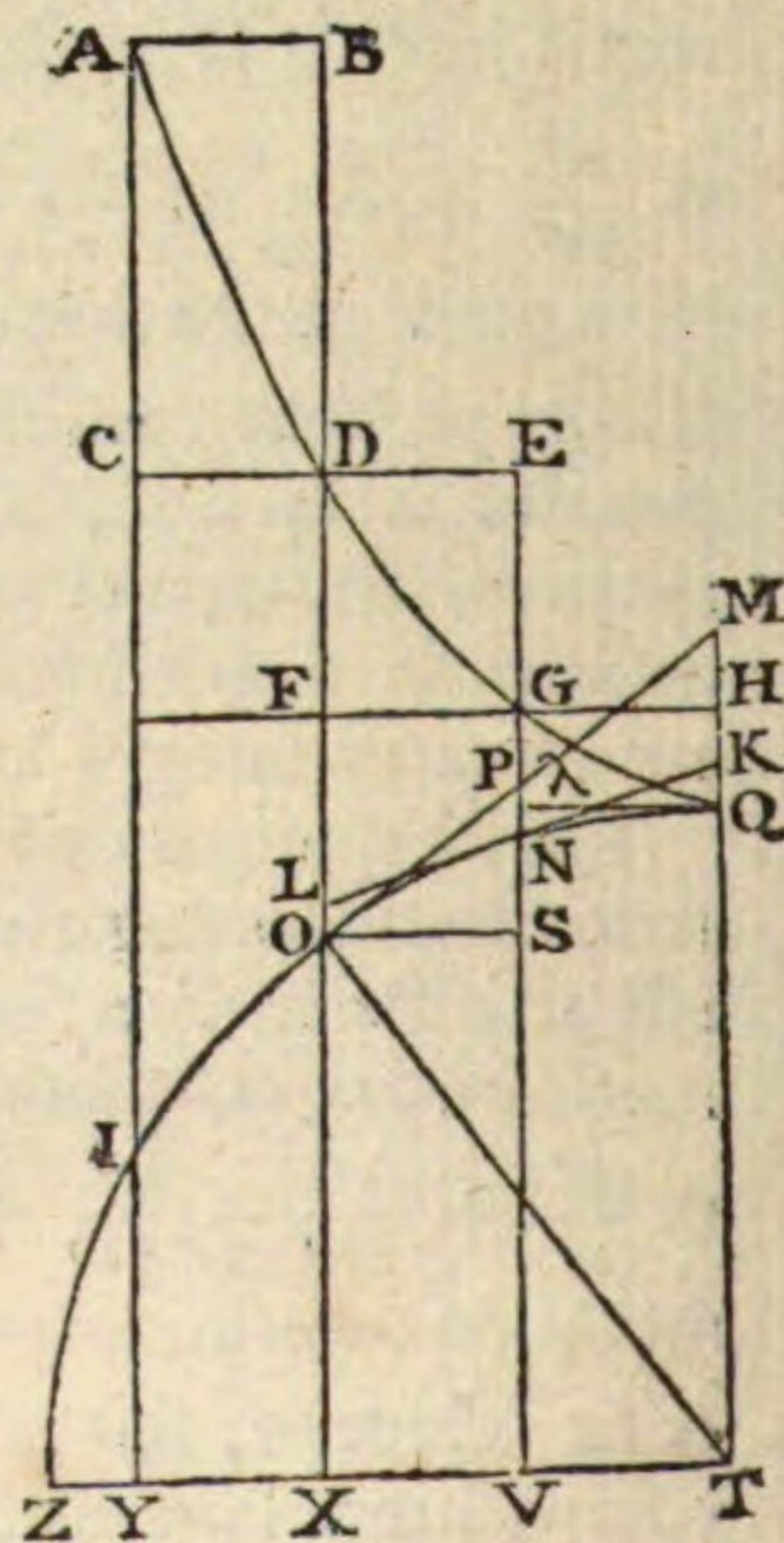
ET

TANGENTES ARTIFICIALES,

GEOMETRICE DEMONSTRATA, &c.

PROP. I. THEOREMA.

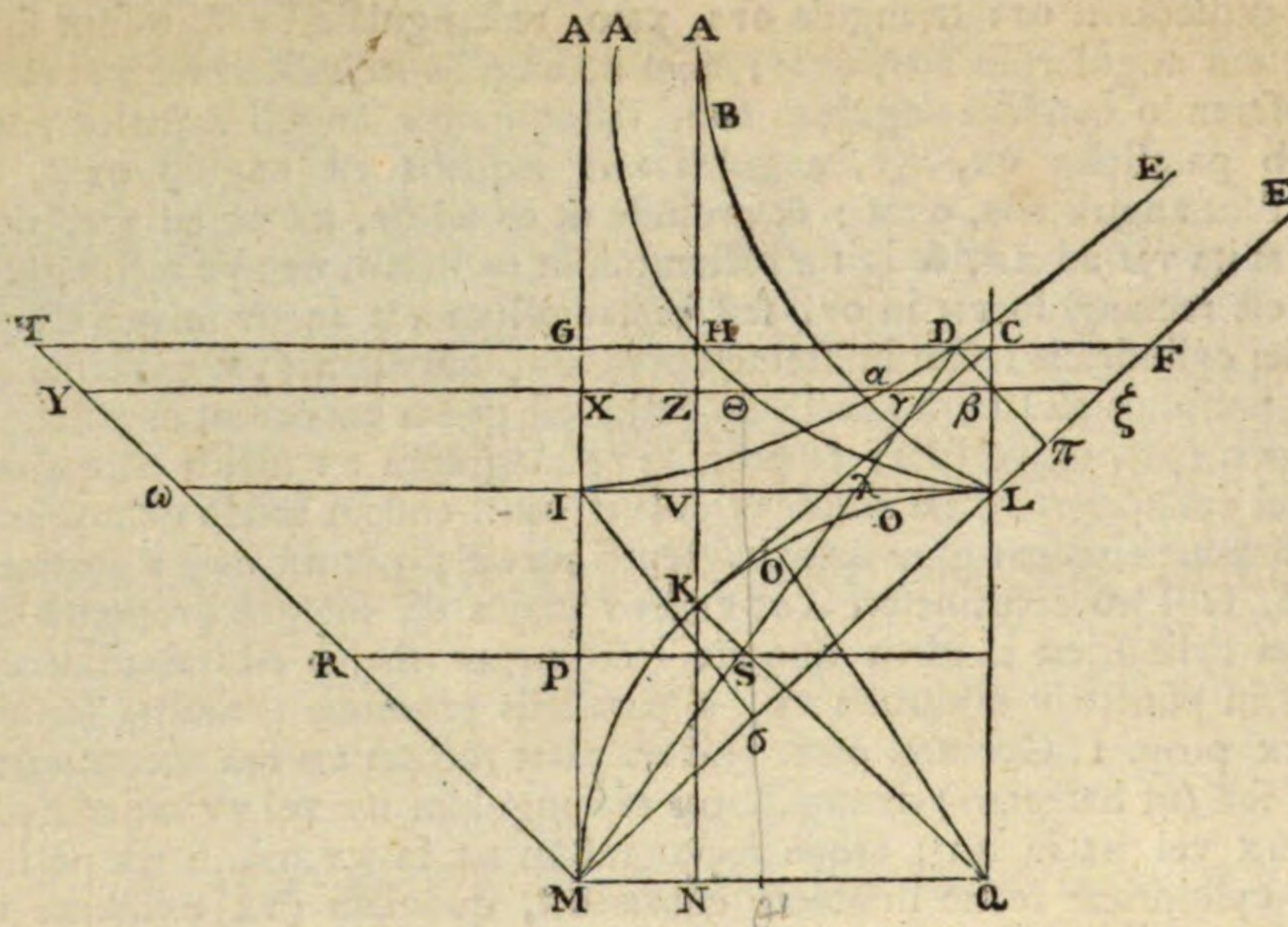
SIT circuli quadrans TQZ , cujus pars sit arcus QI : super arcu QI imaginetur portio superficiei cylindrici recti talis naturæ, ut (sumpto in arcu QI quolibet puncto o) perpendicularis ad planum TZQ ex puncto o ad summam portionis superficiei cylindricæ excitata semper fiat æqualis secanti arcus OQ . Deinde sit mixtilineum $AIYTQ$ talis naturæ, ut (ducta in eo recta XD radio QT parallela & arcum quadrantis secante in puncto ad libitum o) recta DX , secans arcus OQ , & radius TQ sint continuè proportionales. Dico mixtilineum $AIYTQ$ esse æquale dictæ portioni superficiei cylindricæ: si prædictæ figuræ non sunt æquales, sit inter illas differentia α , & dividatur recta YT in tot partes æquales à rectis BX , EV , radio QT parallelis, ut (completis rectangulis AX , DV , GT , CX , FV , λT ,) differentia rectilineorum $ABDEGHTY$, $CDFGLQTY$, sit minor quàm α ; manifestum est enim hoc fieri posse ab indefinita divisione recte YT . Ducatur in puncto o recta tangens OPM , sitque in rectam



ΣV perpendicularis os : triangula ops, tmo, rectangula ad s & o sunt similia ob æqualitatem angulorum pos, otm; nam ab angulis æqualibus nempe rectis pot, sox, auferendo eundem angulum sot, relinquuntur anguli æquales pos, xot, atque ob parallelas ox, qt, angulus xot æqualis est angulo otm, & ideo æqualis sunt anguli pos, otm; & proinde ut os ad op, ita ot ad tm, ut autem ot ad tm ita tm ad xd, & igitur rectangulum os in xd, nempe rectangulum dv, æquale est rectangulo tm in op, sed rectangulum tm in op majus est portione superficiei cylindricæ rectæ insistente curvæ on, quoniam (tm existente maxima ejusdem portionis altitudine) recta po major est quàm curva no, ut patet ex prop. 1. Geomet. part. universalis; & proinde rectangulum dv majus est quàm portio superficiei cylindricæ rectæ insistentis curvæ on: eodem modo demonstratur gt majus portione ejusdem superficiei insistente curvæ nq & ax majus portione super curva io, & ideo rectilineum abdeghty majus est integrâ propositâ portione superficiei cylindricæ; idem quoque rectilineum majus est mixtilineo aqty. Tangens in puncto n ducatur lnk, à parallelis proximis utrinque terminata in l, k; ex prop. 1. Geomet. part. univer. patet rectam ln seu nk minorem esse arcu on, sed (ut hætenus) demonstratur rectangulum gt vel fv æquale esse rectangulo nk vel nl in kt; atque rectangulum nl in kt minus est portione superficiei cylindricæ rectæ insistente curvæ on, quoniam (tk existente minima ejusdem portionis altitudine) recta ln minor est curva on: & proinde rectangulum fv minus etiam est quàm prædicta portio: eodem modo demonstratur rectangulum lt minus esse portione super curva nq & rectangulum cx minus portione super curva io, & ideo rectilineum cdfglqty minus est integrâ propositâ superficiei cylindricæ rectæ portione, idem quoque rectilineum minus est mixtilineo aqty: ex prædictis ergo manifestum est quatuor esse quantitates, quarum maxima & minima sunt rectilinea abdeghty, cdfglqty, intermediæ autem superficiei cylindricæ portio proposita & mixtilineum aqty; & ideo differentia intermediarum minor est quàm differentia maximæ & minimæ, differentia autem maximæ & minimæ ex constructione minor est quàm α , & proinde differentia intermediarum nempe superficiei cylindricæ & mixtilinei aqty multo minor est quàm α , quod est absurdum, ponitur enim α , non igitur differunt mixtilineum aqty & proposita superficiei cylindricæ portio; & ideo æqualia sunt, quod demonstrare oportuit.

PROP. II. THEOREMA.

Sit circuli quadrans qml, sitque mixtilineum ahlqm talis naturæ, ut (ducta recta ad libitum hn radio lq parallela & quadrantis arcum secante in k) recta hn æqualis sit secanti arcus lk, sitque mixtilineum ablqma talis naturæ, ut (productâ arbitrariâ nh in b) rectæ lq, nh, nb, sint continuè proportionales: deinde sit femihyperbola ide cujus axis ma, vertex i & asymptoton mle: ducatur ad libitum radio lq parallela recta nb, curvas lkm, lha, lba, secans in punctis k, h, b; & per punctum h ducatur radio mq recta parallela hf hyperbolæ occurrens in puncto d. Dico sectorem hyperbolicum imd æqualem esse femissi figuræ blqn, quæ figura (ut in antecedente demonstratum est) æqualis est



est superficiei cylindricæ conflatae ex omnibus secantibus arcuum infinitorum OL
 plano LMQ in debitis suis punctis o normaliter insistentibus. Ex *Gregorii à S.
 Vincentio* doctrina ductuum, truncus cylindrici recti super figura LHNQ resectus à
 plano basin seminormaliter secante in recta QM, æqualis est semissi cylindrici
 recti cujus basis est figura LBNQ & altitudo recta LQ, quoniam LQ semper est ad
 NH ut NH ad NB: manifestum autem est eundem truncum oriri ex ductu trianguli
 rectanguli isoscelis TGM in figuram LHNQ, hoc est ex ductu trianguli rectanguli
 isoscelis ωIM in rectangulum VLQN & ex ductu trapezii TGIω in figuram HLV; atque
 solidum factum ex ductu trianguli rectanguli ωIM in rectangulum VLQN æquale
 est prisinati cujus altitudo LQ & basis triangulum IAM, quod sic probo; $MG^2 : MI^2 :: MI^2 : KN^2$, & per conversionem rationis $MG^2 : MG^2 - MI^2 = GD^2 :: MI^2 : MI^2 - KN^2 = NQ^2$, & ideo $MG : GD :: MI : NQ$, ut autem MG ad DG, ita quæ-
 libet MP = PR ad ductam PS ipsi GD parallelam, & ideo ut PR ad PS ita MI ad
 NQ, cùmque hoc semper fiat, manifestum est prisma cujus basis IAM & altitudo
 IM æquale esse solido ex ductu trianguli ωIM in rectangulum VLQN: item soli-
 dum factum ex ductu trapezii TGIω in figuram HLV æquale est cylindrico cujus
 basis est figura IDλ & altitudo IM, quod sic probo; $GM : GD :: MI : QN$, &
 per conversionem rationis $GM : DF :: MI : GH$; ducatur recta ad libitum Yξ
 radio MQ parallela & curvas secans ut in figura, eritque $GM : DF :: XM : γξ$, &
 ideo $XM = XY : γξ :: MI : GH = XZ$, & ideo $XY \times XZ = MI \times γξ$; ut autem
 antè demonstratum est GM esse ad DF ut MI ad GH, eodem modo nunc demon-
 strari potest $XM = XY$ esse ad αξ ut MI ad xθ, & ideo $XY \times xθ = MI \times αξ$,
 cùmque $XY \times XZ = MI \times γξ$, si ab æqualibus æqualia auferantur, $XY \times zθ = MI \times αγ$,
 cùmque hæc æqualitas semper contingat, manifestum est trapezium TGIω
 ductum in figuram HLV efficere solidum æquale cylindrico cujus basis IDλ & alti-
 tudo IM; & proinde totus truncus super figura HLQN ortus ex ductu trianguli TGM
 in

in eandem figuram $HLQN$ est æqualis cylindrico cujus basis est sector hyperbolicus MID & altitudo MI , atque idem truncus æqualis est dimidio cylindrici cujus basis est figura $BLQN$ & altitudo eadem IM , & ideo semiffis cylindrici super figura $BLQN$ cum altitudine IM æqualis est cylindrico cujus basis MID & altitudo eadem IM , est igitur figura $BLQN$ dupla sectoris hyperbolici MID , quod demonstrare oportuit.

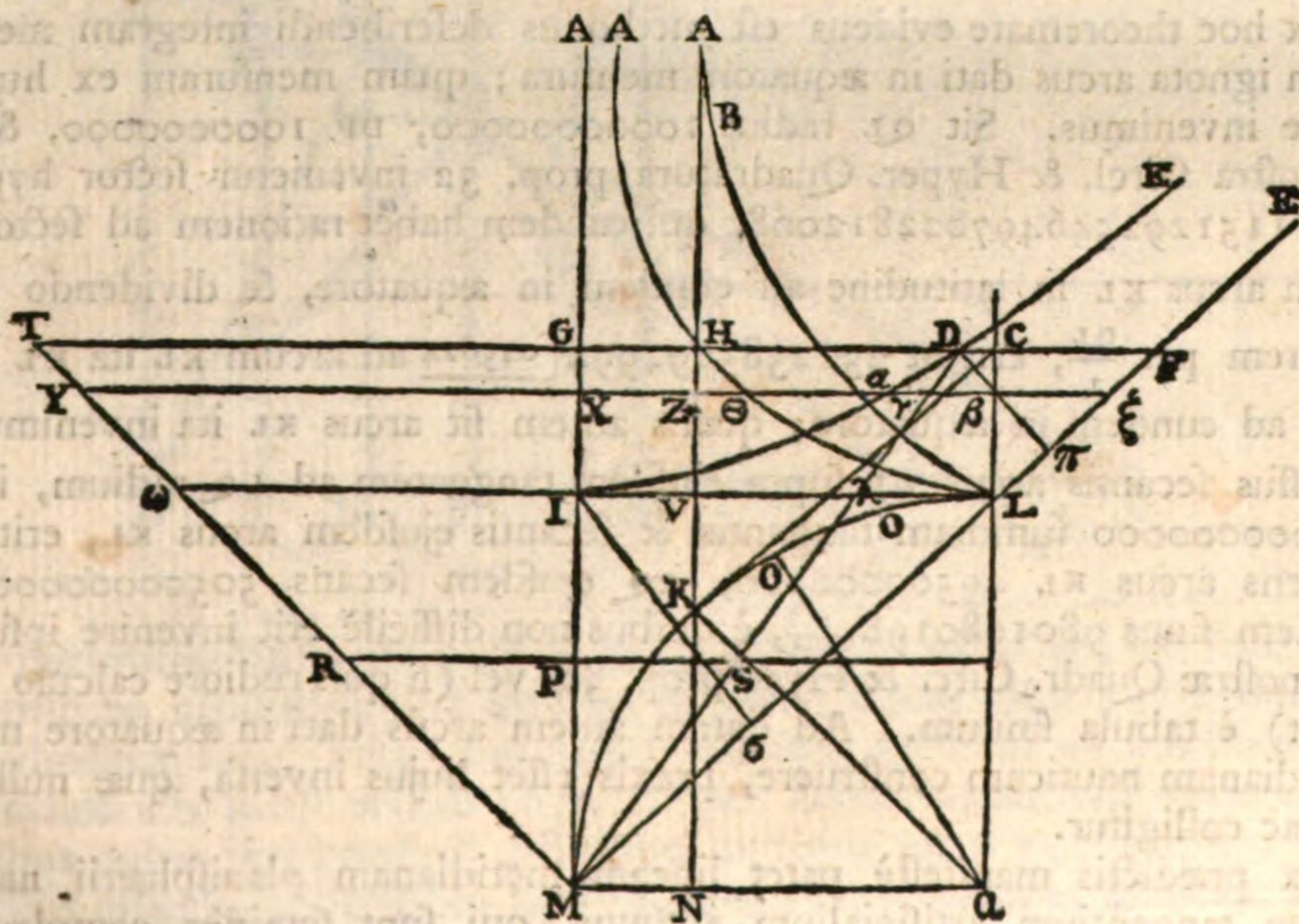
CONSECTARIUM.

Hinc sequitur, quod figura $BLQN$ semper sit dupla logarithmi differentiae inter tangentem & secantem arcus KL , posito radio LI loco unitatis, quod sic probo. Ex punctis I, D , in asymptoton ME demittantur perpendiculares $IO, D\pi$; ex demonstratis in Circ. & Hyperb. Quad. manifestum est sectorem MID esse æqualem figuræ $ID\pi O$, item figuram $ID\pi O$ esse logarithmum rectæ $D\pi$ positâ IO unitate; ut autem IO ad $D\pi$ ita IL radius ad DF differentiam inter tangentem & secantem, & ideo posita IL unitate erit idem sector MID logarithmus rectæ DF , nempe excessus quo secans arcus KL superat ejusdem tangentem.

PROP. III. THEOREMA.

Linea Meridiana Planisphaerii Nautici est Scala Logarithmorum Excessuum, quibus Secantes Latitudinum superant earundem Tangentes, posito Radio Loco Unitatis.

Suppono ex scriptoribus nauticis in eorum planisphærio arcum LK in æquatore esse ad eundem LK in latitudine, ut rectangulum ex LQ in KL ad portionem superficiëi cylindricæ conflatæ ex omnibus secantibus arcuum infinitorum OL .



plano LMQ in debitis suis punctis o normaliter insistentibus, seu figuram $BLQN$; demonstratum autem est in antecedente sectorem hyperbolicum MID seu logarithmum rectæ DF (posita QL unitate) differentiæ inter tangentem & secantem arcus KL esse semissem figuræ $BLQN$, patet quoque sectorem circulare QKL esse semissem rectanguli LQ in KL ; & ideo LK in æquatore est ad LK in latitudine ut sector QKL ad logarithmum excessus quo secans arcus KL superat ejusdem tangentem; & ideo solidum ex LK in logarithmum dicti excessus æquale est solido ex LK in latitudine in sectorem QKL , & utrumque solidum applicando ad LK , logarithmus dicti excessus æqualis est rectangulo ex LK in latitudine $\frac{QL}{2}$; eodem modo demonstratur logarithmum excessus, quo secans arcus cujuscunque OL excedit ejusdem tangentem, æqualem esse rectangulo ex OL in latitudine in $\frac{QL}{2}$; & proinde logarithmus excessus quo secans arcus KL superat ejusdem tangentem, est ad rectangulum ex LK in latitudine in $\frac{QL}{2}$, ut logarithmus excessus quo secans arcus OL superat ejusdem tangentem ad rectangulum ex OL in latitudine in $\frac{QL}{2}$, & permutando, & utrumque rectangulum applicando ad $\frac{QL}{2}$; logarithmus excessus quo secans arcus KL superat ejusdem tangentem, est ad logarithmum excessus quo secans arcus OL superat suam tangentem, ut arcus KL in latitudine ad arcum OL in latitudine ab æquatore planisphærii nautici in linea recta computatis; et ideo linea meridiana planisphærii nautici analoga est logarithmis excessuum, quibus secantes latitudinum superant suas tangentes, quod demonstrare oportuit.

SCHOLIUM.

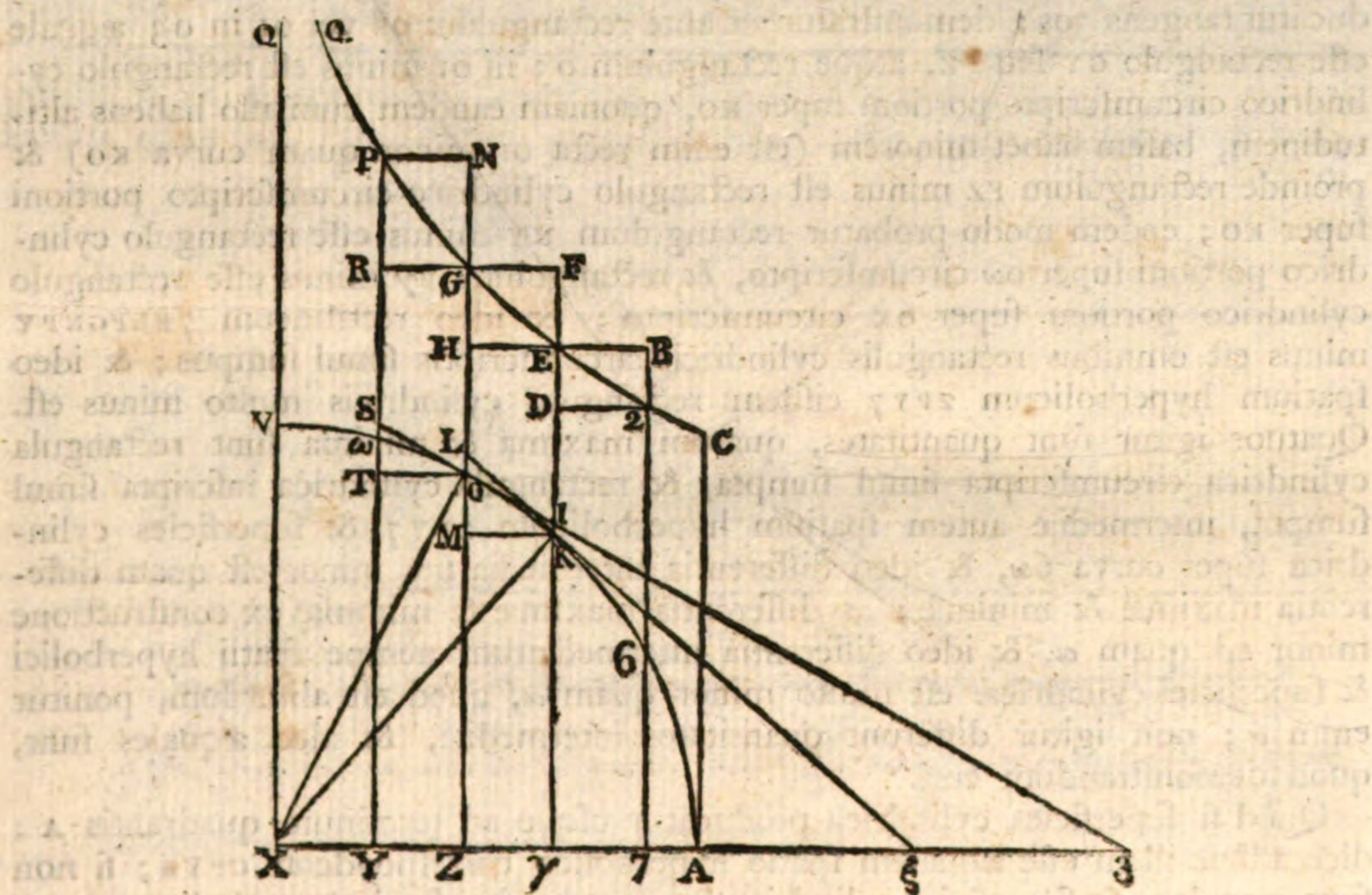
Ex hoc theoremate evidens est methodus describendi integram meridianam, etiam ignota arcus dati in æquatore mensura; quam mensuram ex hujus 2 tali praxe invenimus. Sit QL radius 10000000000, DF 10000000000, & proinde ex nostra Circl. & Hyper. Quadratura prop. 32 invenietur sector hyperbolicus MID 1151292546497022812008, qui eandem habet rationem ad sectorem QKL quam arcus KL in latitudine ad eundem in æquatore, & dividendo utrumque sectorem per $\frac{QL}{2}$, erit ut 23025850929941045624 ad arcum KL ita KL in latitudine ad eundem in æquatore; qualis autem sit arcus KL ita invenimus, ut DF excessus secantis arcus KL supra ejusdem tangentem ad LQ radium, ita LQ ad 100000000000 summam tangentis & secantis ejusdem arcus KL , erit ergo KC tangens arcus KL 49500000000, CQ ejusdem secans 50500000000, & NQ ejusdem sinus 9801980198 $\frac{2}{10000000000}$, è quibus non difficile erit invenire ipsum arcum ope nostræ Quadr. Circ. & Hyp. prop. 30, vel (si quis rudiori calculo contentus fuerit) è tabula sinuum. Ad datam autem arcus dati in æquatore mensuram, meridianam nauticam construere, praxis esset hujus inversa, quæ nullo negotio ex hac colligitur.

Ex prædictis manifestè patet lineam meridianam planisphærii nautici esse scalam tangentium artificialium arcuum, qui sunt semisses complementorum latitudinum,

latitudinum, posito radio loco unitatis, quoniam (ut patet ex trigonometria) predictæ differentiæ sunt eadem cum dictis tangentibus. Si autem complementorum secantes radio insistant, erit figura ex illis conflata nempe $AHLQMA$ ad quadratum radii ut quadrans circumferentiæ ad radium, ut patet ex nostræ Geom. part. universali prop. 2dâ.

PROP. IV. THEOREMA.

Sit circuli quadrans VOA , cujus sit pars 6ω : super arcu 6ω imaginetur superficies cylindrici recti talis naturæ, ut (sumpto in arcu 6ω quolibet puncto o) perpendicularis ad planum VAX ex puncto o ad summitatem superficiæ cylindricæ excitata semper fiat æqualis tangenti arcus OA . Sit hyperbola CPQ cujus vertex c (nempe suppositâ Ac parallelâ & æquali radio xv) & asymptotæ XA , xv ; ducantur rectæ $P\omega Y$, 267 , radio VX parallelæ; dico spatium hyperbolicum $2PY7$ æquale esse dictæ superficiæ cylindricæ; si dictæ figuræ non sint æquales, sit inter illas differentia α , & dividatur recta $7Y$ in tot partes æquales à planis



rectæ $7Y$ perpendicularibus & spatium hyperbolicum in rectis EY , GZ , secantibus, item superficiem cylindricam in diversas portiones dividendibus, ut omnia rectangula cylindrica simul hisce portionibus inscripta differant ab omnibus rectangulis cylindricis simul eisdem circumscriptis minore quantitate quàm α , hoc enim absque dubio fieri potest ab indefinita divisione rectæ $7Y$; intelligo autem rectangulum cylindricum portioni inscriptum esse superficiem cylindricam rectam

cujus eadem est basis cum portione cui inscribitur, & altitudo ubique eadem cum minima altitudine portionis, item circumscriptum cui eadem etiam est basis cum portione, & altitudo eadem cum portionis altitudine maxima. Compleantur rectangula 2γ , $B\gamma$, EZ , FZ , GY , NY ; & ducatur in puncto κ tangens $LK\xi$, fitque in rectam LZ perpendicularis KM . Triangula LKM , $K\xi x$, rectangula ad κ & M sunt similia ob angulos æquales $K\xi x$, LKM ; & ideo ut KM ad KL , ita $K\xi$ ad $x\xi$, & ideo $KL \times K\xi = KM \times x\xi$, atque γE est æqualis rectæ $x\xi$, quod sic probo; $x\xi$ est ad xK vel AC , ut xK ad $x\xi$, sed ob hyperbolam $x\gamma$ est ad AC ut AC ad γE , sunt ergo æquales γE , $x\xi$, & ideo rectangulum ex KL in $K\xi$ æquale est rectangulo ex KM in $E\gamma$ nempe EZ , atque rectangulum ex KL in $K\xi$ majus est rectangulo cylindrico inscripto portioni super κo , quoniam eandem cum illo habens altitudinem nempe $K\xi$ basem habet majorem (est enim recta KL major quam curva κo) & ideo rectangulum EZ majus est rectangulo cylindrico inscripto portioni super κo ; eodem modo probatur rectangulum GY majus esse rectangulo cylindrico inscripto portioni super $o\omega$, & rectangulum 2γ majus esse rectangulo cylindrico inscripto portioni super 6κ ; & proinde rectilineum $72DEHGRY$ majus est omnibus rectangulis cylindricis inscriptis simul sumptis, & ideo spatium hyperbolicum $2PY7$ eisdem rectangulis cylindricis multò majus est. In puncto o ducatur tangens $3os$; demonstratur ut antè rectangulum os vel oi in $o3$ æquale esse rectangulo GY seu FZ , atque rectangulum $o3$ in oi minus est rectangulo cylindrico circumscripto portioni super κo , quoniam eandem cum illo habens altitudinem, basem habet minorem (est enim recta oi minor quam curva κo) & proinde rectangulum FZ minus est rectangulo cylindrico circumscripto portioni super κo ; eodem modo probatur rectangulum NY minus esse rectangulo cylindrico portioni super $o\omega$ circumscripto, & rectangulum $B\gamma$ minus esse rectangulo cylindrico portioni super 6κ circumscripto; & ideo rectilineum $7BEFGNPR$ minus est omnibus rectangulis cylindricis circumscriptis simul sumptis; & ideo spatium hyperbolicum $2PY7$ eisdem rectangulis cylindricis multo minus est. Quatuor igitur sunt quantitates, quarum maxima & minima sunt rectangula cylindrica circumscripta simul sumpta, & rectangula cylindrica inscripta simul sumpta, intermediæ autem spatium hyperbolicum $2PY7$ & superficies cylindrica super curva 6ω , & ideo differentia intermediarum minor est quam differentia maximæ & minimæ; at differentia maximæ & minimæ ex constructione minor est quam α , & ideo differentia intermediarum nempe spatii hyperbolici & superficiæ cylindricæ est multo minor quam α , quod est absurdum, ponitur enim α ; non igitur differunt quantitates intermediæ, & ideo æquales sunt, quod demonstrandum erat.

Quòd si superficies cylindrica producaturs usque ad terminum quadrantis A : dico adhuc illam esse æqualem spatio hyperbolico correspondenti $CPYA$; si non sunt æquales, sit superficies cylindrica super $A\omega$ major spatio hyperbolico $CPYA$, & abscindatur plano $B67$ rectæ AX normali portio superficiæ cylindricæ, ita ut relicta nempe superficies cylindrica super curva 6ω sit æqualis spatio hyperbolico $CPYA$, atque superficies super 6ω æqualis est spatio hyperbolico $2PY7$, ex hactenus demonstratis; spatia igitur hyperbolica $CPYA$, $2PY7$, sunt æqualia, quod est absurdum; superficies ergo cylindrica super $A\omega$ non est major spatio hyperbolico $CPYA$; sit (si fieri potest) minor & à spatio hyperbolico $CPYA$ auferatur rectâ 27 ipsi CA parallelâ spatium $c27A$, ita ut relictum $2PY7$ fiat æquale superficiæ

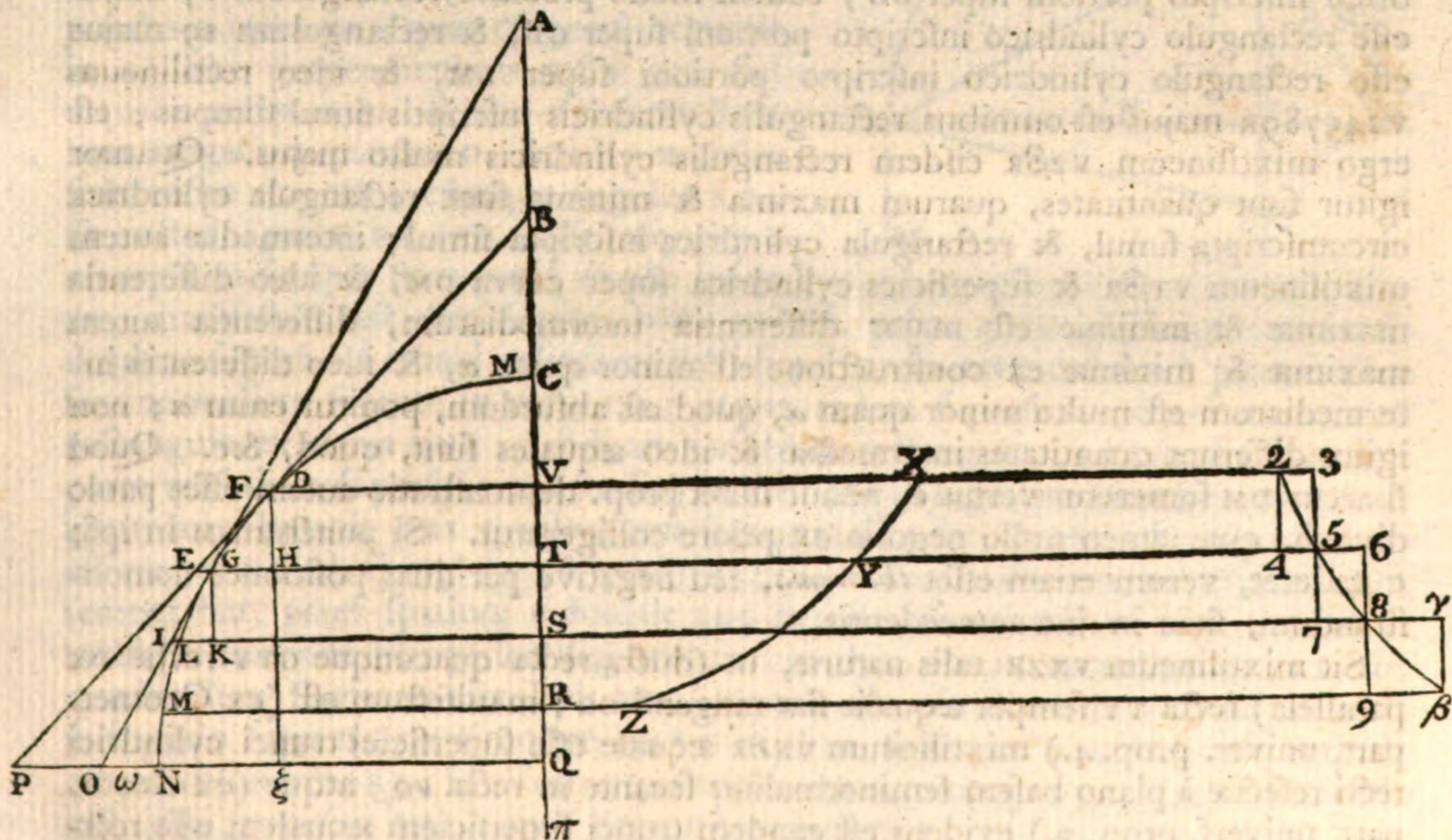
ficiei cylindricæ super $A\omega$, atque spatium hyperbolicum $2PY7$ æquale est superficiei cylindricæ super 6ω ; & ideo superficies cylindricæ super 6ω & super $A\omega$ inter se sunt æquales, quod est absurdum; & ideo superficies cylindrica super $A\omega$ non est minor spatio hyperbolico $CPYA$, sed (ut ante demonstratum est) nec est major; restat igitur, ut illi sit æqualis, quod demonstrandum erat.

CONSECTARIUM.

Hinc manifestum est superficiem cylindricam ex tangentium summa conflata (v. g. super curva $A\omega$) esse secantem artificialem ejusdem arcus $A\omega$, posito radio loco unitatis; nam (ut demonstratur in nostra Circuli & Hyp. Quadr.) posita CA seu radio loco unitatis, erit spatium hyperbolicum $ACPY$ logarithmus rectæ YP secantis arcus $A\omega$.

PROP. V. THEOREMA.

Sit circuli quadrans $QC\omega$ bifariam divisus in puncto D . Sit arcus DM minor quam $D\omega$, super quo imaginetur superficies cylindrici recti talis naturæ, ut (sumpto in arcu DM quolibet puncto G) perpendicularis ad planum $QC\omega$ ex puncto G ad summitatem superficiei cylindricæ excitata semper fiat æqualis secanti arcus GC . In radium QC demittantur rectæ perpendiculares DV , MR ,



quæ producantur in 2 & β ; fiatque mixtilineum $v2\beta R$ talis naturæ, ut (ductâ rectâ quâcunque $GT5$ rectæ DV parallelâ) tangentes arcuum GC , $G\omega$, simul, nempe recta AO , æquales sint rectæ $T5$. Dico mixtilineum $v2\beta R$ æquale esse superficiei cylindricæ super curva DM : si predictæ figuræ non sint æquales, sit inter

inter illas differentia α , & dividatur recta VR in tot partes æquales planis rectæ VR perpendicularibus, & mixtilineum $v2\beta R$ in rectis $\tau 5$, $s8$, secantibus & superficiem cylindricam in diversas portiones dividantibus, ut omnia rectangula cylindrica simul, hisce portionibus inscripta, differant ab omnibus rectangulis cylindricis simul, eisdem circumscriptis minore quantitate quàm α ; hoc enim absque dubio fieri potest ab indefinita divisione rectæ VR. Compleantur rectangula $v4$, $v5$, $\tau 7$, $\tau 8$, $s9$, $s\beta$, & ducatur in puncto G tangens AGO, sitque in rectam LS perpendicularis GK. Triangula IGK, OAQ, sunt similia, & ideo ut GK vel VT ad GI vel GE, ita QA ad AO seu $\tau 5$, & proinde rectangulum VT in $\tau 5$ nempe $v5$ æquale est rectangulo EG in AQ, atque rectangulum EG in AQ minus est rectangulo cylindrico circumscripto portioni super GD, quoniam eandem cum illo habens altitudinem nempe AQ, basem habet minorem; & ideo rectangulum $v5$ minus est rectangulo cylindrico circumscripto portioni super GD; eodem modo demonstratur rectangulum $\tau 8$ minus esse rectangulo cylindrico circumscripto portioni super GI, & rectangulum $s\beta$ minus esse rectangulo cylindrico circumscripto portioni super LM; & ideo rectilineum $v3568\gamma\beta R$ minus est omnibus rectangulis cylindricis circumscriptis simul sumptis, & igitur mixtilineum $v2\beta R$ eisdem rectangulis cylindricis multo minus est. Ob similia triangula DFH, BPQ; DH vel VT est ad DF ut BQ ad BP vel $v2$, & ideo rectangulum $v4$ æquale est rectangulo DF in BQ; at rectangulum DF in BQ majus est rectangulo cylindrico portioni super DG inscripto, quoniam eandem cum illo habens altitudinem BQ, basem habet majorem; & ideo rectangulum $v4$ majus est rectangulo cylindrico inscripto portioni super DG; eodem modo probatur rectangulum $\tau 7$ majus esse rectangulo cylindrico inscripto portioni super GL, & rectangulum $s9$ majus esse rectangulo cylindrico inscripto portioni super LM, & ideo rectilineum $v245789R$ majus est omnibus rectangulis cylindricis inscriptis simul sumptis; est ergo mixtilineum $v2\beta R$ eisdem rectangulis cylindricis multo majus. Quatuor igitur sunt quantitates, quarum maxima & minima sunt rectangula cylindrica circumscripta simul, & rectangula cylindrica inscripta simul; intermediæ autem mixtilineum $v\tau\beta R$ & superficies cylindrica super curva DM, & ideo differentia maximæ & minimæ est major differentiâ intermediarum, differentia autem maximæ & minimæ ex constructione est minor quam α , & ideo differentia intermediarum est multo minor quam α , quod est absurdum, ponitur enim α ; non igitur differunt quantitates intermediæ & ideo æquales sunt, quod, &c. Quod si arcus DM sumeretur versus c, adhuc stare prop. demonstratio autem esset paulo diversa, quæ tamen nullo negotio ex priore colligeretur. Si punctum M in ipso c caderet, verum etiam esset *theorema*, sed negativè per duas positiones demonstrandum, sicut in fine antecedentis.

Sit mixtilineum $v\chi z R$ talis naturæ, ut (ducta recta quacunque GTY rectæ DX parallela) recta TY semper æqualis fiat tangenti GO; manifestum est (ex Geomet. part. univ. prop. 4.) mixtilineum $v\chi z R$ æquale esse superficiei trunci cylindrici recti resectæ à plano basem seminormaliter secante in recta PQ, atque (ex Geom. part. univ. prop. 3.) evidens est eandem trunci superficiem æqualem esse rectangulo $n\zeta$ in cQ, & ideo mixtilineum $v\chi z R$ eidem rectangulo est æquale. Ex prædictis evidens est mixtilineum $x2\beta z$ talis esse naturæ ut $v5$ semper sit æqualis tangenti GA; & proinde mixtilineum $x2\beta z$ unà cum portione quadrantis DVRM æquale est spatio conchoïdali resecto à rectis DV, MR, cujus conchoïdis

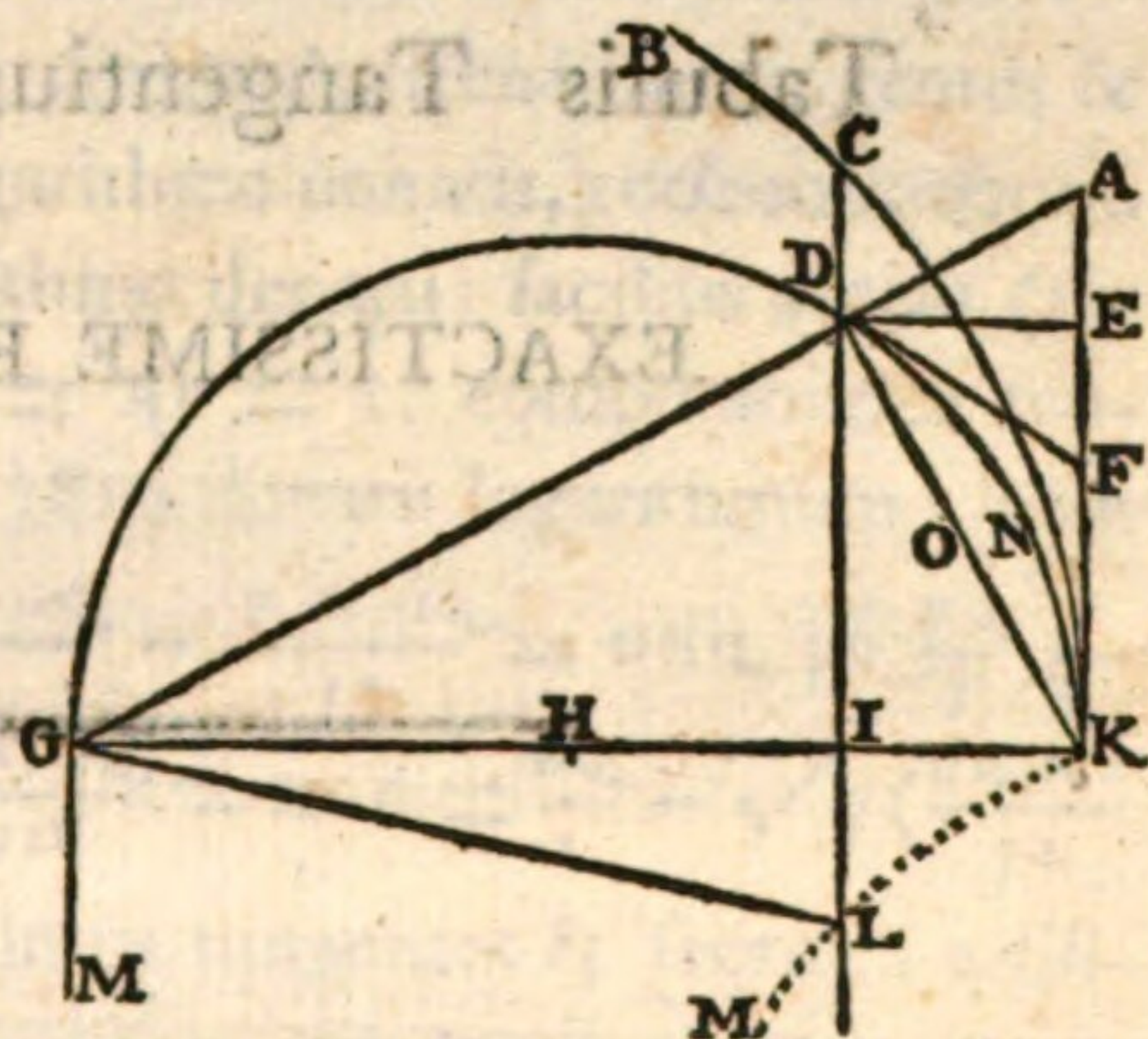
choïdis vertex est c , norma pq , polus π , $cq = q\pi$: ex hac prop. & hujus 2 evidens est sequens confectarium spatio conchoïdali quadrando satis expeditum.

CONSECTARIUM.

Si in prædicta conchoïde accipiatur spatium contentum à curva conchoidali & rectis $CT, T6$; erit prædictum spatium equale duplo spatii hyperbolici (cujus asymptotæ QA, QP , semiaxis QC) contenti à curva hyperbolica una asymptotâ & rectis $QC, QA - GA$, alteri asymptotæ parallelis, una cum semifragmento circulari CCT dempto rectangulo QC in GT . Aliarum conchoïdeon spatia (nempe quando vertex & polus non æqualiter distant à normali) possunt mensurari per Analogiam à *Wallisio* in Epist. Com. pag. 171 demonstratam.

PROP. VI. THEOREMA.

Sit cissois KLM cujus asymptoton GM, semicirculus GDK. Diametro GK sit normalis DIL, & jungantur rectæ DK, GL; dico spatium cissoïdale GKL triplum esse segmenti circularis DNK. Ducatur curva KCB talis naturæ, ut (ductâ ad libitum à puncto G recta GDA quæ tangenti occurrat in A) CDI perpendicularis rectæ GK, fiat æqualis rectæ KA. Recta DF semicirculum tangat in puncto D, & ideo æquales sunt rectæ FD, FK; cùmque angulus KDA sit rectus, patet FA, FD, esse æquales; cùmque CI semper sit æqualis duplæ ipsius DF, manifestum est (ex Geom. part. univ.



DE, manifestum est (ex Geom. part. univer. prop. 4) mixtilineum CKI esse duplum superficiei trunci cylindrici recti super curva DNK resectæ à plano basem seminormaliter secante in recta KA; atque eadem superficiei trunci æqualis est rectangulo ex curva DNK in radium KH ablato rectangulo ex DI in HK (ut satis patet versatis in superficierum studio) hoc est duplo segmenti circularis DNK; est igitur mixtilineum CKI quadruplum segmenti circularis DNK. Ut DI ad IK ita est IK vel DE ad EA vel CD, & proinde æquales sunt rectæ CD, IL, cùmque hoc semper fiat, patet spatium cissoïdale IKL æquale esse mixtilineo DNKC, & idem utrinque addendo nempe semifegmentum circulare DIKN, mixtilineum CKI, seu quadruplum segmenti circularis DNK, æquale est mixtilineo DNKL, & utrinque auferendo segmentum circulare DNK; triplum segmenti circularis DNK æquale est mixtilineo DOKL, seu (ob æqualitatem triangulorum DKI, GIL) ipsi spatio cissoïdali proposito GKL, quod demonstrandum erat.

Hic supponitur arcus DNK quadrante minor, quod si quadrante esset major, nullo negotio variari potest demonstratio ut illi inserviat: at KND existente semicircumferentiâ, multò facilior esset demonstratio, nempe quòd spatium ciffoidale infinitè extensum æquetur semicirculi triplo.

METHODUS

M E T H O D U S

COMPONENDI

Tabulas Tangentium & Secantium Artificialium

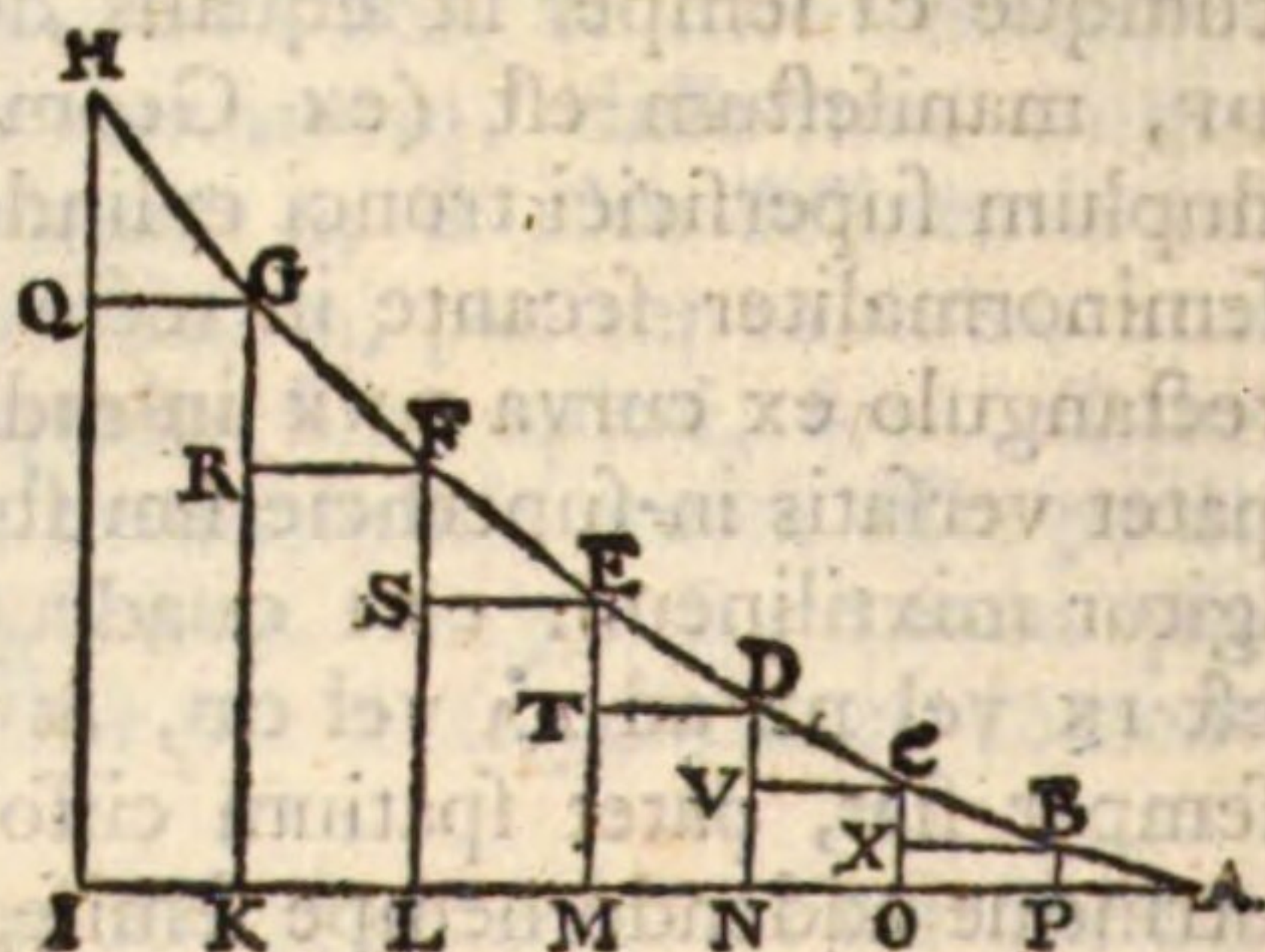
EX

Tabulis Tangentium & Secantium Naturalium

EXACTISSIME ET MINIMO CUM LABORE.

SIT AI arcus quadrantis in lineam rectam extensus, sitque figura AHI conflata ex tangentibus naturalibus singulorum arcuum à puncto A, in debitis suis punctis, rectis AI normaliter insistentibus: sit AP pars minima, in cui æquales dividitur quadrans, nempe $\frac{1}{80}$ vel $\frac{1}{100}$ gradus, sintque illi æquales PO, ON, NM, &c, & ducantur rectæ AI perpendiculares PB, OC, ND, ME, &c, manifestum est ex conspectu 4 hujus, figuram ABP esse secantem artificialem arcus AP item figuram

Aco esse secantem artificialem arcus AO (positâ cyphrâ loco radii artificialis), &c, manifestum est rectangula BO, CN, DM, &c, inveniri ex multiplicatione minimæ partis quadrantis AP in singulas tangentes naturales; at in mensurandis figuris ABP, BCX, CDV, paulò major est difficultas; primo igitur si tangentes conveniant in differentiis primis, non differunt lineæ AB, BC, CD, &c, à rectis, & ideo figuræ ABP, BCX, CDV, &c, erunt triangula rectangula, & proinde, e. g.



GHQ

$GHQ = \frac{HQ \times GQ}{2}$: quod si differentiae secundae fuerint aequales, erunt dictae figurae portiones trilineorum quadraticorum, e. g. erit GHQ portio trilinei quadratici, cujus axi HQ est parallela, differentiae illae inter se aequales sint z ; & proinde erit $GHQ = \frac{HQ \times GQ}{2} - \frac{z \times GQ}{12}$; si autem differentiae tertiae fuerint aequales, erunt dictae figurae portiones trilineorum cubicorum, eritque e. g. $GHQ = \frac{HQ \times GQ}{2} - \sqrt{q} \left(\frac{HQ \times z \times GQ^2}{72} - \frac{z^2 \times GQ^2}{1728} \right)$: quando differentiae quartae sunt aequales, erunt dictae figurae portiones trilineorum Q -quadraticorum, & differentiae quartae erunt aequales 24^{uplo} Q -quadrati ipsius GQ diviso per cubum lateris recti, item quando differentiae quintae sunt aequales, erunt dictae figurae portiones trilineorum sursolidorum, & differentiae quintae erunt aequales 120^{plo} sursolidi ipsius GQ diviso per Q -quadratum lateris recti, & sic in infinitum. Quae hic diximus de compositione secantium artificialium ex tangentibus naturalibus eodem modo intelligi velim de compositione tangentium artificialium ex secantibus naturalibus secundum hujus tertiam. Animadvertendum tangentes & secantes artificiales suprâ computari, posito 0 logarithmo unitatis, 1000000000000 radio, & 2302585092994045624017870 logarithmo denarii: facilius autem (nimirum solâ additione) posito $\frac{1}{60}$ grad. = $GQ = AP = 1$, computabimus tangentes & secantes artificiales ad 7915704467897819 denarii logarithmum; nam hac ratione $\frac{HQ \times GQ}{2} = GHQ = \frac{HQ}{2}$, item $\frac{HQ \times GQ}{2} - \frac{z \times GQ}{12} = GHQ = \frac{HQ}{2} - \frac{z}{12}$, item $\frac{HQ \times GQ}{2} - \sqrt{q} \left(\frac{HQ \times z \times GQ^2}{72} - \frac{z^2 \times GQ^2}{1728} \right) = HGQ = \frac{HQ}{2} - \sqrt{q} \left(\frac{HQ \times z}{72} - \frac{z^2}{1728} \right)$: & tandem unicâ solâ divisione invenimus tangentes & secantes artificiales ad logarithmum denarii 1000000000000000, posito semper radio unitatis loco, quae sunt differentiae tangentium & secantium artificialium in tabula ab ipso radio artificiali; & proinde divisoris multiplices, ad facilitandam operationem, in tabella subsidiaria hic reponimus. Quod si $\frac{1}{60}$ grad. = GQ , tangentes & secantes artificiales debentur logarithmo denarii 13192840779829703, cujus etiam multiplices in subiecta tabella notantur. Quod, si quis velit hos numeros potius repraesentare radium artificialem quam denarii logarithmum, addatur cyphra & habebit intentum. — Notandum hos numeros convenire radio naturali 1000000000000, hinc enim patebit numerus notarum in adscriptis artificialibus.

1	7915704467897819	13192840779829703
2	15831408935795638	26385681559659406
3	23747113403693457	39578522339489109
4	31662817871591276	52771363119318812
5	39578522339489095	65964203899148515
6	47494226807386914	79157044678978218
7	55409931275284733	92349885458807921
8	63325635743182552	105542726238637624
9	71241340211080371	118735567018467327

Ex Epistola D. Jacobi Gregorii ad D. Collins, 15 Feb. Anno 167 $\frac{1}{2}$ data, cujus habetur Autographon.

EX quo epistolam ad te dedi, tres a te accepi, unam Dec. 15, alteram Dec. 24, tertiam 21 *Januarii* nuper elapsi datam.

Quod attinet *Newtoni* methodum universalem, aliqua ex parte, ut opinor, mihi innotescit, tam quoad geometricas quam mechanicas curvas. Nihilo tamen minus ob series ad me missas gratias habeo, quas ut remunerem mitto quæ sequuntur.

Sit radius = r , arcus = a , tangens = t , secans = s ,

$$\text{Et erit } a = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \frac{t^9}{9r^8} \&c.$$

$$\text{Eritque } t = a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} \&c.$$

$$\text{Et } s = r + \frac{a^2}{2r} + \frac{5a^4}{24r^3} + \frac{61a^6}{720r^5} + \frac{277a^8}{8064r^7} \&c.$$

Sit nunc tangens artificialis = t , & secans artificialis = s , & integer quadrans = q ,

$$\text{Erit } s = \frac{a^2}{2r} + \frac{a^4}{12r^3} + \frac{a^6}{45r^5} + \frac{17a^8}{2520r^7} + \frac{62a^{10}}{28350r^9} \&c.$$

$$\text{Sit } 2a - q = e, \& \text{ erit } t = e + \frac{e^3}{6r^2} + \frac{e^5}{24r^4} + \frac{61e^7}{5040r^6} + \frac{277e^9}{72576r^8} \&c.$$

Sit nunc secans artificialis 45 gr. = s , fitque $s + l$ secans artificialis ad libitum, erit ejus arcus = $\frac{1}{2}q + l - \frac{l^2}{r} + \frac{4l^3}{3r^2} - \frac{7l^4}{3r^3} + \frac{14l^5}{3r^4} - \frac{452l^6}{45r^5} \&c,$

$$\text{Eritque } 2a - q = t - \frac{t^3}{6r^2} + \frac{t^5}{24r^4} - \frac{61t^7}{5040r^6} + \frac{277t^9}{72576r^8} \&c.$$

Hic

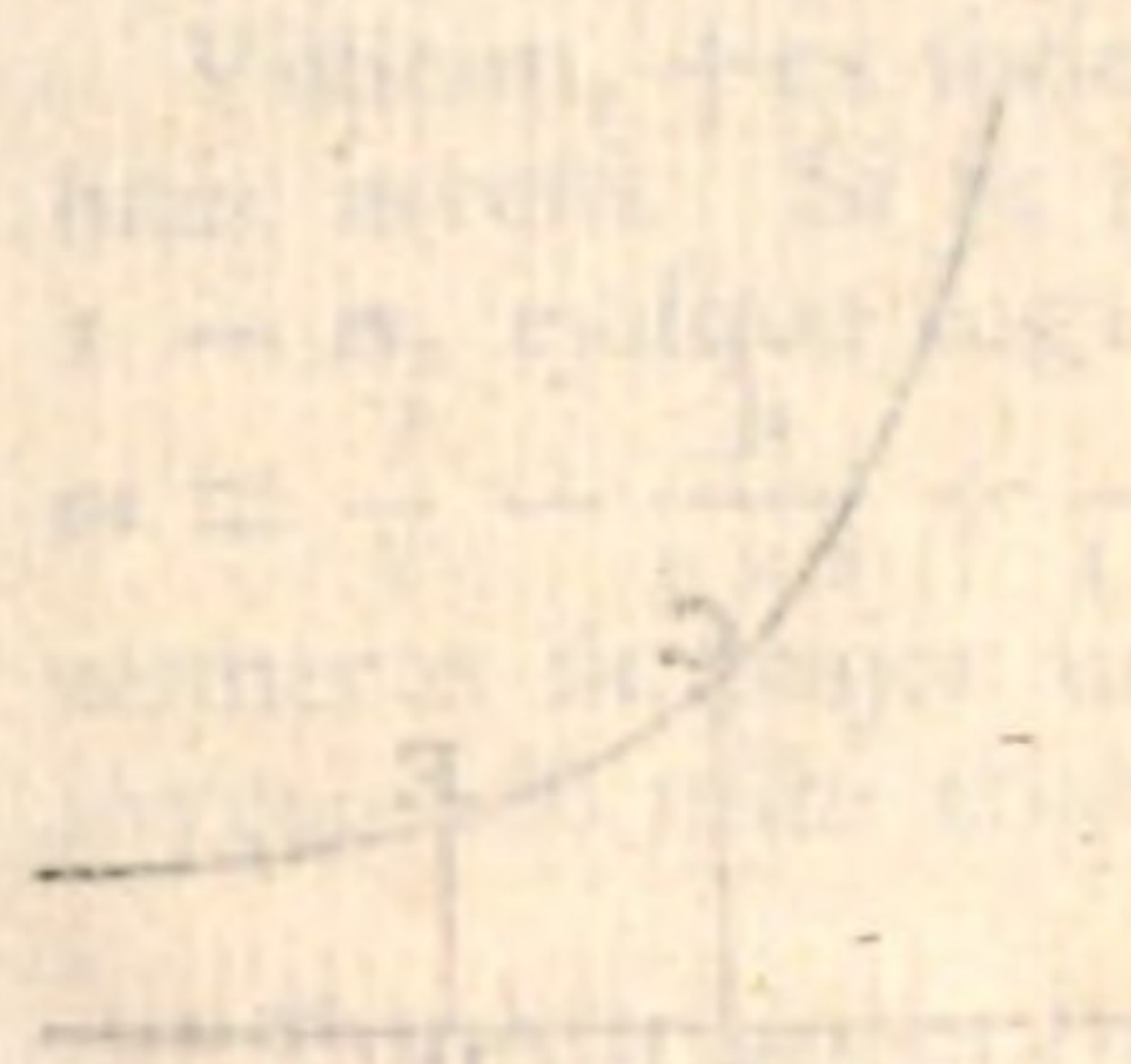
Hic animadvertendum est radium artificialem esse 0; & ubi inveneris q majorem quam $2a$, five artificialem secantem 24 gr. majorem esse data secante, mutanda esse signa, & pergendum secundum vulgaris algebrae præcepta.

Sit ellipsis cujus alter femiaxium $= r$, alter $= c$; ex quolibet curvæ ellipticæ puncto demittatur in femiaxem r recta perpendicularis $= a$: erit curva elliptica perpendiculari a adjacens $= a + \frac{r^2 a^3}{6c^4} + \frac{4r^2 c^2 a^5 - r^4 a^5}{40c^8} + \frac{8c^4 r^2 a^7 + r^6 a^7 - 4c^2 r^4 a^7}{112c^{12}} + \frac{64c^6 r^2 a^9 - 48c^4 r^4 a^9 + 24c^2 r^6 a^9 - 5r^8 a^9}{1152c^{16}}$ &c.

Si determinetur ellipseos species, series hæc simplicior evadet. Ut si $c = 2r$, foret curva prædicta $= a + \frac{a^3}{96r^4} + \frac{3a^5}{2048r^4} + \frac{113a^7}{458752r^6} + \frac{3419a^9}{75497472r^8}$ &c.

Reliquis vero manentibus, si curva prædicta esset hyperbola, prædicta quoque series ei inserviret; si modo omnium terminorum partes affirmantur, & negentur totus terminus tertius, totus quintus, septimus, &c, in locis imparibus.

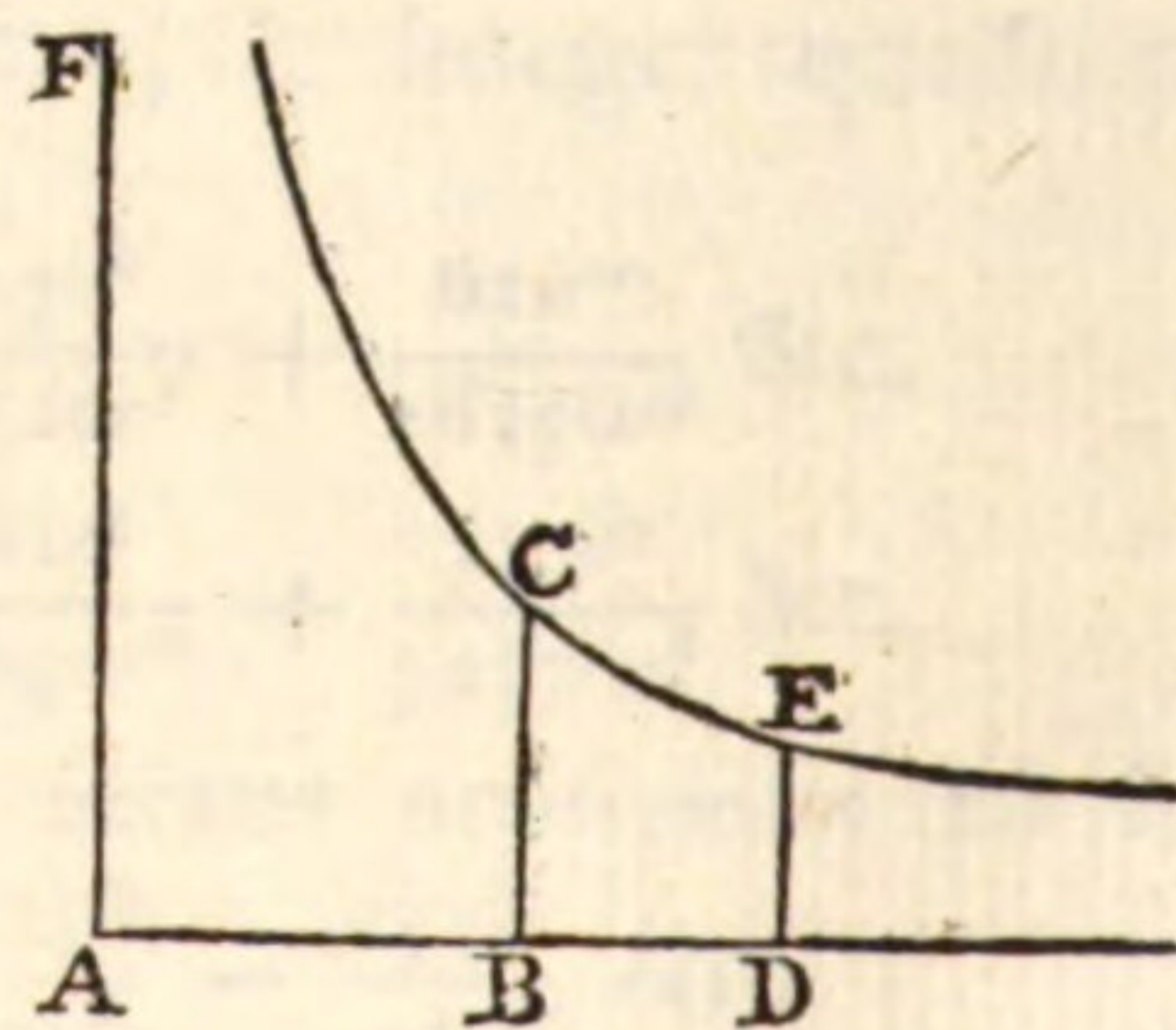
END OF EXTRACTS FROM GREGORY.



OTHER
EXTRACTS
FROM
COLLINS'S COMMERCIIUM EPISTOLICUM;
RELATING TO
LOGARITHMS.

1. *An Extraet from Mr. Isaac Newton's First Epistle to Mr. Henry Oldenburgh, Secretary to the Royal Society of London; with a Direction to communicate the Contents of it to Mr. Godfrey William Leibnitz; dated on the 13th Day of June, in the Year 1676, from Cambridge, where Mr. Newton (who was afterwards Sir Isaac Newton, Knight) was at that Time Professor of the Mathematicks upon Dr. Lucas's Foundation: containing a Discovery relating to Logarithms.*

PRÆTEREA, si sit CE hyperbola, cujus asymptoti AD, AF, rectum angulum FAD constituent; & ad AD erigantur utcunque perpendiculara BC, DE occurentia hyperbolæ in c & E: & AB dicatur a , Bcb, & area BCED z ; erit $BD = \frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} + \&c.$ Ubi coefficientium denominatores prodeunt multiplicando terminos hujus arithmeticae progressionis, 1, 2, 3, 4, 5, &c, in se continuo. Et hinc ex logarithmo dato potest numerus ei competens inveniri.



2. *An Extract from an Epistle of Mr. Godfrey William Leibnitz to Mr. Henry Oldenburgh, Secretary of the Royal Society of London; with a Direction to communicate the Contents of it to Mr. Isaac Newton; dated the 27th Day of August, in the Year 1676: containing a Passage relating to Logarithms.*

OMNIUM verò possibilium circuli, & sectoris conici centrum habentis cujuslibet, per series infinitas quadraturarum, simplicissimam hanc esse dicere ausim quam nunc subjicio.

Sit QAF sector, duabus rectis in centro Q concurrentibus & curvâ conicâ AF, ad verticem A five axis extremum perveniente, comprehensus. Tangenti verticis AT occurrat tangens FT. Ipsum AT vocemus t ; & rectangulum sub semi-latere recto in semi-latus transversum sit unitas. Erit sector hyperbolæ, circuli vel ellipseos, per semi-latus transversum divisus, $= \frac{t}{1} \pm \frac{t^3}{3} + \frac{t^5}{5} \pm \frac{t^7}{7} \&c.$ Signo ambiguo $\pm$ valente $+$ in hyperbola, $-$ in circulo vel ellipsi. Unde, posito quadrato circumscripto 1, erit circulus $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \&c.$ Quæ expressio, jam triennio abhinc & ultrâ à me communicata amicis, haud dubiè omnium possibilium simplicissima est maximeque afficiens mentem.

Unde duco harmonium sequentem *;

$$\begin{array}{cccccccccccccccc} \frac{1}{3} & + & \frac{1}{8} & + & \frac{1}{15} & + & \frac{1}{24} & + & \frac{1}{35} & + & \frac{1}{48} & + & \frac{1}{63} & + & \frac{1}{80} & + & \frac{1}{99} & + & \frac{1}{120} & \&c. & = & \frac{3}{4} \\ \frac{1}{3} & & & & \frac{1}{15} & & & & \frac{1}{35} & & & & \frac{1}{63} & & & & \frac{1}{99} & & & & \&c. & = & \frac{2}{4} \\ & & \frac{1}{8} & & & & \frac{1}{24} & & & & \frac{1}{48} & & & & \frac{1}{80} & & & & \frac{1}{120} & & & & \&c. & = & \frac{1}{4} \\ & & & & & & & & \frac{1}{35} & & & & & & \frac{1}{99} & & & & & & & & \&c. & & & \\ & & & & & & & & & & \frac{1}{48} & & & & & & & & \frac{1}{120} & & & & \&c. & & & \end{array}$$

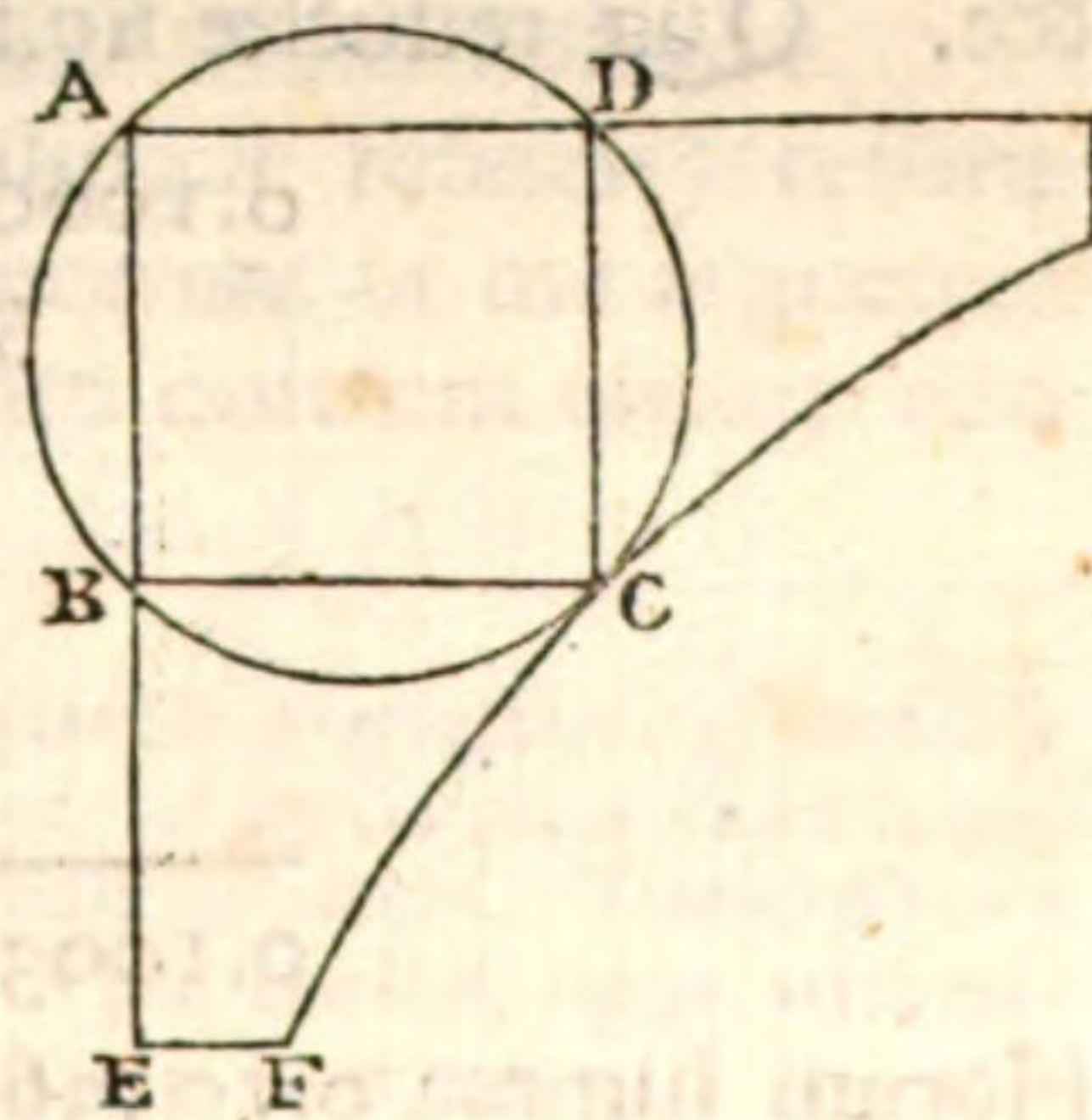
Ubi $\frac{1}{3} + \frac{1}{15} + \frac{1}{99} \&c.$ exprimit aream circuli ABCD, & $\frac{1}{8} + \frac{1}{48} + \frac{1}{120} \&c.$ aream hyperbolæ æquilateræ BCEF, cum sit BC dupla ipsius EF, & quadratum inscriptum $= \frac{1}{4}$. Numeri 3, 8, 15, 24, &c, sunt quadrati unitate minuti.

Vissicim, † ex seriebus regressuum pro hyperbola hanc inveni. Si sit numerus aliquis unitate minor $1 - m$, ejusque logarithmus hyperbolicus l . Erit $m = \frac{l}{1} - \frac{l^2}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} - \frac{l^4}{1 \times 2 \times 3 \times 4} \&c.$ Si numerus sit major unitate, ut $1 + n$; tunc pro eo inveniando mihi etiam † prodiit regula, quæ in *Newtoni* Epistolâ expressa est;

* Vide Acta Lipsica Feb. 1682.

† N. B. Methodum perveniendi ad has series *Leibnitius* à *Newtono* jam modo acceperat, idque ex ipsius rogatu. Imo series ipsas à *Newtono* una cum methodo perveniendi ad easdem jam modo acceperat, & pro hyperbola signum tantum mutavit; pro circulo sinum versum à *Newtono* acceptum subduxit à radio, ut haberet sinum complementi.

scilicet,

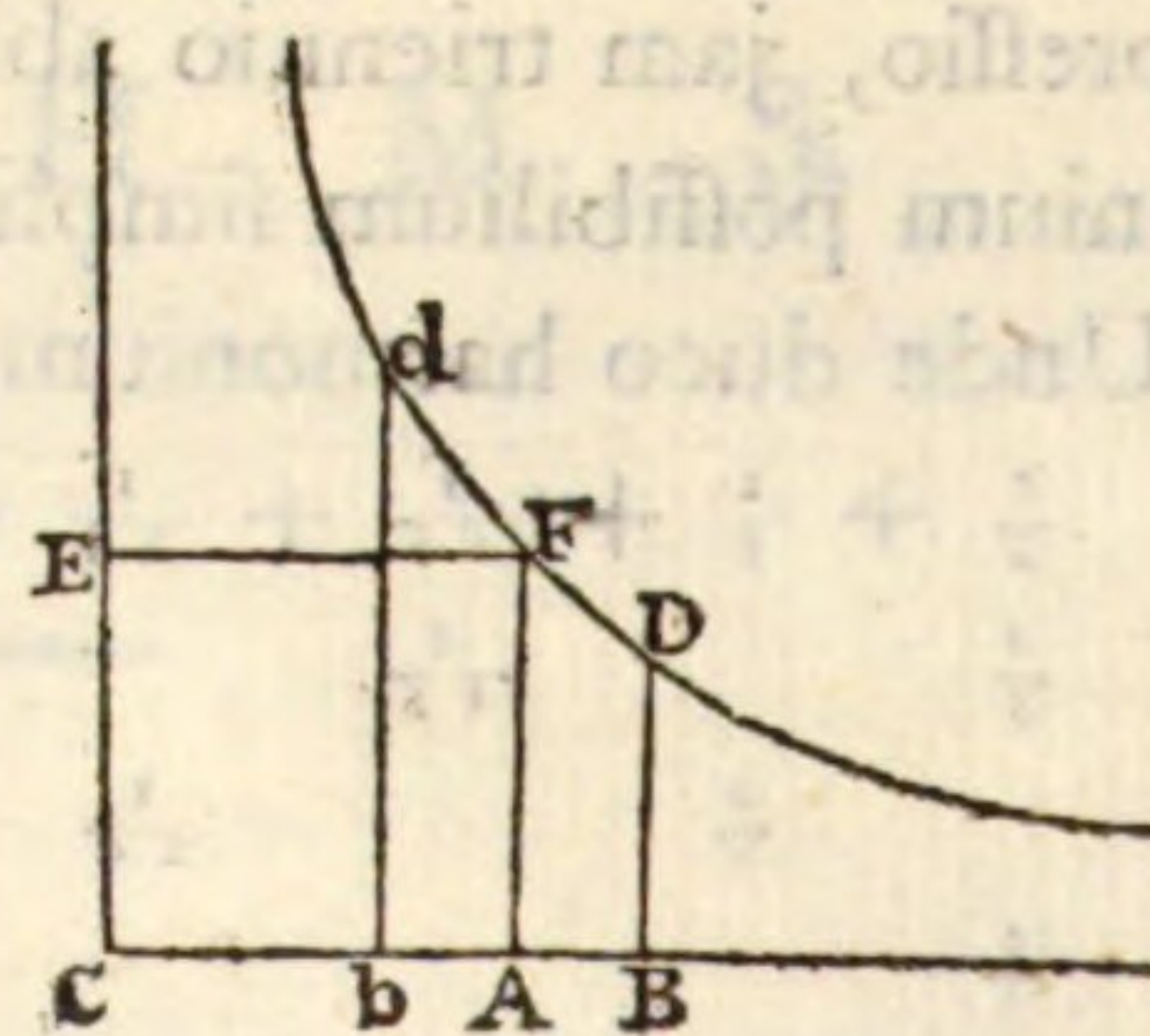


scilicet, erit $n = \frac{1}{1} + \frac{1^2}{1 \times 2} + \frac{1^3}{1 \times 2 \times 3} + \frac{1^4}{1 \times 2 \times 3 \times 4} \&c.$ Prior tamen cele-
rius appropinquat. Ideoque efficio ut ea possim uti, etiam cum major est unitate
numerus $1 + n$. Nam idem est logarithmus pro $1 + n$ & pro $\frac{1}{1+n}$. Unde, si
 $1 + n$ fit major unitate, erit $\frac{1}{1+n}$ minor unitate. Fiat ergo $1 - m = \frac{1}{1+n}$,
ac inventa m , habebitur & $1 + n$ numerus quaesitus.

3. *An Extract from Mr. Isaac Newton's Second Epistle to Mr. Henry Oldenburgh,
Secretary of the Royal Society of London; with a Direction to communicate the Con-
tents of it to Mr. Godfrey William Leibnitz; dated the 24th Day of October, in
the Year 1676: containing some Discoveries relative to Logarithms.*

EO tempore pestis ingruens (quæ contigit annis 1665, 1666), coegit me hinc
fugere, & alia cogitare. Addidi tamen subinde condituram quandam lo-
garithmorum ex area hyperbolæ, quam hic subjungo.

Sit dFD hyperbola, cujus centrum e , vertex
 F , & quadratum interjectum $CAFE = 1$. In AC
cape AB , Ab hinc inde $= \frac{1}{10}$ seu 0.1 : Et, erectis
perpendicularis BD , bd ad hyperbolam terminatis,
erit semi-summa spatiorum AD & $Ad = 0.1 +$
 $\frac{0.001}{3} + \frac{0.00001}{5} + \frac{0.0000001}{7} \&c.$ & semi-diffe-
rentia $= \frac{0.01}{2} + \frac{0.0001}{4} + \frac{0.000001}{6} + \frac{0.00000001}{8}$
&c. Quæ reductæ sic se habent,



0.10000000000000	0.00500000000000
3333333333	2500000000
2000000000	16666666
142857	12500
1111	100
9	1
0.1003353477310	0.0050251679267

Horum summa 0.1053605156577 est Ad ; & differentia 0.0953101798043 est
 AD . Et eadem ratione positis AB , Ab hinc inde $= 0.2$, obtinebitur $Ad =$
 0.2231435513142 , & $AD = 0.1823215567939$. Habitis sic logarithmis hy-
perbolicis numerorum quatuor decimalium 0.8 , 0.9 , 1.1 , & 1.2 ; cum sit
 $\frac{1.2}{0.8} \times \frac{1.2}{0.9} = 2$; & 0.8 & 0.9 , sint minores unitate: adde logarithmos eorum
ad duplum logarithmi 1.2 , & habebis 0.6931471805597 logarithmum hyper-
bolicum

bolicum numeri 2. Cujus triplo adde log. 0.8 (siquidem fit $\frac{2 \times 2 \times 2}{0.8} = 10$) & habebis 2.3025850929933 logarithmum numeri 10: Indeque per additionem simul prodeunt logarithmi numerorum 9 & 11: adeoque omnium primorum horum 2, 3, 5, 11 logarithmi in promptu sunt. Insuper, ex sola depreffione numerorum superioris computi per loca decimalia & additione, obtinentur logarithmi decimalium 0.98, 0.99, 1.01, 1.02; ut & horum 0.998, 0.999, 1.001, 1.002. Et inde per additionem & subductionem prodeunt logarithmi primorum 7, 13, 17, 37, &c. Qui unà cum superioribus, per logarithmum numeri 10 divisi, evadunt veri logarithmi in tabulam inferendi. Sed hos postea propius obtinui.

Pudet dicere ad quot figurarum loca has computationes, otiosus eo tempore, produxi. Nam tunc sane nimis delectabar inventis hisce. Sed ubi prodiit ingeniosa illa * *Nicolai Mercatoris Logarithmotechnia* (quem suppono sua primum invenisse), cœpi ea minus curare; suspicatus, vel eum nosse extractionem radicum æque ac divisionem fractionum; vel alios saltem, divisione patefacta, inventuros reliqua, prius quam ego ætatis essem maturæ ad scribendum.

Eo ipso tamen tempore quo liber iste prodiit, communicatum est per amicum D. Barrow (tunc Matheseos Professore Cantab.) cum D. Collinio, † compendium quoddam methodi harum serierum; in quo significaveram areas & longitudines curvarum omnium, & solidorum superficies & contenta, ex datis rectis; & vice versa, ex his datis rectas determinari posse: & methodum ibi indicatum illustraveram diversis seriebus.

Suborta deinde inter nos epistolari consuetudine; D. Collinius, vir in rem mathematicam promovendam natus, non destitit suggerere ut hæc publici juris facerem. Et ante annos quinque (1671) cum suadentibus amicis consilium ceperam edendi tractatum de Refractione Lucis, & Coloribus, quem tunc in promptu habebam; cœpi de his seriebus iterum cogitare; & ‡ tractatum de iis etiam conscripsi, ut utrumque simul ederem.

Sed, ex occasione telescopii catadioptrici, epistolâ ad te missâ qua breviter explicui conceptus meos de natura lucis, inopinatum quiddam effecit ut mei interesse sentirem ad te festinanter scribere de impressione istius epistolæ. Et subortæ statim per diversorum epistolas (objectionibus aliisque refertas) crebræ interpellationes me prorsus à consilio deterruerunt; & effecerunt ut me arguerem imprudentiæ, quod umbram captando, eatenus perdideram quietem meam, rem prorsus substantialem.

* Mathematici priores invenerunt hoc theorema, quod *summa terminorum progressionis geometricæ in infinitum pergentis est ad terminorum primum & maximum, ut hic terminus ad differentiam duorum terminorum primorum*. Idem demonstratur arithmetice multiplicando extrema & media. Demonstravit Wallisius dividendo rectangulum sub mediis per extremum ultimum. Vide Wallisii opus arithmeticum anno 1657 editum, cap. 33 § 68. Per Wallisii divisionem Mercator demonstravit & auxit quadraturam hyperbolæ à D. Brounker prius inventam. Et Gregorius idem demonstravit geometricè. Sed horum nemo methodum generalem quadrandi curvas per divisionem invenit. Mercator hoc nunquam professus est. Gregorius ejusmodi methodum, licet vir acutissimus & literis Collinii admonitus, vix tandem invenit. Newtonus invenit per interpolationem serierum, & postea divisionibus & extractionibus radicum, ut notioribus, usus est.

† Analysin intelligit per æquationes infinitas.

‡ Hujus tractatus meminit D. Collins in epistolis duabus impressis, pag. 101, 102, *Commerc. Epistol.* Et Newtonus in epist. impressa, p. 205 ibid.

Sub eo tempore *Jacobus Gregorius*, ex unica quadam serie è meis, quam D. *Collinius* ad eum transmiserat, post multam considerationem (ut ad *Collinium* rescripsit) pervenit ad eandem methodum, & tractatum de eâ reliquit quem speramus ab amicis ejus editum iri. Siquidem, pro ingenio quo pollebat, non potuit non adjicere de suo nova multa, quæ rei mathematicæ interest ut non pereant.

Ipse autem tractatum meum non penitus absolveram, ubi destiti à proposito; neque in hunc diem mens rediit ad reliqua adjicienda. Deerat quippe pars ea qua decreveram explicare modum solvendi problemata, quæ ad quadraturas reduci nequeunt; licet aliquid de fundamentis ejus posuissem. Cæterum in tractatu isto, series infinitæ non magnam partem obtinebant.

Neque majori labore eruitur area totius circuli ex segmento cujus sagitta est quadrans diametri. Ejus computi specimen, siquidem ad manus est, visum fuit apponere; & unâ adjungere aream hyperbolæ quæ eodem calculo prodit.

Posito axe transverso = 1, & sinu verso seu segmenti sagitta = x ; erit semi-segmentum hyperbolæ } = $x^{\frac{1}{2}}$ in $\frac{2}{3} x \pm \frac{xx}{5} - \frac{x^3}{28} \pm \frac{x^4}{72}$ &c. Hæc autem series
circuli }
sic in infinitum producitur, sit $2x^{\frac{3}{2}} = a$, $\frac{ax}{2} = b$, $\frac{bx}{4} = c$, $\frac{3cx}{6} = d$, $\frac{5dx}{8} = e$, $\frac{7ex}{10} = f$, &c. Et erit semi-segmentum hyperbolæ } = $\frac{a}{3} \pm \frac{b}{5} - \frac{c}{7} \pm \frac{d}{9} - \frac{e}{11}$
circuli } $\pm \frac{f}{13}$ &c. Eorumque semi-summa $\frac{a}{3} - \frac{c}{7} - \frac{e}{11} - \dots$ & semi-differentia $\frac{b}{5} + \frac{d}{9} + \frac{f}{13} + \dots$. His ita præparatis, suppono $x = \frac{1}{4}$, quadrantem nempe axis; & prodit $a (= \frac{1}{4}) = 0.25$; $b (= \frac{ax}{2} = \frac{0.25}{1 \times 8}) = 0.03125$; $c (= \frac{bx}{4} = \frac{0.03125}{2 \times 8}) = 0.001953125$; $d (= \frac{3cx}{6} = \frac{0.001953125}{8}) = 0.000244140625$. Et sic procedo usque dum venero ad terminum depressissimum, qui potest ingredi opus. Deinde hos terminos per 3, 5, 7, 9, 11, &c, respective divisos dispono in duas tabulas: ambiguos cum primo in unam; & negativos in aliam; & addo ut hic vides,

0.0833333333333333
6250000000000000
2712673611111111
5135169396
144628917
4954581
190948
7963
352
16
1

0.0896109885646618

0.0002790178571429
34679066051
834465027
26285354
961296
38676
1663
75
4
0.0002825719389575

Tunc

Tunc à priori summa aufero posteriorem, & restat 0.0893284166257043 area semi-segmenti hyperbolici. Addo etiam eas summas, & aggregatum aufero a primo termino duplicato 0.1666666666666666, & restat 0.0767731061630473 area semi-segmenti circularis. Huic addo triangulum istud quo completur in sectorem, hoc est $\frac{1}{3}\sqrt{3}$, seu 0.0541265877365274, & habeo sectorem 60 graduum, 0.1308996938995747, cujus sextuplum 0.7853981633974482 est area totius circuli: quæ divisa per $\frac{1}{4}$ five quadrantem diametri, dat totam peripheriam 3.1415926535897928. Si alias artes adhibuissim, potui per eundem numerum terminorum seriei pervenisse ad multo plura loca figurarum, puta viginti quinque aut amplius: sed animus fuit hic ostendere, quid per simplex seriei computum præstari posset. Quod sane haud difficile est, cum in omni opere multiplicatores ac divisores magna ex parte non majores quam 11, & nunquam majores quam 41 adhibere opus sit.

Neque observasse videtur [clarissimus *Leibnitius*] morem meum generaliter usurpandi literas pro quantitatibus cum signis suis + & - affectis, dum dividit

hanc seriem $\frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \&c.$ Nam

cum area hyperbolica BE, hic significata per z , sit affirmativa vel negativa, prout jaceat ex una vel altera parte ordinatim applicatæ BC; si area illa in numeris data sit l , & l substituatur in serie pro z , orietur vel $\frac{l}{b} + \frac{ll}{2abb} + \frac{l^3}{6aab^3} + \frac{l^4}{24a^3b^4} \&c.$ vel $-\frac{l}{b} +$

$\frac{ll}{2abb} - \frac{l^3}{6aab^3} + \frac{l^4}{24a^3b^4} \&c.$ prout l sit affirmativa vel negativa. Hoc est posito

$a = 1 = b$, & l logarithmo hyperbolico; numerus ei correspondens erit $1 +$

$\frac{l}{1} + \frac{ll}{2} + \frac{l^3}{6} + \frac{l^4}{24} \&c.$ si l sit affirmativus; & $1 - \frac{l}{1} + \frac{ll}{2} - \frac{l^3}{6} + \frac{l^4}{24}$

&c, si l sit negativus. Hoc modo fugio multiplicationem theorematum, quæ alias in nimiam molem crescerent. Nam v. g. illud unicum theorema, quod supra posui pro quadratura curvarum, resolvendum esset in 32 theoremata, si pro signorum varietate multiplicaretur.

Præterea, quæ habet vir clarissimus de inventione numeri unitate majoris per

datum logarithmum hyperbolicum, ope seriei $\frac{l}{1} - \frac{ll}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} - \frac{l^4}{1 \times 2 \times 3 \times 4}$

+ &c, potius quam ope seriei $\frac{l}{1} + \frac{ll}{1 \times 2} + \frac{l^3}{1 \times 2 \times 3} + \frac{l^4}{1 \times 2 \times 3 \times 4} + \&c.$ non

dum percipio. Nam si unus terminus adjiciatur amplius ad seriem posteriorem

quam ad priorem, posterior magis appropinquabit. Et certe minor est labor com-

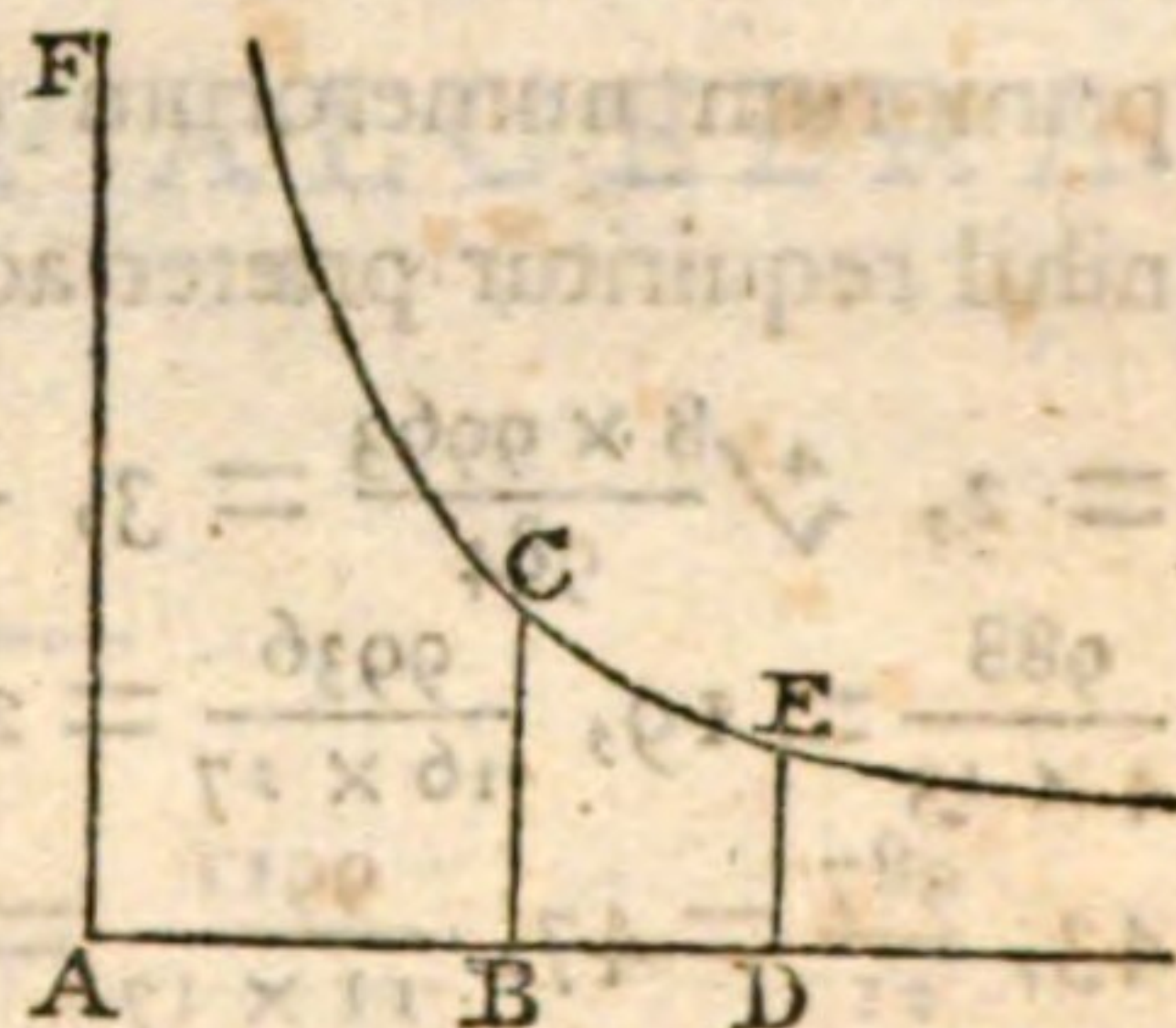
putare unam vel duas primas figuras adjecti hujus termini, quam dividere unitatem

per numerum prodeuntem ex logarithmo hyperbolico ad multa figurarum loca

extensum, ut inde habeatur numerus quæsitus unitate major. Utraque igitur

series (si duas dicere fas sit) officio suo fungatur. Potest tamen $\frac{l}{1} + \frac{l^3}{1 \times 2 \times 3} +$

$\frac{l^5}{1 \times 2 \times 3 \times 4 \times 5} \&c.$ series, ex dimidia parte terminorum constans, optime adhiberi;



siquidem hæc dabit femi-differentiam duorum numerorum, ex qua et rectangulo dato uterque datur. Sic & ex serie $1 + \frac{n}{1 \times 2} + \frac{n^2}{1 \times 2 \times 3 \times 4}$ &c, datur femi-summa numerorum, indeque etiam numeri. Unde prodit relatio serierum inter se, qua ex unâ datâ dabitur altera.

Constructionem logarithmorum non aliunde peti debere credetis forte, ex hoc simplici processu qui ab istis pendet. Per methodum supra traditam quærantur logarithmi hyperbolici numerorum 10, 0.98, 0.99, 1.01, 1.02: id quod fit spatio unius & alterius horæ. Dein divisis logarithmis quatuor posteriorum per logarithmum numeri 10, & addito indice 2, prodibunt veri logarithmi numerorum 980, 99, 100, 101, 102, in tabulam referendi. Hi per dena intervalla interpolandi sunt, & exhibunt logarithmi omnium numerorum inter 980 & 1020: & omnibus inter 980 & 1000 iterum per dena intervalla interpolatis, habebitur tabula eatenus constructa. Tunc ex his colligendi erunt logarithmi omnium primorum numerorum & eorum multiplicium, minorum quam 100: ad quod nihil requiritur præter additionem & subtractionem. Siquidem fit $\sqrt[10]{\frac{9984 \times 1020}{9945}}$

$= 2$, $\sqrt[8]{\frac{9963}{984}} = 3$, $\frac{10}{2} = 5$, $\sqrt{\frac{98}{2}} = 7$, $\frac{99}{9} = 11$, $\frac{1001}{7 \times 11} = 13$, $\frac{102}{6} = 17$,
 $\frac{988}{4 \times 13} = 19$, $\frac{9936}{16 \times 27} = 23$, $\frac{986}{2 \times 17} = 29$, $\frac{992}{3^2} = 31$, $\frac{999}{27} = 37$, $\frac{984}{24} = 41$, $\frac{989}{23} =$
 43 , $\frac{987}{21} = 47$, $\frac{9911}{11 \times 17} = 53$, $\frac{9971}{13 \times 13} = 59$, $\frac{9882}{2 \times 81} = 61$, $\frac{9949}{3 \times 49} = 67$, $\frac{994}{14} = 71$,
 $\frac{9928}{8 \times 17} = 73$, $\frac{9954}{7 \times 18} = 79$, $\frac{996}{12} = 83$, $\frac{9968}{7 \times 16} = 98$, $\frac{9894}{6 \times 17} = 97$. Et habitis sic logarithmis omnium numerorum minorum quam 100, restat tantum hos etiam semel atque iterum per dena intervalla interpolare.

END OF EXTRACTS FROM COLLINS.

THE

T H E

T W E L F T H C H A P T E R

O F

D R . W A L L I S ' S T R E A T I S E O F A L G E B R A .

Of Logarithms : Their Invention and Use.

THE other improvement which I mentioned (as added to the algorisme of the Arabs since we borrowed it from them), is that of Logarithms; an improvement of our own age and nation.

This was first of all invented (without any example of any before him, that I know of) by John Neper, Baron of Merchiston in Scotland; and by him first published at Edinburgh, in the year 1614: and soon after by himself (with the assistance of Henry Briggs, Professor of Geometry, first at London, in Gresham College, and afterwards at Oxford) reduced to a better form, and perfected.

The invention was greedily embraced (and deservedly) by learned men.

Mr. Briggs, upon the first publication of it, was so pleased with it, that he presently repaired into Scotland, to consult the author, advise with him, and be assistant to him in the perfecting of it, and in calculating tables for it; which was a work of great labour, as well as subtile invention.

And it was embraced and promoted abroad by Benjamin Urfinus, John Kepler, Adrian Ulack, Petrus Crugerus, and others.

And at home by Henry Gellebrand, who perfected the Trigonometria Britannica, which Mr. Briggs began, but died before it was perfected.

So that, in a short time, it became generally known, and greedily embraced in all parts, as of unspeakable advantage; especially for ease and expedition in trigonometrical calculations.

The foundation of it is this:

If to a rank of continual proportionals in a geometrical progression from 1: suppose

1. 2. 4. 8. 16. 32. 64. &c.

We accommodate a rank of exponents in an arithmetical progression, from 0: suppose

0. 1. 2. 3. 4. 5. 6. &c.

It is manifest, that for every multiplication or division of those terms one by another, there is an answerable addition or subduction of the exponents.

For as (in the terms) 4 multiplied by 8 makes 32, so (in the exponents) if to 2 we add 3, it makes 5; and as 32 divided by 8, gives 4: so if from 5 we subtract 3, there remains 2: and so every where.

Terms.	1.	2.	4.	8.	16.	32.	64.
Exponents.	0.	1.	2.	3.	4.	5.	6.

$$4 \times 8 = 32. \quad \frac{32}{8} = 4.$$

$$2 + 3 = 5. \quad 5 - 3 = 2.$$

(Not much unlike to what we before shewed out of Archimedes's Arenarius, concerning his α , β , γ , δ , &c, in continual progression geometrical from 1, attended by a series of exponents in arithmetical progression; the foundation of that and this being all one.)

And the same holds, if between any two of those terms, interpose one or more means proportional; and between their exponents, as many arithmetical means.

As if between 4 and 8 (or between 2 and 16) we interpose a mean proportional $\sqrt{32}$, that is $4\sqrt{2}$; and between 2 and 3 (or 1 and 4) an arithmetical mean, $2\frac{1}{2}$; then as $4\sqrt{2}$ by 8 makes $32\sqrt{2}$ (a mean proportional between 32 and 64): so adding their exponents $2\frac{1}{2}$ and 3, makes $5\frac{1}{2}$, an arithmetical mean between 5 and 6; and so every where.

And universally, (whatever be the values of r , e .) supposing

The terms, 1. r . rr . rrr . r^4 . r^5 . r^6 . &c.

Exponents, 0. e . $2e$. $3e$. $4e$. $5e$. $6e$. &c.

Then, as $rr \times r^3 = r^5$, and $rr\sqrt{r} \times rrr = r^5\sqrt{r}$;

So $2e + 3e = 5e$, and $2\frac{1}{2}e + 3e = 5\frac{1}{2}e$.

And so every where.

And consequently whatever term we interpose between any of those continual proportionals; if we also interpose between their exponents, a like arithmetical mean, as that is a proportional mean (as if that be the first or second of two means proportional, this accordingly is the first or second of two means arithmetical; if that the second of five means proportional, this the second of as many

many arithmetical means, &c) then to every addition or subduction of these one with another, will answer a like multiplication or division of those.

And if for $o, e, 2e, 3e, \&c$, (taking $e = 1$) we put $o, 1, 2, 3, \&c$, then doth this exponent always give us the number of ratios or dimensions in the term to which it belongs.

1. $r. rr. r^3. r^4. r^5. r^6. \&c.$

0. 1. 2. 3. 4. 5. 6. &c.

(as 3 in r^3 , 6 in r^6 , and so every where) or shews how many fold (*quam multiplicata*) the proportion (for instance) of r^6 to 1 is of r to 1. That is, how many ratios or proportions of r to 1 are compounded in r^6 to 1, to wit 6; to which the name Logarithmus fitly answers, that is $\lambda\acute{o}\gamma\omega\nu \acute{\alpha}\rho\iota\theta\mu\acute{o}\varsigma$, the number or proportions so compounded.

Now this foundation being laid, their design in the logarithms is this: Having selected (as most convenient) a rank of continual proportionals, in a decuple progression; to wit,

1. 10. 100. 1000. 10000. 100000. 1000000. &c.

they fit hereunto (as their exponents) in arithmetical progression,

0. 1. 2. 3. 4. 5. 6. &c.

(And consequently the logarithm of any fractions less than 1 is to be a negative number.) And then for each of the numbers interposed between 1 and 10, between 10 and 100, and so of the rest (as 2, 3, 4, &c, 11, 12, 13, &c); they seek out (between 0 and 1, between 1 and 2, &c) an exponent (to be expressed in decimal parts), which is such a mean arithmetical as the other is a mean proportional.

And these exponents they call logarithms, which are artificial numbers, so answering to the natural numbers, as that the addition and subduction of these answers to the multiplication and division of the natural numbers.

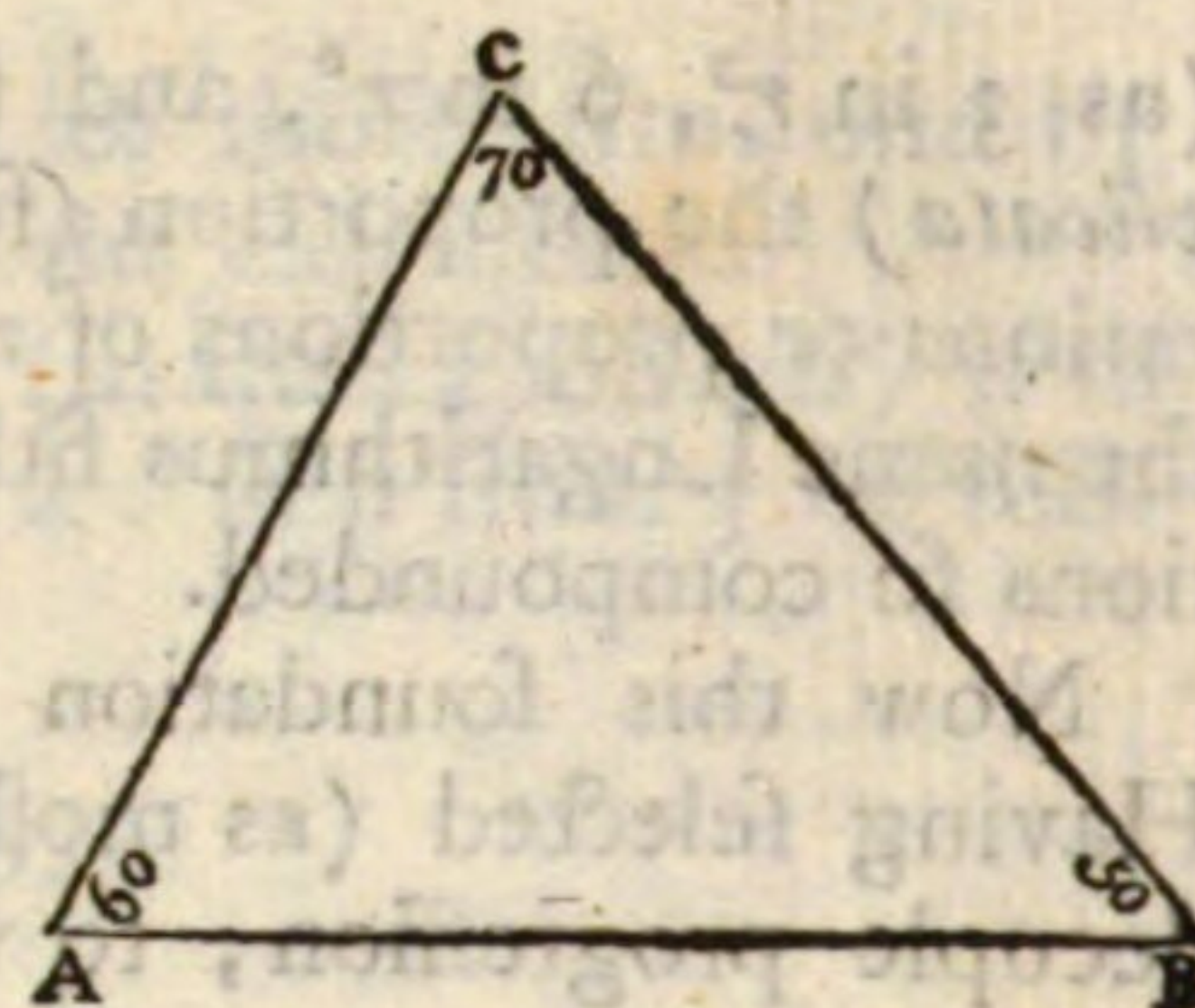
By this means (the tables being once made) the work of multiplication and division is performed by addition and subduction; and consequently that of squaring and cubing, by duplation and triplation, and that of extracting the square and cubick root, by bisection and trisection; and the like in higher powers.

Of these logarithms we have printed tables, for all numbers as far as one hundred thousand, so that if any two numbers (not exceeding 100,000) be proposed to be multiplied or divided one by the other, the logarithms of those numbers (to be found in those printed tables) being accordingly added or subducted, will give the logarithm of that natural number (to be found by those tables), which is the product or quotient of such multiplication or division. And the double or treble of such logarithm, is the logarithm of its square or cube. And the half or third part of it, is the logarithm of its quadratick or cubick root; and the like of higher powers, which, in large numbers, is matter of great expedition.

And

And (because a main end of this design was to facilitate astronomical and other trigonometrical calculations) beside those logarithms for numbers in their natural order, we have also tables of artificial or logarithmical fines, tangents, and secants; the addition and subduction of which answers to the multiplication and division of the natural fines, tangents, and secants: which is a very compendious advantage for expediting such calculations, and is not less accurate than the operation by tables of natural fines, tangents, and secants.

Thus in a plain triangle; supposing the angles given, A 60 degrees, B 50 degrees (and consequently C 70 degrees), and the side AB 31323 paces: for finding the sides AC, or BC, we have this proportion:



As the fine of c, 70 degrees,	9396926
To the fine of B, 50 degrees,	7660444
So is the side AB,	<u>31323</u> paces.
To the side AC.	25535—paces.

For finding which, we are to multiply 7660444 by 31323, and then divide by 9396926; which gives for the side AC (almost) 25535 paces.

And, as the fine of c, 70 degrees,	9396926
To the fine of A, 60 degrees,	8660254
So is the side AB,	<u>31323</u> paces.
To the side BC.	28867 $\frac{1}{2}$ paces.

For finding which, we are to multiply 8660254 by 31323, and divide by 9396926, which gives for the side BC, 28867 $\frac{1}{2}$ paces, *proximè*.

Now (to prevent these tedious multiplications and divisions) by logarithms, we proceed thus:

Log. fine c, 70 degrees,	— 9.9729858
Log. fine B, 50 degrees,	+ 9.8842540
Log. AB, num. 31323,	+ <u>4.4958633</u>
Log. AC, num. 25535,	+ 4.4071315

where subducting the first logarithm from the sum of the second and third, gives the fourth, which (the tables tell us) answers to the number 25535, *ferè*. So many paces therefore is the side AC.

Again, Log. fine c, 70 degrees,	— 9.9729858
Log. fine A, 60 degrees,	+ 9.9375306
Log. AB, num. 31323	+ <u>4.4958633</u>
Log. BC, num. 28867 $\frac{1}{2}$	+ 4.4604081

Where

Where subducting the first logarithm from the sum of the second and third, gives the fourth; which (the table tells us) answers to the number $28867\frac{1}{2}$, *proximè*. So many paces therefore is the side BC; which operations are much more expeditious than multiplying and dividing such large numbers.

And in like manner, in spherical triangles, save that there all the logarithms are to be taken out of the tables of sines, tangents, and secants; which in this example are taken partly from thence, partly from the table of numbers; but the expedition is alike in both.

This was first published by the Lord Neper (the first inventor of it), in the year 1614, under the title of *Mirificus Logarithmorum Canon*, with its description and use; but reserving the manner of construction, and its demonstration, to be after published; this being but an essay set forth to see the judgment of learned men concerning this design, and how it was like to be received.

In this we have a canon or table of natural and logarithmical sines for each degree and minute of the quadrant.

And whereas it was at his choice to give to what number he pleased the logarithm 0, and whether to proceed by way of increase or decrease, he chose to make 0 the logarithm of the whole sine 10000000, that so the multiplication or division by the whole sine (frequent in trigonometrical calculation) might be dispatched without trouble, requiring here but the addition or subduction of 0.

And because the use of lesser sines and numbers, less than the radius or whole sine, were likely to be of more frequent use than of tangents, secants, and other numbers greater than the radius, he chose to give to those lesser numbers affirmative logarithms (increasing the logarithms from 0, as the sines decrease), which he calls *abundantes*: and consequently negative logarithms (which he calls *defectives*) to greater numbers. Designing those by +, these by —.

And, by this means, he directs how this table of sines (with the differences there inserted) may serve also for a table of tangents and of secants: so that this canon is a complete canon of natural sines, and of logarithmical sines, tangents, and secants.

He shews also how this table may be applied to the logarithms of absolute numbers: but because with some trouble, he reserves the fuller account hereof to a farther treatise.

In the year 1619, the Lord Neper being then dead, the same was again published by his son, Robert Neper, with some posthumous treatises of his father, concerning the construction of this Logarithmical Canon, and concerning his design (after communication had with Mr. Briggs) of changing the form of logarithms, making 0 to be the logarithm of 1 (of which he had before given notice in the preface to his *Rabdologia*, published in the year 1617), and concerning some things pertaining to trigonometry; with some lucubrations of Mr. Briggs on the same subject.

But, the Lord Neper being dead, the whole work was devolved on Mr. Briggs, who (according to their joint advice) making the logarithm of 1 to be 0, and of 10, 100, 1000, &c, to be 1, 2, 3, &c, which he calls *indices*, or *characteristics*, and which we may repute as integer numbers, with fourteen ciphers annexed, which we may repute as so many places of decimal fractions

below

below the place of units, or of the characteristick : and between these he fits the intermediate logarithms for the intermediate numbers.

And consequently the logarithm of 1 being 0, the logarithm of fractions less than 1, or of numbers intermediate between 1 and 0, must be negative numbers, or numbers less than 0 (which he calls defective logarithms), denoted by — (the note of negation) prefixed.

Now these defective logarithms may be two ways expressed ; either so as that the note of negation shall affect the whole logarithm, or so as to affect only the characteristick (leaving the rest of the logarithm to be understood as affirmative).

As, for example, the fraction $\frac{3}{8}$, or (which is equivalent) 0.375. This fraction supposeth the numerator 3 to be divided by the denominator 8; which in logarithms is to be performed by subtracting the logarithm of 8 from that of 3, and the remainder will be the logarithm of $\frac{3}{8}$, which will then be the negative number, —0.4259687.

$$\begin{array}{r} \text{Log. 3.} \quad 0.4771212 \\ \text{Log. 8.} \quad 0.9030899 \\ \hline \text{Log. } \frac{3}{8}. \quad - 0.4259687 \end{array}$$

Or thus; for as much as the logarithm of 375 (supposing it to be an integer number) is 2.5740312. And the depressing this to the first, second, or third, or further place of decimal fraction, doth (without altering the figures) divide the value by 10, 100, 1000, &c; which in logarithms is done by subtracting 1, 2, 3, &c, from the characteristick or place of integers (1, 2, 3, &c, in that place, being the logarithms of 10, 100, 1000, &c). Such alteration of the value (the figures remaining) is done by altering the characteristick of the logarithm, without varying the other figures in this manner.

$$\begin{array}{r} \text{Log. 3750} \quad 3.5740312 \\ \text{Log. 375} \quad 2.5740312 \\ \text{Log. 37|5} \quad 1.5740312 \\ \text{Log. 3|75} \quad 0.5740312 \\ \text{Log. 0|375} \quad 1.5740312 \\ \text{Log. 0|0375} \quad 2.5740312 \end{array}$$

Which two forms, though they seem different, and some may rather choose the one, some the other; or in some cases the one, in some cases the other; yet they are in substance or value the same. For

$$\begin{array}{r} - 1.0000000 \\ + 0.5740312 \\ \hline \text{is} = - 0.4259687 \end{array}$$

And every one is left to his liberty, whether of the two ways (or what other equivalent thereunto) he shall please to use.

In this method Mr. Briggs hath calculated a table of logarithms (published in the year 1624) for twenty chiliads of absolute numbers (from 1 to 20,000); and again for 10 more (from 90,000 to 100,000), and one chiliad supernumerary (to wit, the hundred and first chiliad), that is 31 chiliads in all.

Before which is prefixed, a large account of the nature and construction of this Logarithmical Canon, and the uses thereof; and direction how to supply the

the intermediate chiliads which are here wanting. The whole intituled *Aritbmetica Logarithmica*.

The same is again published in the year 1628, by Adrian Vlacq (or Flack), with a supplement (as Mr. Briggs directed) of the chiliads before omitted; that is, in all, of 100 chiliads, with one supernumerary. But in shorter numbers, extending but to 10 places below that of the integers, or the characteristick. And he subjoins also a logarithmical canon of sines, tangents, and secants (for degrees and minutes of the quadrant) of as many places.

Mr. Briggs proceeded to calculate a trigonometrical canon logarithmical, suited to that for absolute numbers, to the logarithms extending (as in that other) to 14 places, beside the characteristick. And having before calculated a table of natural sines, tangents, and secants (for degrees and centesmes of degrees) in numbers extending to 15 places, he fitted thereunto a canon of logarithmical sines and tangents (because those of secants might be spared); and a treatise prefixed concerning the construction thereof, with other things pertinent thereunto; intending a farther treatise concerning the use of it.

But dying before this last was finished, or the rest published, Mr. Henry Gellibrand supplied this latter, and published the whole, with the title of *Trigonometria Britannica*, in the year 1633. To which is subjoined another canon of logarithmical sines and tangents, by Adrian Vlacq, for degrees, minutes, and tenth seconds, extending (as his former did) to 10 places, beside the characteristick; and Mr. Briggs's 20 chiliads for logarithms of absolute numbers.

So that the whole doctrine of logarithms was by this time sufficiently perfected, with convenient canons or tables fitted thereunto, in large numbers: of which also Petrus Crugerus gives an account in the preface to his *Trigonometria Logarithmica*, printed in the year 1634, with his logarithmical tables, but in shorter numbers.

And the tables of logarithms above mentioned (for 100 chiliads of absolute numbers, and for sines and tangents to degrees and centesmes) were the same year, 1633, contracted into a lesser form, and more manageable (but in shorter numbers, the former not extending to above seven places, beside the characteristick, but the latter to 10); by Nathaniel Roe; with directions for the use of them (in trigonometry, geometry, astronomy, geography, and navigation), by Edmund Wingate.

In the mean time, Benjamin Ursinus did also publish tables of logarithms in the year 1618, and again in the year 1625, in his *Trigonometria*; and Johannes Keplerus also in the year 1624, in his *Chilias Logarithmorum* (which he applies also to his *Rudolphine Tables*, published in 1627), and Claudius Batschius about the same time, or soon after; and Georgius Ludovicus Frobenius, in the year 1634, and perhaps some others. But all, or most of them, in short numbers, and conformable to the Lord Neper's first design; not to that form, which, upon second thoughts, he and Mr. Briggs agreed upon as most eligible, and which hath since been received in common practice.

Since which time, much hath not been added to the doctrine of logarithms; nor was it necessary, that work having obtained sufficient perfection.

But in case logarithms on any emergent occasion be desirable with greater exactness, and in larger numbers than those printed tables do afford, Mr. Nicholas Mercator, in a small treatise, called *Logarithmotechnia*, printed in the year 1668, shews (with great subtilty) how it may be effected, in numbers of whatever length desirable, with much more ease than heretofore.

Nor shall I need to add more concerning logarithms: those who desire farther, may find it in the authors above mentioned; especially Mr. Briggs's *Arithmetica Logarithmica* and *Trigonometria Britannica*, with Adrian Vlac's additions to both.

Without farther insisting, therefore, on the algorism by numeral figures (with the improvements thereof since we had them from the Arabs), I shall return to what doth more immediately concern Algebra.

END OF THE TWELFTH CHAPTER OF WALLIS'S ALGEBRA.

EXTRACT

E X T R A C T

FROM THE

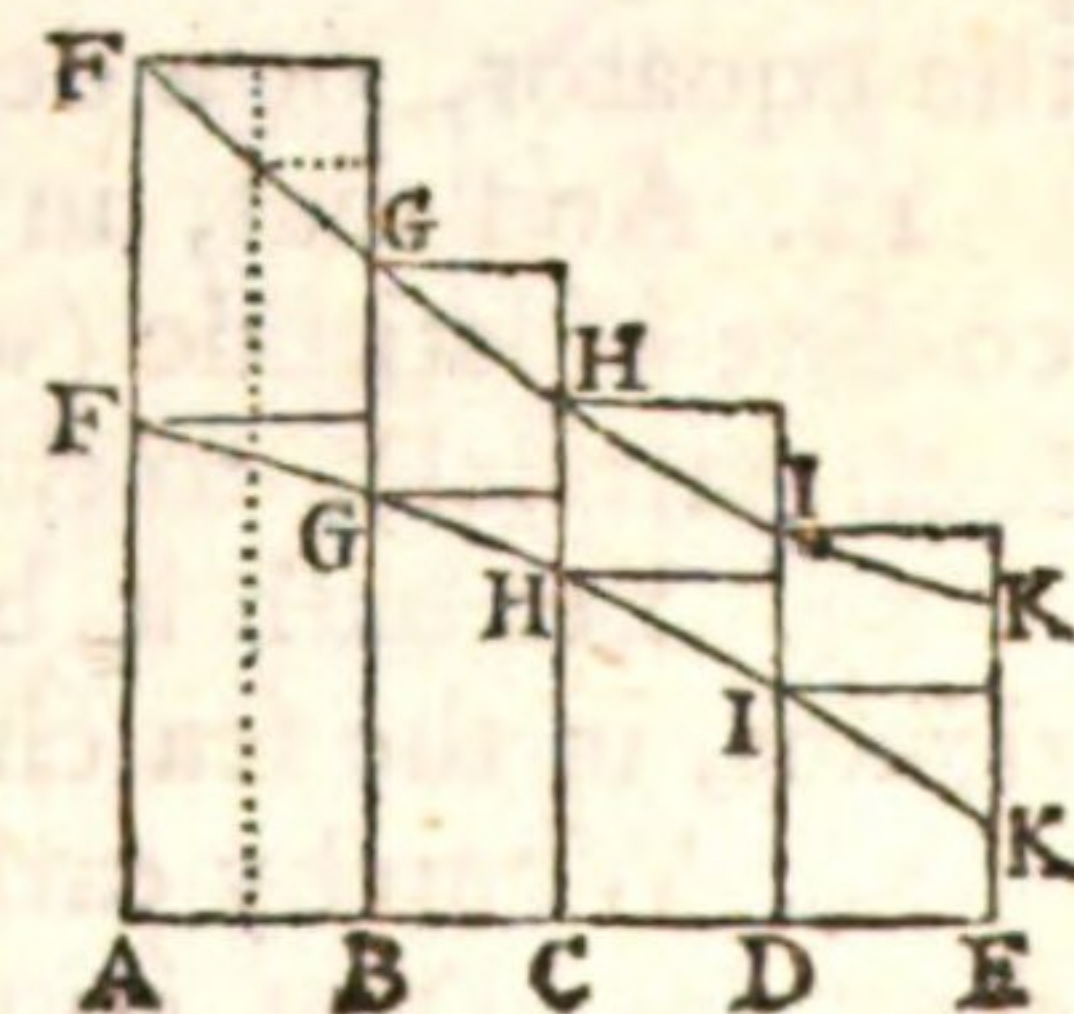
PHILOSOPHICAL TRANSACTIONS.

A Letter from the Reverend Dr. Wallis, Professor of Geometry in the University of Oxford, and Fellow of the Royal Society, London, to Mr. Richard Norris; concerning the Collection of Secants, and the true Division of the Meridians in the Sea Chart.

AN old inquiry (about the sum or aggregate of secants) having been of late moved anew, I have thought fit to trace it from its original, with such solution as seems proper to it; beginning first with the general preparation, and then applying it to the particular case.

GENERAL PREPARATION.

1. Because curve lines are not so easily managed as straight lines: the ancients, when they were to consider of figures terminated (at least on one side) by a curve line (convex or concave), as AFKE, did oft make use of some such expedient as this following (but diversely varied as occasion required), namely,



2. By parallel straight lines, as AF, BG, CH, &c (at equal or unequal distances, as there was occasion), they parted it into so many segments as they thought fit; or supposed it to be so parted.

3. These segments were *so many wanting one*, as was the number of those parallels.

4. To each of these parallels wanting one, they fitted parallelograms, of such breadths as were the intervals (equal or unequal) between each of them respectively, and the next following; which formed an adscribed figure made up of those parallelograms.

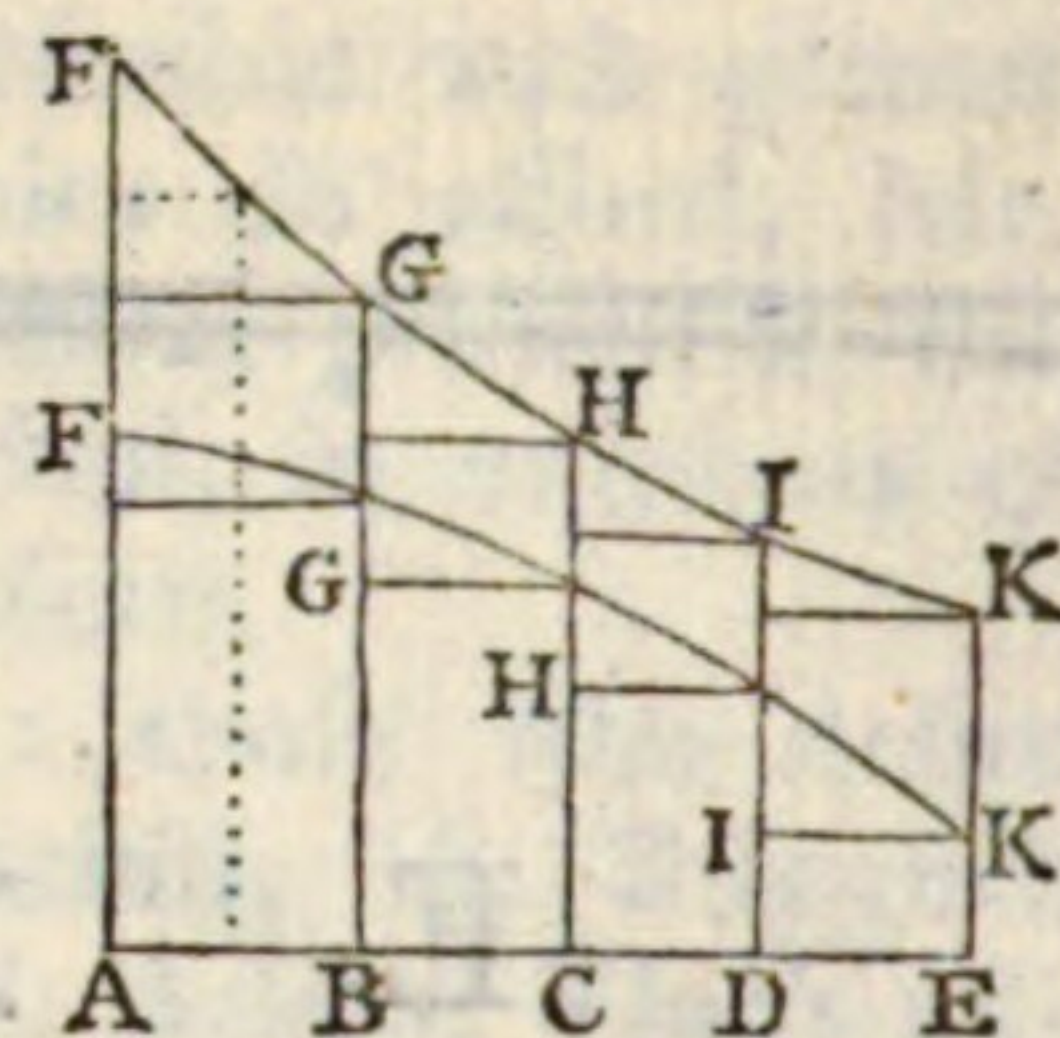
5. And if they began with the greatest (and therefore neglected the least) such figure was circumscribed (as fig. 1), and therefore bigger than the curvilinear proposed.

6. If with the least (neglecting the greatest), the figure was inscribed (as fig. 2), therefore less than that proposed.

7. But, as the number of segments was increased (and thereby their breadths diminished), the difference of the circumscribed from the inscribed (and therefore of either from that proposed) did continually decrease, so as at last to be less than any assigned.

8. On which they grounded their method of exhaustions.

9. In cases wherein the breadth of the parallelograms, or intervals of the parallels, is not to be considered, but their length only; or, which is much the same, where the intervals are all the same, and each reputed = 1; Archimedes (instead of inscribed and circumscribed figures) used to say, "All except the greatest, and all except the least:" as prop. 11 Lin. Spirak.



PARTICULAR CASE.

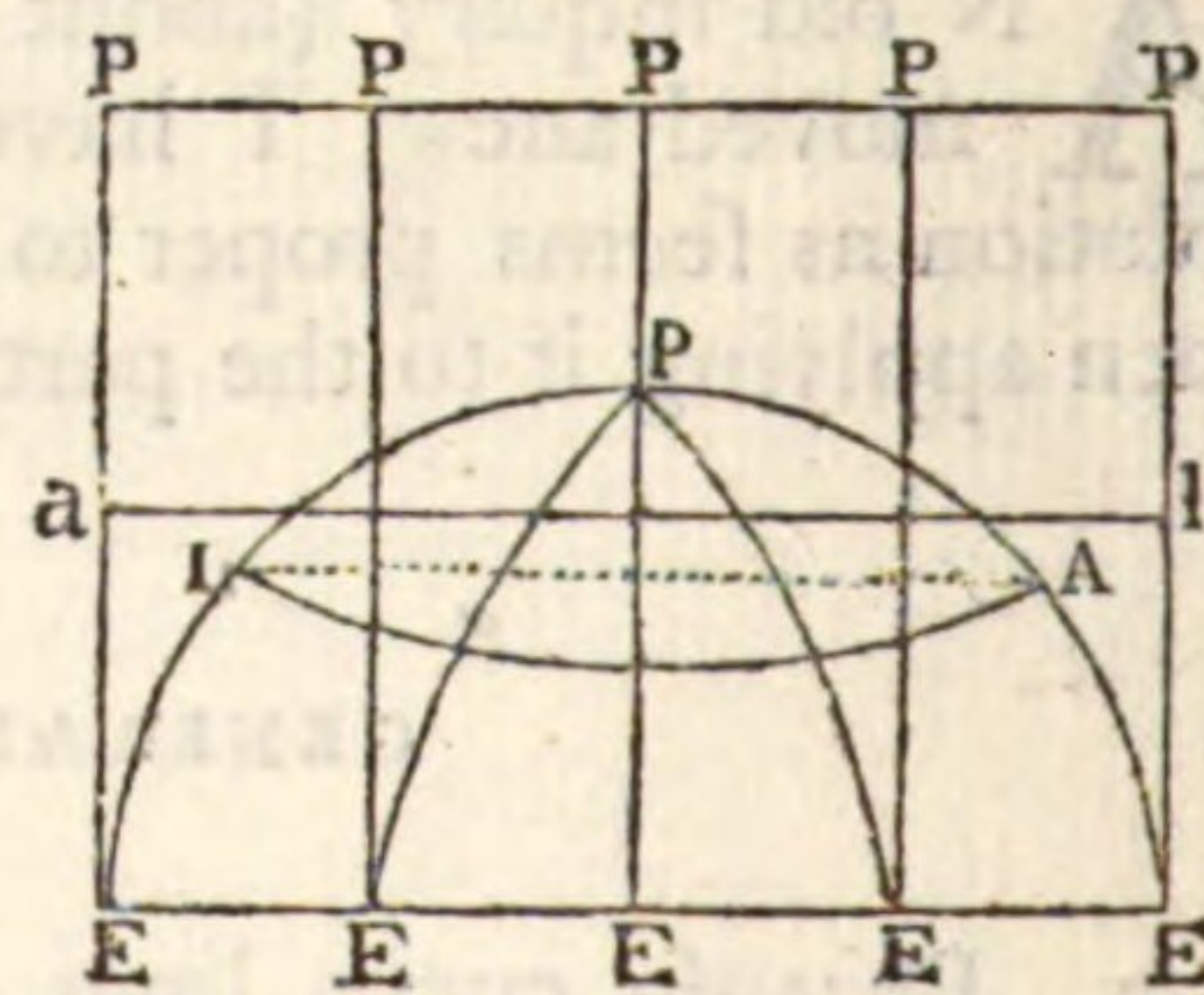
10. Though it be well known, that, in the terrestrial globe, all the meridians meet at the pole, as EP, EP, whereby the parallels to the equator, as they be nearer to the pole, do continually decrease.

11. And hereby a degree of longitude in such parallels, is less than a degree of longitude in the equator, or a degree of latitude.

12. And that, in such proportion, as is the co-sine of latitude (which is the semidiameter of such parallel), to the radius of the globe, or of the equator.

13. Yet hath it been thought fit (for some reasons) to represent these meridians, in the sea chart, by parallel straight lines, as EP, EP.

14. Whereby each parallel to the equator (as LA) was represented in the sea chart (as *la*) as equal to the equator EE, and a degree of longitude therein as large as in the equator.



15. By

15. By this means each degree of longitude in such parallels was increased beyond its just proportion, at such rate as the equator, or its radius, is greater than such parallel, or the radius thereof.

16. But in the old sea charts, the degrees of latitude were yet represented (as they are in themselves) equal to each other, and to those of the equator.

17. Hereby, amongst many other inconveniencies (as Mr. Edward Wright observes, in his *Correction of Errors in Navigation*, first published in the year 1599), the representation of places remote from the equator was so distorted in those charts, as that, for instance, an island in the latitude of 60 degrees (where the radius of the parallel is but half so great as that of the equator) would have its length, from East to West, in comparison of its breadth, from North to South, represented in a double proportion of what indeed it is.

18. For rectifying this in some measure (and of some other inconveniencies), Mr. Wright adviseth, that (the meridians remaining parallel, as before) the degrees of latitude remote from the equator, should, at each parallel, be protracted in like proportion with those of longitude.

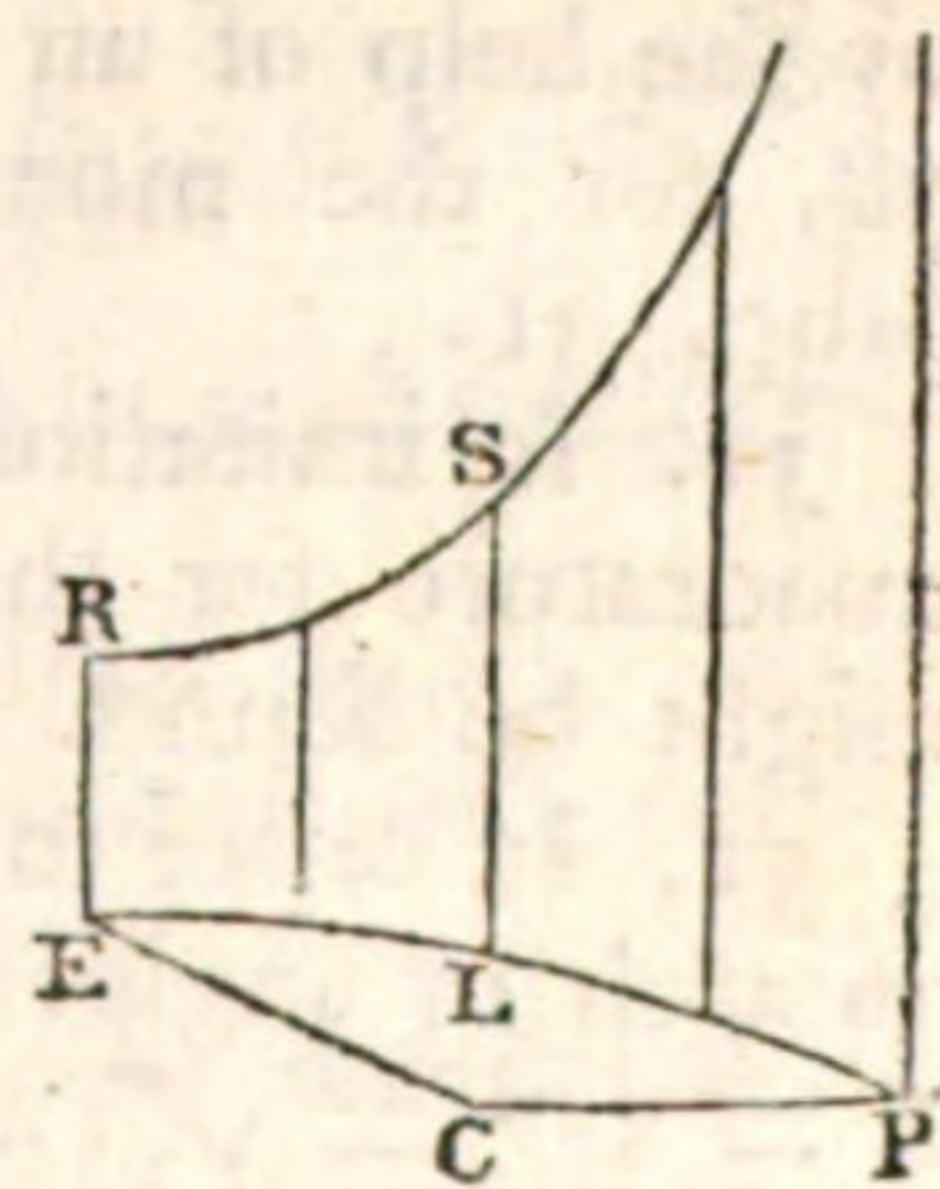
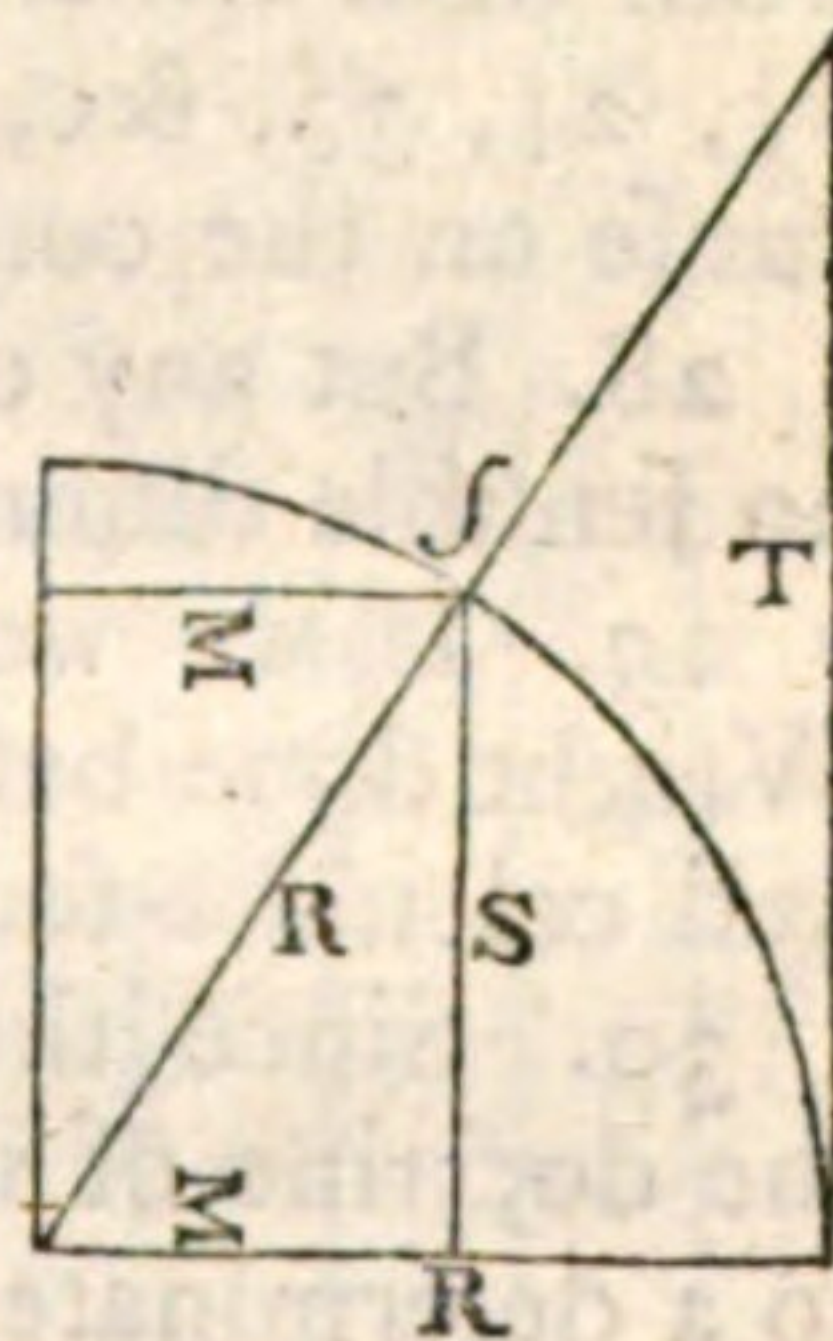
19. That is, as the co-sine of latitude (which is the semi-diameter of the parallel) to the radius of the globe (which is that of the equator), so should be a degree of latitude (which is every where equal to a degree of longitude in the equator) to such a degree of latitude so protracted (at such distance from the equator), and so to be represented in the chart.

20. That is, every where in such proportion as is the respective secant (for such latitude) to the radius. For as the co-sine to the radius, so is the radius to the secant (of the same arch or angle), as $\Sigma . R :: R . f$.

21. So that, by this means, the position of each parallel in the chart should be at such distance from the equator, compared with so many equinoctial degrees or minutes (as are those of latitude), as are all the secants (taken at equal distances in the arch) to so many times the radius.

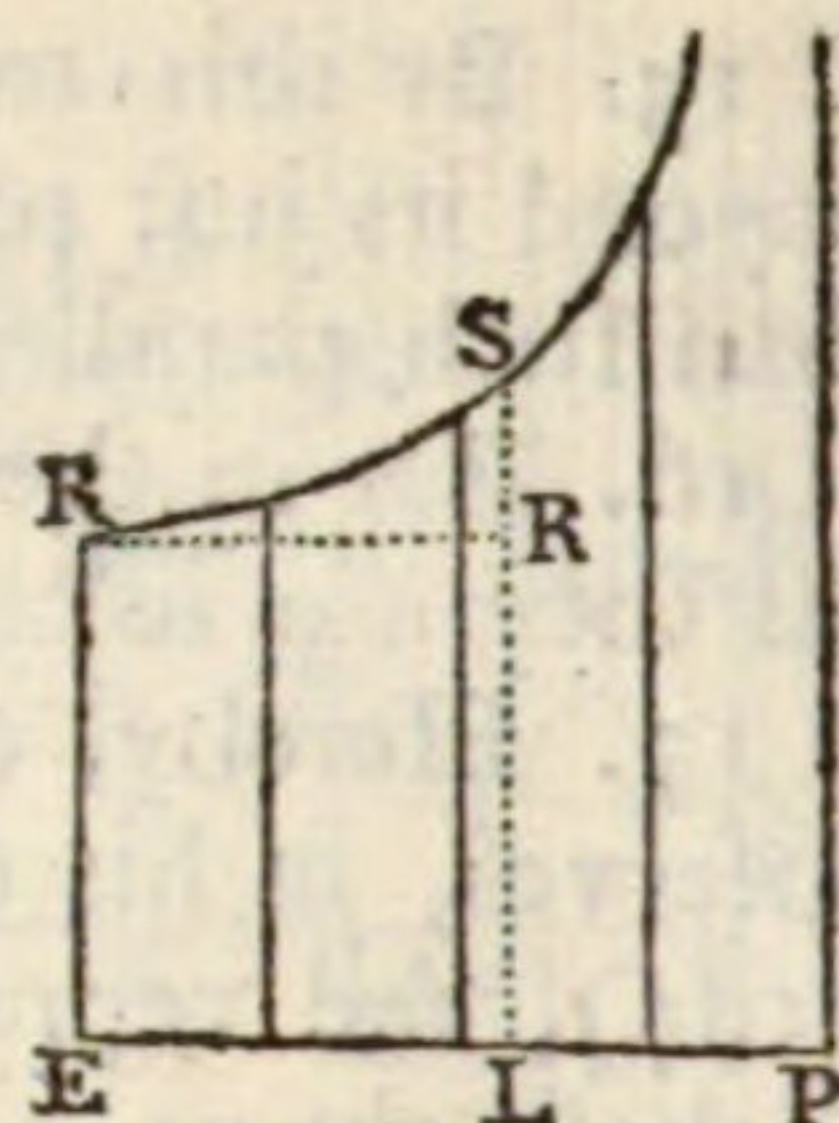
22. Which is equivalent (as Mr. Wright there notes) to the projection of the spherical surface (supposing the eye at the center) on the concave surface of a cylinder, erected at right angles to the plain of the equator.

23. And the division of meridians, represented by the surface of a cylinder erected (on the arch of latitude) at right angles, to the plain of the meridian, or a portion thereof. The altitude of such projection, or portion of such cylindrick surface, being, at each point of such circular base, equal to the secant of latitude answering to such point.



24. This

24. This projection (or portion of the cylindrick surface) if expanded into a plain, will be the same with a plain figure, whose base is equal to a quadrantal arch extended, or a portion thereof, on which, as ordinates, are erected perpendiculars equal to the secants, answering to the respective points of the arch so extended; the least of which (answering to the equinoctial) is equal to the radius, and the rest continually increasing till, at the pole, it be infinite.



ungetheilt. 25. So that as ERS_L (a figure of secants erected at right angles on EL, the arch of latitude extended) to ERRL (a rectangle on the same base, whose altitude ER is equal to the radius), so is EL (an arch of the equator equal to that of latitude) to the distance of such parallel (in the chart) from the equator.

26. For finding this distance answering to each degree and minute of latitude, Mr. Wright (as the most obvious way) adds all the secants (as they are found calculated in the Trigonometrical Canon) from the beginning, to the degree or minute of latitude proposed.

27. The sum of all which, except the greatest (answering to the figure inscribed), is too little; the sum of all, except the least (answering to the circumscribed), is too great (which is that he follows; and it would be nearer to the truth than either, if (omitting all these) we take the intermediates; for min. $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c, or (the doubles of these) min. 1, 3, 5, 7, &c: which yet (because on the convex side of the curve) would be somewhat too little.

28. But any of these ways are exact enough for the use intended, as creating no sensible difference in the chart.

29. If we would be more exact, Mr. Oughtred directs (and so had Mr. Wright done before him) to divide the arch into parts yet smaller than minutes, and calculate secants suiting thereunto.

30. Since the arithmetick of infinites introduced, and, in pursuance thereof, the doctrine of infinite series (for such cases as would not, without them, come to a determinate proportion), methods have been found for squaring some such figures, and particularly the exterior hyperbola (in a way of continual approach), by the help of an infinite series. As in the Philosophical Transactions, Numb. 38, for the month of August, 1668; and my book, De Motu, cap. 5, prop. 31.

31. In imitation whereof, it hath been desired (I find) by some, that a like quadrature for this figure of secants (by an infinite series fitted thereunto) might be found.

32. In order to which, put we for the radius of a circle, R ; the right sine of an arch or angle, s ; the versed sine, v ; the co-sine (or sign of the complement) $\Sigma = R - v = \sqrt{Rq - sq}$: the secant, f ; the tangent t . Fig. 4.

33. Then is $\Sigma \cdot R :: R \cdot f$, that is, $\Sigma) R^2 (\frac{R^2}{\Sigma}$; the secant.

34. And $\Sigma \cdot s :: R \cdot t$, that is, $\Sigma) sR (T = \frac{sR}{\Sigma}$; the tangent.

35. Now if we suppose the radius CP divided into equal parts (and each of them $= \frac{1}{1000} R$), and on these to be erected the co-fines of latitude LA :

36. Then are the fines of latitude in arithmetick progression.

37. And the secants answering thereunto, $Lf = \frac{R^2}{\Sigma}$.

38. But these secants (answering to right fines in arithmetical progression) are not those that stand at equal distance on the quadrantal arch extended. Fig. 6.

39. But standing at unequal distances on the same extended arch; namely, on those points thereof whose right fines (whilst it was a curve) are in arithmetical progression.

40. To find therefore the magnitude of $RELf$, fig. 6, which is the same with this (supposing EL of the same length in both, however the number of secants therein may be unequal), we are to consider the secants, though at unequal distances here, to be the same with those at equal distances in the preceding figure, answering to fines in arithmetical progression.

41. Now these intervals, or portions of the base, are the same with the intercepted arches, or portions of the arch, in the preceding figure; for this base is but that arch extended.

42. And these arches, in parts infinitely small, are to be reputed equivalent to the portions of their respective tangents intercepted between the same ordinates. As in fig. 7 and 9.

43. That is, equivalent to the portions of the tangents of latitude.

44. And these portions of tangents are to the equal intervals in the base, as the tangent of latitude to its fine.

45. To find therefore the true magnitude of the parallelograms, or segments of the figure, we must either protract the equal segments of the base (fig. 7), in such proportion as is the respective tangent to the fine, to make them equal to those of fig. 8.

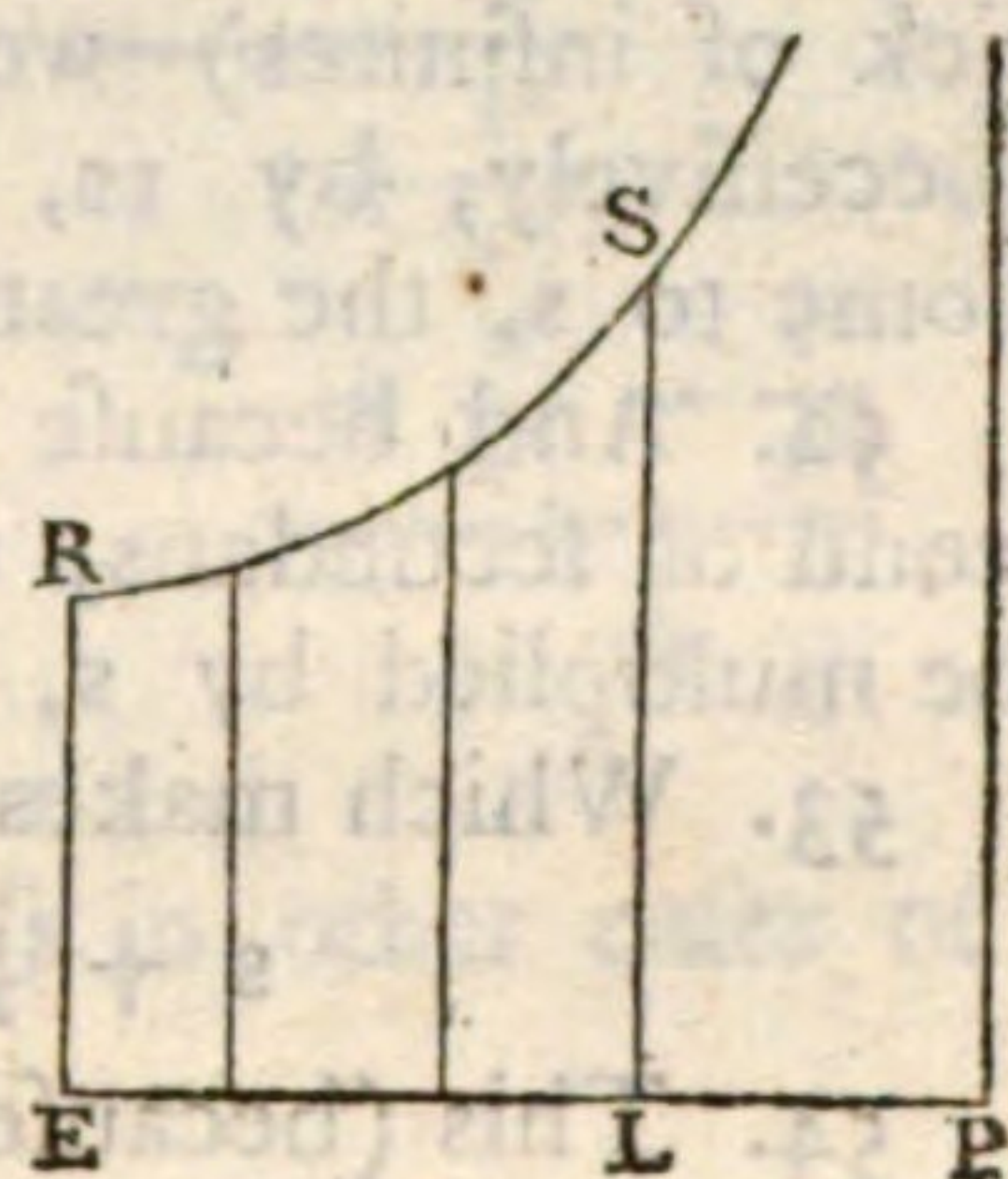
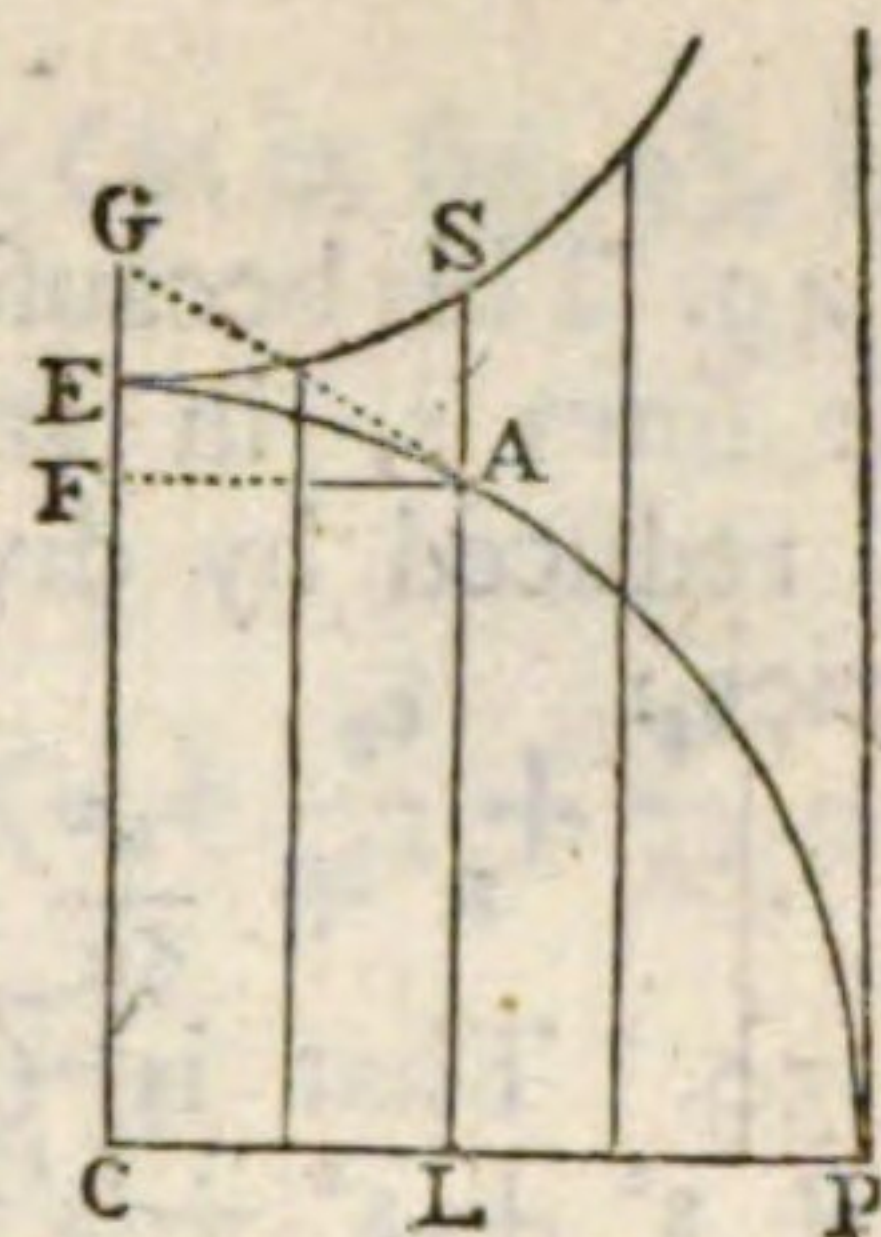
46. Or else, which is equivalent, retaining the equal intervals of fig. 7, protract the secants in the same proportion; for either way the intercepted rectangles or parallelograms will be equally increased; as LM , fig. 9.

47. Namely; as the fine of latitude to its tangent, so is the secant to a fourth; which is to stand on the radius equally divided, instead of that secant.

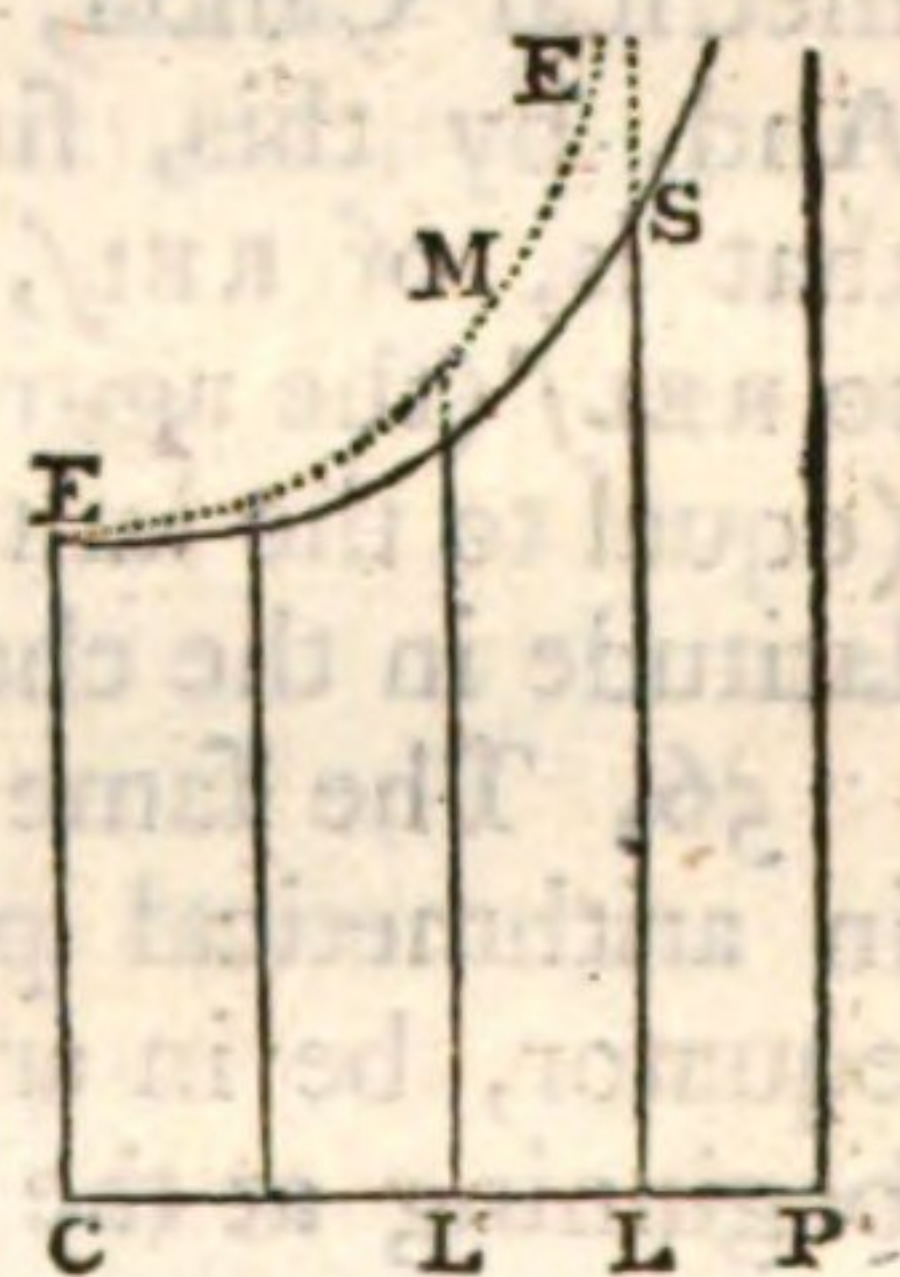
$$s \cdot \frac{SR}{\Sigma} (:: \Sigma \cdot R) :: \frac{R^2}{\Sigma} \cdot \frac{R^3}{\Sigma^2 = R^2 - s^2} = LM, \text{ fig. 9.}$$

48. Which therefore are as the ordinates in (what I call Arith. Infin. prop. 104) Reciproca Secundinorum, supposing Σ^2 to be squares in the order of secundanes.

49. This



8. = 6.



9.

49. This because of $\Sigma^2 = R^2 - s^2$, and the fines s , in arithmetical progression, is reduced by division into this infinite series :

$$R + \frac{s^2}{R} + \frac{s^4}{R^3} + \frac{s^6}{R^5} \&c.$$

50. That is (putting $R = 1$),
 $1 + s^2 + s^4 + s^6 \&c.$

51. Then (according to the arithmetick of infinites) we are to interpret s , successively, by $1s$, $2s$, $3s$, &c, till we come to s , the greatest; which therefore represents the number of all.

52. And because the first member doth represent a series of equals, the second of secundans, the third of quartans, &c; therefore the first member is to be multiplied by s , the second by $\frac{1}{3}s$, the third by $\frac{1}{5}s$, the fourth by $\frac{1}{7}s$, &c.

53. Which makes the aggregate

$$s + \frac{1}{3}s^3 + \frac{1}{5}s^5 + \frac{1}{7}s^7 + \frac{1}{9}s^9 \&c = \text{ECLM, fig. 9.}$$

54. This (because s is always less than $R = 1$) may be so far continued, till some power of s become so small as that it, and all which follow it, may be safely neglected.

55. Now (to fit this to the sea chart, according to Mr. Wright's design) having the proposed parallel of latitude given, we are to find, by the Trigonometrical Canon, the sine of such latitude, and take, equal to it, $CL = s$. And, by this, find the magnitude of ECLM , fig. 9; that is, of REL , fig. 8; that is, of REL , fig. 6. And then, as RRLE , or so many times the radius, to REL (the aggregate of all the secants), so must be a like arch of the equator (equal to the latitude proposed) to the distance of such parallel (representing the latitude in the chart) from the equator; which is the thing required.

56. The same may be obtained, in like manner, by taking the versed sines in arithmetical progression. For if the right sines, as here, beginning at the equator, be in arithmetical progression, as $1, 2, 3$, &c, then will the versed sines beginning at the pole (as being their complements to the radius) be so also.

THE COLLECTION OF TANGENTS.

57. The same may be applied in like manner (though that be not the present business) to the aggregate of tangents, answering to the arch divided into equal parts.

58. For those answering to the radius so divided, are $\frac{sR}{\Sigma}$; taking s in arithmetical progression.

59. And

$$\begin{array}{r} R^2 - s^2 \big) R^3 \left(R + \frac{s^2}{R} + \frac{s^4}{R^3} + \right. \\ \underline{R^3 - s^2 R} \\ + s^2 R - \frac{s^4}{R} \\ \underline{+ s^2 R - \frac{s^4}{R}} \\ + \frac{s^4}{R} - \frac{s^6}{R^3} \\ \underline{+ \frac{s^4}{R} - \frac{s^6}{R^3}} \\ + \frac{s^6}{R^3} \end{array}$$

59. And then enlarging the base (as in fig. 8), or the tangent (as in fig. 9), in the proportion of the tangent to the sine.

$$s \cdot \frac{SR}{\Sigma} (:: \Sigma \cdot R) :: \frac{SR}{\Sigma} \cdot \frac{SR^2}{\Sigma^2} = \frac{SR^2}{R^2 - s^2}.$$

60. We have by division this series,

$$s + \frac{s^3}{R^2} + \frac{s^5}{R^4} + \frac{s^7}{R^6} + \frac{s^9}{R^8} \&c.$$

61. That is (putting $R = 1$),

$$s + s^3 + s^5 + s^7 + s^9 \&c.$$

62. Which (multiplying the respective members by $\frac{1}{2}s, \frac{1}{4}s, \frac{1}{6}s, \frac{1}{8}s, \frac{1}{10}s,$ &c) becomes

$$\frac{1}{2}s^2 + \frac{1}{4}s^4 + \frac{1}{6}s^6 + \frac{1}{8}s^8 + \frac{1}{10}s^{10} \&c.$$

Which is the aggregate of tangents to the arch, whose right sine is s .

63. And this method may be a pattern for the like process in other cases of like nature.

$$\begin{array}{r} R^2 - s^2 \quad SR^2 \left(s + \frac{s^3}{R^2} + \frac{s^5}{R^4} + \right. \\ \hline \frac{SR^2 - s^3}{+ s^3} \\ + s^3 - \frac{s^5}{R^2} \\ \hline + \frac{s^5}{R^2} \\ + \frac{s^5}{R^2} - \frac{s^7}{R^4} \\ \hline + \frac{s^7}{R^4} \end{array}$$

LOGARITHMOTECHNIA:

OR, THE

MAKING OF NUMBERS

CALLED

L O G A R I T H M S

TO TWENTY-FIVE PLACES,

FROM A

G E O M E T R I C A L F I G U R E,

WITH SPEED, EASE, AND CERTAINTY.

BY EUCLID SPEIDELL, PHILOMATH.

PRINTED AT LONDON, IN THE YEAR 1688.

TO THE READER.

HAVING for some years past shewn to several persons the *praxis* of the following Treatise, and also communicated to them somewhat of the doctrine leading thereunto, I was often desired not to let them sleep in oblivion, but to publish the same; which was first promoted by my honoured friend Mr. Peter Hoot, merchant, and seconded by my loving friend Mr. Reeve Williams; or else they had not seen the publick. What I have done therein I desire thee to take in good part, being also at proportional charges myself, besides my composing thereof, to make it communicable to thee, rather than such an easy and certain way to make logarithm numbers (to so many places) should not be known in our native tongue. I have called them Geometrical Logarithms, for that the first inventors of those numbers had not adapted geometrical figures to them. But the scheme hereunto annexed having such properties and affections as logarithm numbers have, hath made me so style them. What I have done herein is to gratify such who have a curiosity to examine logarithm tables, and to make logarithm numbers to so small radiuses as are so often printed for common uses with brevity and exactness. Two sheets of the *praxis* hereof were printed some time before the rest; which having found kind acceptance with divers, induced me also to let the remainder be published. And before the printing thereof, one was writing upon those two sheets, and was so fair to desire my consent to publish it; which I readily gave; for that I knew him able enough to do it, and when to be at leisure myself to attend the publishing of the residue, I knew not. But that not being performed by him, I desire thee to accept of what is done herein, as time and leisure hath permitted. I shall not need to write how needful logarithm numbers are in those great and useful arts of Navigation, Astronomy, Dialling, Fortification and Gunnery, Surveying, Gauging, Interest, and Annuities, &c, when, as there are so many books written and published thereof, not only in our own language but in many others. And truly the first inventors thereof are not a little to be had in reverence for making and perfecting those numbers with so much labour as those methods by which they derived them did require. Here thou mayst make a logarithm to 7 or 8 places readily and easily; but to 25 places would have been very difficult, if not impossible, for the first inventors to have produced after their ways. If any thing herein shall offer whereby thou mayst make farther improvement, let the publick share of the benefit thereof. Thus wishing thee good success in all thy studies, is most earnestly desired by

London,
March 26, 1688.

E. SPEIDELL.

GEOMETRICAL LOGARITHMS.

C H A P. I.

ABOUT the year 167 $\frac{5}{6}$, being in company with Michael Dairy, a citizen of London (who had for most part of his life-time addicted himself to mathematical studies, and hath published divers practical pieces of several parts of the mathematicks, of good use and delight); and discoursing about making hyperbolical logarithms, I desired him to give me a rule to make the hyperbolical logarithm of 10, from the consideration of an hyperbola inscribed within a right-angled cone, who gave me this rule following.

To the number proposed, viz. 10, add an unit, and subtract from it an unit, and there will be a result of $\frac{9}{11}$; then divide 1, or 100000000 &c, by $\frac{9}{11}$, which is 818181818 &c, which cube *in infinitum*, and divide every one of them (which will be a rank of proportional numbers) by the proper indices of their respective powers, that is to say, by 3, 5, 7, 9, 11, &c, then the addition of those quotes will make the logarithm of 10.

Finding, then, that 10 divided by $\frac{9}{11}$ maketh 81818181818181 &c, and to cube it *in infinitum*, was very difficult, I rejected the rule, and thought it then not much more easy than Briggs's way: neither did he tell any reason or demonstration for the said rule; and because in this example I found it so intricate, I did not much care to prosecute it, but neglected it. Not long after he departed this life, and since his death resuming the said thing, and trying if it were serviceable in any other part of the hyperbola, I soon found it a jewel, and could make the hyperbolical logarithm of 10 at twice, that is to say, from two parts numbered in any asymptote, whose fact is 10, with ease, certainty, and delight, and have made the hyperbolical logarithm of 2 to 25 places, in order to see if the learned and laborious Henry Briggs's logarithms were true to 15 places, which were made after a most laborious and difficult way of extracting square roots, and, as I have heard, was the work of eight persons a whole year, and that without any proof but only if any two or more agreed in their extractions, line
by

by line, step by step, it was taken *de bene esse*; which was a work of very great pains and uncertainty. However, they did effect it; and I do find they made the logarithm of 2 to 15 places very true, as by my operation, hereafter following, will appear, being done to 25 places, and afterwards from these hyperbolical logarithms deduced Briggs's logarithms; both which figurative operations were performed and examined by me in 8 hours' time. I took this pains to make the hyperbolical logarithm to 25 places, in order, also, to see if the most ingenious and laborious James Gregory's hyperbolical logarithm would agree with this of mine, which he hath, in his *Quadratura Circuli & Hyperbolæ*, printed at Padua; but I find that his logarithm of 2 corresponds with mine but to 17 places. I must confess, I did not take the pains to raise the logarithm of 2 to 25 places, according to the doctrine he hath delivered in that most learned piece, but am contented that this easy and certain way I deliver here, and by the operation thereof the hyperbolical logarithm of 2 to 25 places, is as true in the last as in any where, and may be examined in a few hours; so that any body, if he please, may be his own examiner and judge, if this way be not easy, certain, and speedy.

Having made several logarithms for digit numbers, and mixt numbers, as for $1\frac{1}{4}$, $1\frac{1}{2}$, which are hereafter inserted, I find the rule delivered by Michael Dairy is of admirable use and benefit in squaring the hyperbola, and making logarithms from it.

Some time since the death of the said Michael Dairy, I shewed unto Mr. John Collins, who I knew had been a great familiar and friend to the said Michael Dairy, the figurative work of my making the hyperbolical logarithms, according to the said Dairy's rule; who seemed very well pleased with it, acknowledging it to be the speediest way could possibly be of squaring the hyperbola, and making the logarithms from it; and, after a little pausing on it, replied, "That Dairy must have had this rule out of the said James Gregory's works." I made answer, "Not from his said *Quadratura Circuli & Hyperbolæ*." He answered "No, from his *Exercitationes Geometricæ*, printed at London, 1668."—A book I had not seen or heard of till then. And as he the said Mr. Collins had been always very frank and free to communicate any mathematical thing to me, so I held myself obliged to acquaint him first with this work. He seemed to admire that Michael Dairy should keep such a thing from him, who had been so great a familiar with him in these studies. Not long after my discovery hereof to the said Mr. Collins, he also departed this life; whose death, all that were mathematical, and knew him, lamented not a little: for he was not only excellent in mathematical arts and sciences, but of a very good, affable, and frank nature to communicate any thing he knew to any lover and inquirer of those things; and hath left behind him those mathematical works which will continue his fame amongst the lovers and students therein. He also, in his lifetime, promoted the publishing of other men's mathematical works, as the elaborate Algebra of the learned John Kersey, who was my father's disciple about 164 $\frac{5}{6}$; and also of the learned Baker's Algebra, and several others. He was a man of great correspondence with mathematical persons in foreign parts, and thereby could give information of any new or old mathematical book; and, till my acquaintance with him, I was ignorant of foreign authors; being
but

but young when my father died, and not then having taken any pains in these studies: so that, by the said Collins's information and means, I have heard of, and seen, some mathematical authors of note and esteem.

After the said Mr. Collins had told me of James Gregory's said *Exercitationes Geometricæ*, sold by Moses Pitts, in St. Paul's Church-yard, I bought there one of them; and do find, that Michael Dairy had deduced this rule from the said book: wherein the said James Gregory hath made the squaring of the hyperbola, an exercise geometrically demonstrating the quadrature of the hyperbola, some time before published by the industrious and lucky Nicholas Mercator; who, by the happy discovery of some properties in the hyperbola, hath made all the ways of squaring the hyperbola flowing from the same, very easy, certain and delightful: and because neither of them have exemplified their doctrine and rules with figurative work, so large as to 25 places. I have here, to illustrate their admirable works, inserted divers figurative operations, whereby the reader and student may see, and have that satisfaction in fact and operation, which is so pleasing and desirable by every one.

I shall not here trouble the reader with any sections of the cone, whereby he may see the rise and geniture of an hyperbola from that body, but content myself to shew him from a square and an infinite company of oblongs on a superficies, each equal to that square, how a curve is begotten which shall have the same properties and affections of an hyperbola inscribed within a right-angled cone: and seeing a curve made after this manner following doth become such an hyperbola, the doctrines and analogies delivered and discovered by those two ingenious artists, Mercator and Gregory, may be applied to this curve as often as need and occasion doth require.

And, not to detain the reader any longer from knowing how to make this curve, we proceed to describe the same accordingly.

There is a square $ABCD$, whose side or root is 10; let DB be prolonged *in infinitum*, and continually divided equally by the root, or DB , and those equal divisions numbered by 10, 20, 30, 40, 50, 60, 70, &c, *in infinitum*: upon these numbers let perpendiculars be erected, which call ordinates, and each of those perpendiculars of that length, that perpendiculars let fall from the aforesaid perpendiculars to the side or base CD (which call complement ordinates), the oblongs made of the ordinate perpendiculars, and complement ordinate perpendiculars, may be ever equal to the square AD , which may easily be done thus; for it is $\frac{100}{20}, \frac{100}{30}, \frac{100}{40}, \frac{100}{50}$, &c, produces the length of the ordinate perpendiculars; for 100 divided by 20, maketh 5 for the length of the ordinate perpendicular 20E. And 100 divided by 30, giveth 3333333 &c for the ordinate perpendicular 30F; and 100 divided by 40, produceth 25 for the ordinate 40G, and so of the rest. And, geometrically, it is as 20D is to BD, so is AB to AH equal to 20E, as before; for that the angle ACH is equal to C20D, and so of the rest. And, for the length of the next ordinate, you say, as 30D to BD, so AB to AK, which is equal to 30F. And, for the ordinate 40G, say, as 40G to BD, so AB to AM, which will be equal to 40G, and so of all the rest; whereby you have all the perpendiculars upon the prolonged side BD, both geometrically and arithmetically. The same proportion is to be observed for any intermediate parts.

Now,

Now, for all the perpendiculars which are let fall from the aforesaid perpendiculars or ordinates to the base CD , which call complement ordinates, the geometrical proportion for NE , equal $D2O$ is as HA to AC , so CD to $CL2O$ equal to NE ; and for the complement ordinate OF equal $D3O$, it is as KA to AC , so CD to $D3O$ equal OF , and so of the rest. Now, for NE arithmetically, say, as 5 to 10, so 10 to 20 equal to NE , equal to $D2O$; and for OF , say, as 33333333 to 10, so 10 to 30 equal OF , which is equal to $CL3O$, and for PG equal to $D4O$, say, as 25 is to 10, so 10 to 40 equal to PG , equal to $D4O$; and so for all the rest of the complement ordinates standing upon the base CD , whereby it doth appear, that all the oblongs made of the ordinates, and complement ordinates, are each of them equal to the square AD , which is here 100; for the oblong ED being made of $E2O$ and $D2O$, is by the 13 of the 6 Euclide, equal to the square AD ; for $Q2O$ is a mean proportional between $D2O$, and $2OR$, and $Q2O$ is found to be equal AB , so is the oblong or parallelogram ED equal to the square AD , and the like demonstration serves for all the oblongs or parallelograms standing upon the base CD , by the tips or angular points of those parallelograms, or from the ends of all the ordinates standing upon 20, 30, 40, 50, 60, 70, *in infinitum*, draw the curve line from A towards E , so shall you describe the curve $AEFGS$, which curve you see is begotten without any consideration or respect to the section of a cone, and yet becomes the same in all respects, to have the same affections and properties of an hyperbola derived from the intersecting of a right-angled cone, as shall be shewed in the next chapter.

You may observe the complement ordinate NE , being equal to $D2O$, is equal to twice radius. And if CD be made the radius of a circle, then is NE equal to $D2O$, equal to the tangent of twice radius; for $D2O$ becometh the tangent of twice radius. Also, it is manifest that the complement of the tangent equal to twice radius is also equal to half the radius; that is, the tangent complement of $D2O$ is $2OE$ equal to 5. And seeing the radius is ever a mean proportion between the tangent and the tangent complement, therefore each oblong is equal to the square AD .

C H A P. II.

IN the former chapter, we have shewed the begetting of a curve, without any regard to the section of any solid body; and now it remaineth to prove that this curve hath the same properties and affections that an hyperbola, deduced from the section of a right-angled cone.

I remember some time before the death of John Collins, he told me, it was a great work of the learned Vincent, or Magnan, to prove, that distances reckoned in the asymptote of an hyperbola, in a geometrical progression, and the spaces that the perpendiculars thereon erected, made in the hyperbola, were equal the one to the other. This property is now very well known; the hyperbola hath, and this curve hath the same property; which is discernible almost *intuitu*. In the hyperbola, they call the prolonged line DB *in infinitum*, from the point B , an asymptote. And here in this prolonged line from B , on 20, 40, 80, 160,

320, 640, &c, let the ordinates touch the curve in EFGS &c; I say, that those trapezias with the curve line (or hyperbolical spaces) are all equal the one to the other. In the right-lined trapezias thereon, it is manifest, they are all equal the one to the other, by several propositions of the 6th book of Euclid: for in the right-lined trapezia ZEAB, the side AB is twice EZ; and, by the former chapter, it was found, that GY is half EZ, by saying, as AB to EZ, so EZ to GY. And the right-lined trapezia ZEAB shall be therefore equal to 75. Now, forasmuch as in the right-lined trapezia YGEZ, the base of that YZ is double to ZB, but the perpendiculars are in the ratio of AB to EZ; for, as before, it is as AB : EZ :: EZ :: GY; therefore the right-lined trapezia YGEZ equal to the right-lined trapezia ZEAB, and so will all right-lined trapezias, so based and perpendiculared, be equal the one to the other. The trapezia LYEZ is equal to the square AD, because $ZY \times ZE$ is equal to $AB \times AB$, as in the foregoing chapter, the oblong GZ is half the parallelogram LZ, and the triangle GEI half the parallelogram LI. Now the parallelogram GZ + the triangle GEI (half the parallelogram LI) is equal to the right-lined trapezia ZEAB, for in numbers $20 \times 25 = 50 +$ half 50 equal to 75.

Thus you see the right-lined trapezias, numbered upon the prolonged side in geometrical proportion, are equal the one to the other. It remaineth now to prove the mixed trapezias; that is, the trapezias standing upon the same basis, but joined aloft with this curve, are also equal the one to the other.

First, let it be observed, that these curvilinear trapezias (or hyperbolical spaces) are ever less than the right-lined trapezias, because all the points in the curvilinear trapezias fall within the right line that joins the right-lined trapezias; and is thus proved in the right-lined trapezia BZEA: let there be in the base ZB upon the point 5 erected a perpendicular to touch EA in T, then is T5 equal to AB less αT , which is half QE, that is, 10 less 2, 5 equal 7, $5 = TS$. But, by the foregoing chapter, if a perpendicular be erected upon the said point 5 to V (to touch the curve in V) so that the parallelogram VD shall be equal to AD, as in the former chapter; then will it be AD divided by $DV = 5V$, which is 100 divided by 15, produceth 6,666666 for the true length of 5V; whereas before 5T is 7,5. By the same means may all the intermediate points in this curve line EVA be found to fall within the right line AE, that is, between the line EA and ZB; and therefore the right-lined trapezia ZETAB greater than the curvilinear trapezia (or hyperbolical space) ZEVAR.

Now, forasmuch as we have proved that the aforesaid right-lined trapezias are ever equal the one to the other, it will now follow, that seeing the curve passing by all those points which are extremities of the right-lined trapezia (as well as the curvilinear spaces, being upon the same basis always), and this curve being generated continually by one and the same ratio, as in the former chapter. That therefore the curvilinear trapezias standing upon geometrical proportional bases, shall be also equal the one to the other; which is the affection and property of the hyperbola. And so the doctrines and precepts delivered by those two famous Geometers, Mercator and Gregory, for the squaring of the hyperbola, be applied to this geometrical curvilinear figure, and from it derived logarithms, which may be called hyperbolical logarithms.

The way and means to find the hyperbolical spaces in numbers shall be shewed in the following chapters.

C H A P. III.

IN this chapter we will consider that most admirable discovery (I suppose Mercator made) upon drawing the diagonal CB , which, by construction, cutteth all the perpendiculars standing upon the base CD at equal angles, and in such distances from the base CD as doth unravel the mystery of his infinite series, and make the quadrature of the hyperbola more easy and certain than any I ever saw or heard of.

The diagonal CB being drawn, doth give the first term of a geometrical progression, or infinite series, between 10 and 20, or 30, 40, 50, 60, 70, 80, 90, &c.

That is to say, would you know the first term of an infinite series (or numbers geometrical proportional continued) between 10 and 20, the sum of all which shall be just 20. Having from z drawn the line zc , to cut BA into H , which taken off and applied to CD , from c to N equal NB , because the angle BCN is equal to the angle CBN ; I say, that NB is the first term of an infinite series between CD equal AC , and the perpendicular NE equal Dz ; which may be done by squaring AC , and dividing it by the side or number given, the complement whereof to 10 is the first mean or term of that infinite series, so shall the first term of the infinite series between 10 and 20 be found 5. Thus in numbers, $10 \times 10 = 100$, $\frac{100}{20} = 5$, the complement whereof to 10 is 5, equal EN , equal NB for the first term of an infinite series between 10 and 20, whose sum is 20, as by the arithmetical work in the margin, where a, b, c, d, e, f , &c are a rank of geometrical progression numbers, whose infinite sum would make but 20, and is demonstrated by the 7th and 8th of Euclid.

And in numbers, thus: as 10 less 5 is to 10, what 10? the quotient will be found 20 for the whole sum of that infinite series between 10 and 20 whose first term is 5.

In like manner, would you know the first term of an infinite series between 10 and 30, divide the square of $ac = 100$ by 30, the quotient will be found 3333333, whose complement to 10 is 6666666; I say, that 6666666 is the first term of an infinite series between 10 and 30, as by the arithmetical operation in the margin; and briefly thus, as $10 :: 3333333 = 6666666 : 10 :: 10 : 30$, so is 30 the whole sum of all those infinite progression numbers between 10 and 30. In the figure you draw the line Φc , which cutteth BA in K ; I say, that BK transferred from c to O equal or is the first term of an infinite series between AC and OF equal $D\Phi$. And Ps be the first term equal 7,5, equal PC between AC and PG equal $DY = 40$ between 10 and 40; for, as before, $10 - 7,5 = 2,5 : 10 :: 10 = 40$, so is 40 the whole sum of an infinite series between 10 and 40, whose first term is 7,5, and so of all the rest. On this great mystery depends much the following squaring of the hyperbola, and hath made it so intelligible and easy.

$a.$	10
$b.$	5
$c.$	2.5
$d.$	1.25
$e.$	625
$f.$	3125
$g.$	15625
$h.$	78125
$i.$	390625
$j.$	1953125
$k.$	9765625
19.	99.0234375

$a.$	10
$b.$	6.666666666
$c.$	4.444444444
$d.$	2.962962962
$e.$	1.975308641
$f.$	1.316872427
$g.$	877914951
$h.$	585276634
$i.$	390184422

Hence

Hence you may note, that if you would know the first mean of an infinite series between any other number, and that number doubled, tripled, quadrupled, &c, you may from this root 10 deduce it; as, for example, let cd be 12, and I would know the first term of an infinite series between 12 and 24, the double of 12, I say, as 10 is to 5 what 12 *facit* 6; for, supposing $cd = 10$ it was before found 5. Therefore between 12 and 24 you say $10 : 5 :: 12 : = 6$ for the first mean of an infinite geometrical progression between 12 and 24; and is thus, by the former analogy, proved by saying, as $12 - 6 = 6 : 12 :: 12 : = 24$, so is 24 found to be the total sum of an infinite series between 12 and 24, as by the operation in the margin will appear.

Also, if it were required to find the first mean between 12 and three times that number, viz. 36, say, as $10 : 6666666 :: 12 : 8$. And so by tabulating or working this example as you do the former, the total sum of that infinite series between 12 and 36 (the first term being found 8), will amount to 36; and, for proof, you say, $12 - 8 = 4 : 12 :: 12 : = 36$, so is 36 the whole sum of that infinite series, or geometrical progression, between 12 and 36, the first term being 8, as was desired.

If it shall be required to know an infinite series between 10 and any other number; as, to know the first term between 10 and 15, that is, between dc and $d5$, draw $5c$, and it cutteth BA in E ; I say, that BE is the first term of an infinite series between 10 and 15, as by the operation in the margin, and as before taught, $\frac{100}{15} = 6666666$; which subtract from 10, leaveth 3333333 for the first term. And by the former rule is proved thus: as $10 - 3333333 = 6666666 : 10 :: 10 : = 15$. Thus may you find the first of any term of an infinite series between 10 and any other number.

And if it should be desired to know the first term of an infinite series between any two other numbers; as, for example, I would know the first term of an infinite series between 12 and 16: to do it geometrically, you must suppose $BA = 12$; and then counting 16 from D in the line DB , prolonged by that point, and c draw a line which will cut the line BA (now representing 12) in a point, which taken from B , will be the length of that line geometrically; and arithmetically, it will be found by saying, as 16 to DC , so $16 = 4$ to BK in the line BA , numbered from B to A when BA is 12. In numbers the proportion stands

H 2

thus :

a.	12
b..	6
c..	3
d..	1.5
e....	75
f....	375
g....	1875
h.....	9375
i.....	46875
k.....	234375
l.....	1171875
m.....	5859375
	23,994140625

a.	10
b..	3333333333
c..	1111111111
d...	370370370
e...	123456793
f....	41152264
g....	13717421
h.....	4572473
i.....	1524157
k.....	508052
l.....	169350
m.....	56450
	14,999971774

thus: $16 : 12 :: 4 : 3$, which 3 is the first term of an infinite series between 12 and 16, as was required, and is manifest by the operation in the margin; which to prove by the foregoing rules, you say, as $12 - 3 = 9 : 12 :: 12 : 16$; which is to say, as 9 to 12, what 12 *facit* 16 for the whole sum of that infinite series between 12 and 16, the first term being found, as before, to be 3.

Thus have you that great mystery unfolded, of finding geometrically and arithmetically the first term of a geometrical progression with the whole sum of that infinite progression, or series, between any two numbers; which is the main thing I conceive that famous Mercator was so lucky in discovery thereof, and doth unravel the mystery of squaring the hyperbola, as will be manifest in the next chapter following.

a. 12
b.. 3
c... 75
d... 1875
e.... 46875
f..... 1171875
g..... 29296875
h..... 732421875
i..... 18310546875

15,99993896484375

C H A P. IV.

I Observe from the said learned Gregory's *Exercitationes Geometricæ*, he giveth three quantities, or spaces, contiguous to the vertex A, which shall be all equal the one to the other, which is very true and perspicuous; and then shews how to find the areas of them severally, as in page 9, 10, 11, and 12, of said book.

And here we must consider them all three before we come to understand Dairy's rule, which is but a deduction from these, as will appear hereafter. And now I begin to consider the said three several quantities, or spaces, all contiguous to the vertex A, and of a different form, and yet equal the one to the other.

Let it therefore now be shewn those three several spaces differing in form, and yet equal the one to the other, contiguous to the vertex A, which shall represent the curvilinear trapezia, or hyperbolical space for 2. The first curvilinear trapezia, or hyperbolical space for 2, let be ZBAVE, which is intelligible *intuitu*: the second let be AVENC, which is equal to the former ZBAVE, by the 43 of the 1 of Euclid, because the parallelogram EZBH (BH being equal to ZE) is equal to the parallelogram HACN (HA being equal to HB). Now, forasmuch as the curvilinear triangle AVEH is common to both the said curvilinear trapezias, or hyperbolical spaces, it remaineth therefore, that these two curvilinear trapezias, or hyperbolical spaces, ZEVAB and AVENC, are equal the one to the other.

And now to find out the third curvilinear space contiguous to the vertex A, and yet equal to either of the other two, but differing in form, doth require a little further consideration, which from him is directed thus: And is manifest by the figure, divide BZ in two equal parts in 5; then, as before taught, will it be as $5D : DC :: 5B : BK = DX$ make CX equal to CII, or find DII, it is as $D5 : DZ :: DC : DII$ upon II erect a perpendicular to touch the return or continuation

tinuation of the curve on the other side of A, from A towards Θ in Σ ; so is this curvilinear figure, or hyperbolical space, $\Pi\Sigma AVX$ (differing in form) equal to either of the other two $ZEVA\Theta$ or $CAVN$. And for finding the area of this curvilinear figure, or hyperbolical space, $\Pi\Sigma AVX$, is derived Dairy's rule, which is but a deduction from the finding of the areas of the other two, as will hereafter appear.

Arithmetically, DX is found by saying, as 15 to 10; what 10? *facit* 66666666, and 10 less 66666666 rest 33333333 for $c\Pi$, so is $x\Pi$ equal to 66666666; or $d\Pi$ may be found thus: as $D5 : DZ :: DC : d\Pi$, which in numbers is, as 15 : 20 :: 10 : $\equiv 1.33333333$ for $d\Pi$.

James Gregory, in the 4th proposition, page 10 and 11, of his *Exercitationes Geometricæ*, doth contemplate first, the second of these three curvilinear trapezias, or hyperbolical spaces; that is to say, the curvilinear trapezia, or hyperbolical space, $CAVN$; and in that fourth proposition, after a long and learned demonstration, doth prove the space $CAVN$ to be equal to his supposed quantity ω ; and then refers you to Cavalerius's method of indivisibles; a book I have not yet seen. Which briefly, I conceive, may be thus easily demonstrated.

It is manifest, that all the perpendiculars let fall from the ordinates (standing upon the prolonged side DB) to the base CD , doth not only describe the curve, but would also fill the whole hyperbolical space, were number, and the curve definitive. And those perpendiculars let fall from the ordinates (standing upon BD prolonged, numbered 20, 30, 40, 50, 60, &c), doth divide the base DC in $\frac{1}{2} \frac{1}{3} \frac{1}{4} \frac{1}{5} \frac{1}{6}$ &c. And whereas the diagonal CB , by crossing all those perpendiculars, doth give the first term of an infinite series between the root, or side, $AC (= NH)$, and the length of each of those perpendiculars standing upon the base CD . Therefore, to know the hyperbolical space $CAVN$, divide the parallelogram CH , making it the first term of an infinite series by the ratio of $NB = NC$ to NH *in infinitum*, and each of those quotes, or proportional numbers, by 1, 2, 3, 4, 5, 6, 7, 8, &c, also *in infinitum*. The quotes of all the last divisions added, will give you the area of the hyperbolical space $CAVN$, and so for any other curvilinear or hyperbolical space standing upon the base CD , as by the calculation following.

And before I handle any of the other two curvilinear spaces, differing in form, and yet equal to this hyperbolical space $CAVN$, we will exemplify this demonstration in operation; and the figurative work thereof shall be the work of the next chapter.

C H A P. V.

LET it be required to calculate the area of the curvilinear trapezia, or hyperbolical space, $CAVN$ to 15 places, and hereafter you will see it done to 25 places, according to Dairy's rule, in chapter VII. but with greater dispatch.

Thus by the calculation do you find the area of the curvilinear trapezia, or hyperbolical space, $CAVN$ to the 15 place, to be 693147180559945 equal to
the

the curvilinear trapezia $ZE\bar{V}AB$, and also equal to the curvilinear trapezia, or hyperbolical space, $\Pi\Sigma AVX$; the calculation of the areas of any part of these two latter shall be shewn hereafter, which will differ in operation, yet bring out the same number; and in calculating the last, we shall use Dairy's directions.

It having been before shewn, that the hyperbolical space $ZAVAB$, equal to the curvilinear trapezia, or hyperbolical space, $CAVEN$, is equal to the curvilinear trapezia, or hyperbolical space $YGFEZ$; that therefore the said $ZE\bar{V}AB$ is a space, or quantity, to represent the logarithm of 2. So then the aforesaid number 693147180559945 is an hyperbolical logarithm of 2; and having the logarithm of 2, you have also the logarithm of all the powers of 2.

And, by this calculation, you have not only gotten the logarithm of 2, but gained also the logarithm of 3: for if you add all the quotes marked with this asterisk (*), the addition of them shall be half the sum of the hyperbolical logarithm of 3, agreeable to the 4th confectary of the 4th proposition, and first inference on the 5th proposition of said James Gregory's *Exercitationes Geometricæ*; from whence, 'tis plain, that Michael Dairy had his rule, as will appear more manifest after we have contemplated the two other curvilinear trapezias, or hyperbolical spaces, $ZDVAB$ and $\Pi\Sigma AVX$.

The addition of the quotes marked with (*), make 549306144334055, which doubled, is 1098612288668110 for the logarithm of 3; and now, having gotten the logarithm of 3, you have also the logarithm of all the powers of 3, and of all the composites of 2 and 3.

Again, if you shall from 50 subtract 125, and to that add 41666666666666, and from that subtract 15625, and so on throughout, you shall have the logarithm of the difference between 2 and 3, or the logarithm of 1 and $\frac{1}{2}$, or 1 and $\frac{1}{10}$, correspondent to the inference on the 5th prop. of James Gregory; all which shall be fully exemplified hereafter.

The calculation of the logarithm of 2, according to the method before going, is the ground-work of all the calculations following; and I shall only give the calculation of one more space to represent the logarithm of 3, after that method, though we have, you see, gotten already the logarithm thereof; but supposing we had not, and were to find that first according to the said method.

From F you let fall a perpendicular, as FO ; so is the curvilinear trapezia, or hyperbolical space, $AEFOC$ equal to the curvilinear trapezia, or hyperbolical space, $AEF\Phi B$, for the oblong $AEOC$, as before shewn, will be equal to the parallelogram $EF\Phi B$, and the curvilinear triangle $AEFE$ common to both parallelograms. Therefore the curvilinear trapezia $AEFOC$ equal to the curvilinear trapezia, or hyperbolical space, $AEF\Phi B$, to calculate whose area, which will be the logarithm of 3, you proceed as followeth, which work is but partly done, to shew the way thereof, the logarithm of 3 being hereafter done to 25 places, but with far greater dispatch than this method will permit.

6666666666666666	:	I	6666666666666666
4444444444444444	:	II	2222222222222223
296296296296296	:	III	.98765432098765
197530864197530	:	IV	.49382716049382
131687242798352	:	V	.26337448559671
.87791495198902	:	VI	.14631915866484
.58527663465934	:	VII	.08361094780848
.39018443310630	:	VIII	.04877305288829
.26012294873752	:	IX	.02890254985973
.17341529915834	:	X	.01734152991583
.11561019943888	:	XI	.01051001813081
.07707346629258	:	XII	.00642278885771

I have but gone twelve steps in the calculation of the logarithm of 3 after said method, which will, if it were added, but give the logarithm of 3 to 5 places. I have left it unfinished for the exercise of those who shall take delight herein, and finish it throughout, to the intent of making the logarithm of 3 to 15 places, according to this method. By adding the quotes of so much as is done, the first five figures will be 10986, correspondent to the logarithm of 3: this method being somewhat slow, I shall not calculate the logarithms of any other numbers according to it. And, by these two examples, the reader may see enough to calculate any other curvilinear trapezia, or hyperbolical space, standing only in or upon the base cd , equal to an hyperbolical space; reckoned in the prolonged side db .

And so we will contemplate, in the next chapter, the affections and properties of the hyperbolical space, or curvilinear trapezia, $A\Sigma\Theta\Upsilon c$, equal to the curvilinear trapezia, or hyperbolical space, $ABZE\Upsilon$.

C H A P. VII.

BEFORE we shew how to calculate any part of the curvilinear trapezia, or hyperbolical space, $A\Sigma\Theta\Upsilon c$, equal to the curvilinear trapezia, or hyperbolical space, $ABZE\Upsilon$, we will insert tables to illustrate the 1, 2, and 3 propositions of James Gregory's *Exercitationes Geometricæ*.

THE FIRST TABLE OF ILLUSTRATION.

I	A B C D E F G H I K L											
	5	25	125	625	3125	15625	78128	390625	1953125	9765625	48828125	
II												
Z = 9999999999												

The rule to find z,

$$5 - 2,5 = 2,5 : 5 :: 5 : = 10 = z.$$

III	A C E G I				
	5	125	375	5	5
The rule to find z is					
5 - 125 = 375 : 5 :: 5 : =					
66666666 impares = z.					

IV	B D F H K				
	25	625	1875	25	25
The proportion to find z,					
25 - 625 = 1875 : 25 :: 25 : =					
33333333 pares = z.					

V	A C E G I				
	1	1	1	1	1
The rule is					
25 - 625 = 1875 : 25 ::					
25 : = 33333333					
Equalis paribus.					

VI	+ + + + + + + + + + + +											
	A	B	C	D	E	F	G	H	I	K	L	N
The proportion is												
5 + 25 = 75 : 5 :: 5 : = 33333333												
Equales excessui imparium supra omnes pares.												

VII	+ + + + + + + + + + + +											
	A	B	C	D	E	F	G	H	I	K	L	N
The proportion is												
5 - 125 = 375 : 5 :: 5 : = 66666666												
Equalis paribus.												

VIII.	+ + + + + + + + + + + +											
	A	B	C	D	E	F	G	H	I	K	L	N
The proportion is												
5 - 125 = 375 : 5 :: 5 : = 66666666												
Equalis paribus.												

$$2A + 2C + 2E + 2G + 2I +$$

THE EXPLANATION OF THE FOREGOING TABLE.

This table consisteth of eight columns. The first is a supposed literal rank of quantities continually proportional. The second is of numbers correspondent to the first in a ratio, as 2 is to 1; or 5 to 2,5: What 2,5? And so supposed continued *in infinitum*; with the rule how to find out the whole sum of those numbers so continually proportional.

The third column sheweth how to find the whole sum of the odd quantities or numbers.

The fourth teacheth how to know the whole sum of the even quantities or numbers.

The fifth telleth how to find the whole sum of the difference of the quantities of the first column.

The sixth supposeth $A - B + C - D + E$, and teacheth how to find the solution thereof.

The seventh supposeth $A + B - C + D - E$, and giveth a rule to resolve the same.

The eighth and last supposeth the whole rank, first affirmative, and the second evenly or alternately less; and giveth a solution thereof.

A further explanation of this table will be when we come to calculate the area of any part of the curvilinear trapezia $A\Sigma\Theta YC$.

THE SECOND TABLE.

Complement ordinates standing upon CD (more than radius) or perpendi- culars let fall from	10	10	Ordinates or perpendi- culars standing upon DB prolonged.	0	Arithmetical complements.	What proportion the second column hath to radius.
	20	5		5		
	30	3333333333		6666666666		
	40	2,5		7,5		
	50	2		8,		
	60	1,666666666		8,333333333		
	70	1,42877143		8,57142857		
	80	1,25		8,75		
	90	1,111111111		8,888888888		
	100	1.		9.		

This table consisteth of four columns. The first is equal spaces, numbered in the side DB prolonged, or the tangents greater than radius.

The second sheweth the length of the perpendiculars standing upon the side DB prolonged; which are tangents less than radius; and by the tops pass the curve or hyperbolic line.

The third column is the arithmetical complements.

The fourth column sheweth what proportion the second column hath to radius.

The rectangle, or parallelogram, of the first and second column, is equal always to the square AD .

This table is of use to find points to describe the curve or hyperbolical line, or to examine if the curve pass through such points as the table mentions.

The making of this table hath been formerly shewn, when it was taught how to describe the curve.

We now come to shew, how to make a table to find the length of the bases of the compound curvilinear trapezias, or hyperbolical spaces.

We call that a compound curvilinear trapezia, or hyperbolical space, when AC is in the middle of that base.

So AC standing upon the middle of Πx hath perpendiculars or sides $\Pi\Sigma$ and xv , so is the curvilinear trapezia $\Pi\Sigma AVx$, to be hereafter understood a compound curvilinear trapezia, or hyperbolical space, and will be shewn as followeth to be equal to the aforesaid spaces $CAVEN$, and to $AVEZB$ for the logarithm space of 2.

And the compound curvilinear trapezia $\Delta\Theta AVEN$ will be equal to the curvilinear trapezia $AVFOC$, and to $AVF\Phi B$ for the logarithm space of 3.

The compound curvilinear trapezia, or hyperbolical space $\Pi\Sigma AVx$, we may prove to be equal to $AVEZB$, thus: by the 43 of the first of Euclid, the parallelogram CK is equal to $K\zeta$, and the curvilinear triangle AVK common to both; so then is $AVKXC$ equal to $AV\zeta B$. And the parallelogram $\Pi\zeta$ equal to the square vz , and the curvilinear triangle $\Sigma\zeta A$ equal to the curvilinear triangle $v\zeta E$; and so the compound curvilinear trapezia $\Pi\Sigma AVx$ equal to the curvilinear trapezia $AVEZB$ for the logarithm space of 2. For, by the 4th table following, look what proportion the perpendiculars or sides of the compound curvilinear trapezias have one to the other, the like proportion have the sides or perpendiculars of the other two curvilinear trapezias.

So in this compound curvilinear trapezia $\Pi\Sigma$ and xv , the sides or perpendiculars are in a proportion as 2 is to 1 descending, or as 1 is to 2 ascending; so likewise in the curvilinear trapezia $CAVEN$ (equal to the aforesaid compound curvilinear trapezia $\Pi\Sigma AVx$), the side or perpendicular NE is double to CA . And also in the curvilinear trapezia $AVEZB$ (equal, as before, to the compound curvilinear trapezia $\Pi\Sigma AVx$), the side, or perpendicular, BA , twice ZE , as before taught.

Thus, by the ratio of the 2 tables following, you may make a compound curvilinear trapezia equal to either of the other two curvilinear trapezias, or hyperbolical spaces; and the calculating the area of the compound curvilinear trapezias, will be found to be of far greater dispatch than the former method; by which we shall make use of Dairy's rule, or rather the learned James Gregory's, from his first inference on his 5th proposition.

We come now to insert the third table, which is a table of ratios, to find the length of the bases of the compound curvilinear trapezias.

You may note, that in all the three different sorts of curvilinear trapezias, or hyperbolical spaces, equal the one to the other, if on the middle of their bases you shall erect perpendiculars to touch the curve, the greater part or segments in each is equal to either greater segment of the other, and so is the lesser part or segment of the one equal to the lesser segment of either of the other.

THE THIRD TABLE.

Being a Table of Ratios to find the Length of the Bases of the compound curvilinear Trapezias, or hyperbolical Spaces.

I	II	III	IV	V	VI
$\{ 5D : DC :: DC : DX$					
$\{ 15 : 10 :: 10 : ,666666666$				For 2	Length of the base
$\{ 5D : ZD :: DC : D\Pi$					666666666
$\{ 15 : 20 :: 10 : 1,333333333$					
$\{ ZD : DC :: DC : DN$					
$\{ 20 : 10 :: 10 : 5$				3	10
$\{ ZD : \Phi D :: DC : D\Delta$					
$\{ 20 : 30 :: 10 : 15$					
$\{ 25 : 10 :: 10 : 4$				4	12
$\{ 25 : 40 :: 10 : 16$					
$\{ 30 : 10 :: 10 : 333333333$				5	133333333
$\{ 30 : 50 :: 10 : 1,666666666$					
$\{ 35 : 10 :: 10 : 285714285$				6	1428571430
$\{ 35 : 60 :: 10 : 1,714285715$					

This table consisteth of six columns. The first four shew the proportion or ratio to find the lengths of the bases; and the number in the sixth column is the length of the base for so many spaces as the fifth column signifies.

And, by the same reason, you may find the lengths of the bases for any other curvilinear trapezia, or hyperbolical space.

Thus is 666666666, of the sixth column (the difference of the two first numbers in the fourth column), the length of the base, for the curvilinear compound trapezia, or hyperbolical space, to represent the logarithm of 2.

And 10 the length of the base for 3, so is 12 for 4; and 1,333333333 for 5; and so is 1428571430 for the length of the base for the hyperbolical space for 6. And thus may you do for any other space or number.

The numbers in the fourth column for 2, 3, 4, 5, 6, &c, are in proportion as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c; and added are equal to twice radius, or $20 = D\delta$.

We proceed next to shew how to make a table of ratios, to find the lengths of both the perpendiculars, or sides, of the compound curvilinear trapezias.

THE FOURTH TABLE.

Being a Table of Ratios to find the Length of both the Perpendiculars, or Sides, of the compound curvilinear Trapezias, or hyperbolical Spaces.

	I	II	III	IV	V
$\left\{ \begin{array}{l} CD - CX \\ IO - 333333333 \\ CD + c\Pi (= CX) \\ IO + 333333333 \end{array} \right.$	$\begin{array}{l} = XD \\ = 666666666 \\ = D\Pi \\ = 133333333 \end{array}$	$\begin{array}{l} : DB :: DB : D5 \\ : IO :: IO : 15 \\ : DB :: DB : D\Sigma \\ : IO :: IO : 7,5 \end{array}$	$\begin{array}{l} = XV \\ = 15 \\ = D\Sigma \\ = 7,5 \end{array}$	$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} 2$	
$\left\{ \begin{array}{l} CD - CN \\ IO - 5 \\ CD + c\Delta (= CN) \\ IO + 5 \end{array} \right.$	$\begin{array}{l} = ND \\ = 5 \\ = D\Pi \\ = 15 \end{array}$	$\begin{array}{l} : DB :: DB : DZ \\ : IO :: IO : 20 \\ : DB :: DB : \Delta\odot \\ : IO :: IO : 666666666 \end{array}$	$\begin{array}{l} = NE \\ = 20 \\ = \Delta\odot \\ = 666666666 \end{array}$	$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} 3$	
$\left\{ \begin{array}{l} CD - c\mathfrak{D} \\ IO - 6 \\ CD + c\mathfrak{X} (= D\mathfrak{D}) \\ IO + 6 \end{array} \right.$	$\begin{array}{l} = \mathfrak{D}D \\ = 4 \\ = D\mathfrak{X} \\ = 16 \end{array}$	$\begin{array}{l} : DB :: DB : DR \\ : IO :: IO : 25 \\ : DB :: DB : \mathfrak{X}8 \\ : IO :: IO : 625 \end{array}$	$\begin{array}{l} = \mathfrak{D}\mathfrak{R} \\ = 25 \\ = \mathfrak{X}8 \\ = 625 \end{array}$	$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} 4$	
$\left\{ \begin{array}{l} CD - c\Omega \\ IO - 666666666 \\ CD + c\mathfrak{X} (= c\Omega) \\ IO + 666666666 \end{array} \right.$	$\begin{array}{l} = \Omega D \\ = 333333333 \\ = D\mathfrak{X} \\ = 166666666 \end{array}$	$\begin{array}{l} : DB :: DB : D\Phi \\ : IO :: IO : 30 \\ : DB :: DB : \mathfrak{X} \\ : IO :: IO : 6 \end{array}$	$\begin{array}{l} = F\Omega \\ = 30 \\ = \mathfrak{X} \\ = 6 \end{array}$	$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} 5$	

This table consisteth of five columns. The first contains the quantities and numbers of the first term in the proportion. The second column, the quantities and numbers of the second term in the proportion. The third column, the quantities and numbers of the third term in the proportion. The fourth column, the quantities and numbers of the fourth proportional number or term; wherein are numbers for the length of both the perpendiculars for 2, 3, 4, 5, &c. The fifth column, is the numerical order of the compound curvilinear trapezias, or hyperbolical spaces, of 2, 3, 4, 5, &c.

And, by the same ratio, you may find the lengths of both the perpendiculars for any other compound curvilinear trapezias to represent the logarithm of any other number.

By the fourth column, you may perceive the perpendiculars, or sides, of the compound curvilinear trapezias, or hyperbolical spaces, are in such proportion the one to the other, as the number they represent are to unity.

That is to say, in the compound curvilinear trapezia $\Pi\Sigma AVX$ to represent the logarithm space of 2, the perpendicular xv is to $\Pi\Sigma$ as 2 is to 1.

And in the compound curvilinear trapezia $\Delta\odot AVEN$ to represent the logarithm space of 3, the perpendicular NE is in proportion to $\Delta\odot$ as 3 is to 1.

And so the perpendiculars of the fourth logarithm space as 4 to 1; and of the fifth space as 5 to 1, &c; as by the fourth column of this fourth table appeareth.

And

And the perpendiculars of both the other sorts of the curvilinear trapezia, or hyperbolical spaces, are likewise in the very same proportion the one to the other; as you may note from what hath been said before of them.

By these tables, and by what hath been said formerly, these three curvilinear trapezias have the same properties and affections as those have in an hyperbola derived from the section of a right-angled cone.

We shall now, therefore, come to calculate some part of this latter hyperbolical space, before we shew how to do it all at once; that is, of the hyperbolical space $\Delta \odot A V E N$ to calculate the area of the space $\Delta \odot A C$, which is equal, as before shewn, to the space of $5 V A B$. And when we have shewn to calculate this part, we shall, from this, and what hath been taught how to do the other part, come to derive Dairy's rule, or rather James Gregory's, which is comprised in the first inference on his 5th proposition.

C H A P. VII.

WE have, in the fifth chapter, calculated the area of the curvilinear trapezia, or hyperbolical space, $CAVEN$ equal to $AVEZB$ for the logarithm of 2.

In this curvilinear trapezia $CAVEN$ all the perpendiculars standing upon the base CN , are each more than radius CA (or greater than the tangent of $45^\circ 00'$), being still ascending and affirmative; and therefore, by the first table, to be continually adding; as by the calculation thereof is also manifest.

We are now come to calculate the curvilinear trapezia $c\Delta \odot A$, part of the curvilinear trapezia $c\mathcal{V} \odot \odot A$ equal to $AVEZB$.

In this curvilinear trapezia $c\Delta \odot A$ (equal to $B5VA$) all the perpendiculars standing upon the base CA , are each less than the radius CA , being still descending and negative; and, therefore, to be handled by the first table accordingly.

The base $c\Delta$ is equal to CN of that space, calculated, as before, in the 5th chapter.

If by the vertex A you draw a parallel to the diagonal CB as $ZA\mathcal{V}$, it is a tangent to the curve touching it in the point A , and AB doth cut all the perpendiculars contrarywise to CB . For $CX = x^m$ is not equal to $\mathcal{V}\Pi = \Pi\delta$, but $\Pi\delta$ is equal to $K\mathcal{M} = KB$, because $C\Pi$ is equal to CX , and the angle $\Pi\mathcal{V}\delta$ equal to the angle $KB\mathcal{M}$; so, therefore, by the 1 and 4 tables, all the perpendiculars standing upon $c\mathcal{V}$ are less than radius. And seeing, by the sixth column of the first table, and also by the fourth of the fourth table, we may find the length of $\Delta \odot$. Therefore, to know the area of $c\Delta \odot A$, making $c\odot$ the first term of an infinite series in a continual proportion, as CA is to $c\Delta$, that is, as 50 to 25: What 25? *Facit* 125, as in the infinite series of numbers continually proportional for the calculation of the logarithms of 2, in chap. 5. You do therefore, as there said, from 50 (of the second part, under the title, The Quotes to be added)

added) subtract 125, and to that add 4166666666666666, and from that subtract 15625, and so on throughout, you shall have 405465108108165 for the area of the space $c\Delta\Theta A$ equal to $BAV5$. And thus may you find any other part of $c\Gamma\Upsilon\eta A$.

We shall shew how to do it for $c\eta\epsilon A$ and $cxvA$, because from them we shall derive Dairy's rule, or rather James Gregory's; for from them we have derived and calculated the logarithm for two to twenty-five places, as by the calculation following next after this will appear.

Now to calculate the area of the curvilinear trapezia, or hyperbolical spaces, $c\eta\epsilon A$ and $cxvA$, you make cx , or $c\eta$, the first term of any infinite series, and the second term in such a ratio as πc is to ca for your proportionals of your infinite series, and so proceed on, as in chap. V. and as here appeareth.

*The Infinite Series of Numbers
proportional.*

a 333333333333333.
 aa 111111111111111.
 aaa .37037037037037.
 a^4 .12345678012345.
 a^5 ..4115226337448.
 a^6 ..1371742112483.
 a^7 ...457247370827.
 a^8 ...152315790275.
 a^9 50805263425.
 a^{10} 16935087808.
 a^{11} 5645029269.
 a^{12} 1881676423.

The Quotes to be added for $cxvA$.

I $A + 333333333333333 + A$
 II $B - .55555555555555 + B$
 III $C + .12345679012345 + C$
 IV $D - ..3086419753086 + D$
 V $E + ...823045267489 + E$
 VI $F - ...228623685414 + F$
 VII $G +65321052927 + G$
 VIII $H -19051973784 + H$
 IX $I +5645029269 + I$
 X $K -1693508781 + K$
 XI $L +513184479 + L$
 XII $M -156806369 + M$

We have but gone through twelve steps of this calculation, to shew the manner thereof; but should you proceed to go through it till it works off, as in chap. V. you may have both the segment $cxvA$, and $c\eta\epsilon A$: for, if you finish the calculation, and add up all the quotes, that sum will be the area of $cxvA$, and be found 405465108108165, as in chap. V. and is equal to the greater segment $WNHE \triangleright$ in the curvilinear trapezia, or hyperbolical space, $c\eta\epsilon A$, and also $WNHE \triangleright$ is equal to $c\Delta\Theta A$ equal to $AB5V$.

And if from 333333333333333 you shall subtract the second number 555555555555555, and to that add the third number 2345679012345, and then from that subtract the fourth, and so add and subtract according to the signs $+$ and $-$ throughout, you will have ..287682072451780 for the area of the lesser segment of the compound curvilinear trapezia $wn\epsilon A \triangleright$, that is, the area of $c\eta\epsilon A$, equal to $cWDA$, equal to $V5ZE$.

And you have not only gotten, by this calculation, the area of each segment separately, and so, consequently, the area of the whole space, by addition of these two, but you have also obtained the half of the whole area at once; for if you shall (correspondent to the column of the first table) add the numbers with $+$ affirmative, they will give you half the area of the compound curvilinear trapezia

trapezia $\chi\pi\tau\alpha\nu$ for the logarithm of 2, which you will see presently exemplified, and done to 25 places: and this is the sum of James Gregory's inference on his fifth proposition of his *Exercitationes Geometricæ*; and so agreeable to the rule delivered to me, as before declared, by Michael Dairy. Having acquainted several persons with Dairy's rule, in page 45, and shewn to them some figurative work thereupon, in order to make a logarithm, I was, notwithstanding, some time, through inadvertency, almost discouraged of ever knowing how to cube *in infinitum* such a number as there spoken of; neither did any of those to whom I had communicated the same take any such notice thereof (that I know of), so as to do it. And now I come to shew, how I overcame that difficulty of cubing a range of figures for 25 places, which he told me I must do *in infinitum*, before I could make the logarithm of so many places; and to remove the stumbling block, I do confess, took up some time; for Dairy had not then told me a word of such an author as James Gregory, and I had not known his works but for John Collins, some years after Dairy's death. But, before I ever met with Gregory's book, I had obtained my desire to cube *in infinitum* twenty-five figures; that is, twenty-five 3, by dividing by 9 continually, as in the calculation following, to find the logarithm of 2 all at once; which manner dispatcheth the calculation much more speedy than the method of calculation in the fifth chapter.

And now the reason of cubing twenty-five 3, by dividing only by 9, doth follow.

Forasmuch as Dairy's rule, before declared, to make the logarithm of 2, doth bid you to 2, add 1, and from 2 subtract 1, so shall there be a result or fraction of $\frac{1}{3}$; and then divide 1, or 100,000,000,000,0000, by $\frac{1}{3}$, whose quotient is 333333333333333, which cube *in infinitum*, it had been as much as if he had said cube $\frac{1}{3}$ fractionally, which is $\frac{1}{27}$, and divide 1000000000000000000 by $\frac{1}{27}$, the quotient will be 37037037037037 for the cube of a , or $\frac{1}{3}$, as in the operation beforegoing. Now, forasmuch as you would cube the number for $\frac{1}{3}$, viz. 333333333333333 (which is 1, or 100,000,000,000,0000, divided by 3), it is as if you should say, as 27 to 1000000000000000000: What 1? the quotient will be 37037037037037 for the cube of a , or 333333333333333, as before. Now, if you shall, as in the operation beforegoing, set down 333333333333333 (which is equal to $\frac{1}{3}$), you have no more to do but to divide by 9, for that $\frac{1}{9}$ of $\frac{1}{3}$ is equal to $\frac{1}{27}$; and, therefore, dividing 333333333333333 by 9, the quotient will be 370370370370370, as before, for the cube of $\frac{1}{3}$, or a ; and seeing $\frac{1}{3} \times \frac{1}{3}$ is equal to $\frac{1}{9}$, you have no more to do but to divide continually by 9, and they shall all be proportional numbers, by 7th of the 8th of Euclid, and consequently correspondent to the odd powers; for if the root be multiplied by the square that begets the cube, and the cube again by the square that begets the fifth power, and so on. So here, forasmuch as dividing by 9 doth beget the third power; if you shall therefore continually divide by 9, you shall have the respective odd powers accordingly, as is also manifest by the last figurative calculation; and all for that a 1 doth neither multiply nor divide, and that $\frac{1}{3}$ of $\frac{1}{3}$ is equal to $\frac{1}{27}$; and if you shall divide $\frac{1}{27}$ of 1 by 9, the quotient will be 333333333333333, which is equal to a for the first number or root, as before.

Now, forasmuch as to make a compound curvilinear trapezia equal to an uncompounded; as, for instance, to make the compound curvilinear trapezia $w\pi r A$ to be equal to the uncompounded $CAVEN$, equal to $ABZE = A \odot r$ for the logarithm of 2, and to find the length of the base, and both the perpendiculars, hath been discoursed, and may be seen, as in the third and fourth tables foregoing. We come to handle and calculate the area of this compounded curvilinear trapezia $w\pi r A$, for to make the half logarithm of 2 at once.

Seeing, by the sixth column of the third table, the base $w\pi$ is 666666666666666666, whose half is $\dots 333333333333333$ for cw , or $c\pi$, equal to the first term in the former operation (and also the same as Dairy's result, or fraction, of $\frac{1}{3}$), and that I must divide in the ratio of AC to $c\pi$, or cw , *in infinitum*, as in the fifth chapter, and also as in this is shewn and taught, for to make the infinite series of numbers proportional. It will appear, that if I do divide 333333333333333 by 9, it will give me the cube of the first term, and so dividing continually by 9, will produce the numbers appertaining to the odd powers, as by the large calculation to 25 places, next following. And seeing I am, by Dairy's rule, or rather James Gregory's, to divide each of the numbers of the infinite series by the indices of the odd powers, it is manifest, that this rule of Dairy's is derivable from the 8th column of the first table;

$$\text{For } A + B + C + D + E + F + G + H + I$$

$$\text{And } A - B + C - D + E - F + G - H + I$$

$$\text{Doth make } {}^2A + {}^2C + {}^2E + {}^2G + {}^2I + {}^2L + {}^2N;$$

And therefore every other line of the quotes to be added in the former operation, doth make half the logarithm of 2.

In making the infinite series, as in page 66, in order to make the half logarithm of 2 to 25 places, be very careful to set the figures in their due places, and to make that series you are to divide continually by 9, which being done throughout, you may then prove your work by multiplication, in multiplying each line by 9; and if those multiplications produce the foregoing numbers, you may conclude that part of the work to be well prepared. And seeing, by the direction over the figurative work in the second column of page 66, you are to divide each of the numbers in the first by 1, 3, 5, 7, 9, &c, you must so order the quotes of the second, that they may lie in the same line, or range with the respective dividends or numbers in the first. For the better preventing mistakes, the letter figures do represent the divisors proper to each line. And, would you make the logarithm of 2, according to the following method, for 7 or 8 places only, you may very well produce it in half an hour's time, as by that calculation is very perceptible. And some that have had those two sheets I formerly printed as a specimen hereof, have told me, they have done the same; and were very solicitous I would, as soon as I could, publish the remainder; which, at length, as time and leisure hath permitted, is done. And though I have not here inserted many examples, yet, by what are herein done, you may perceive how to proceed for any other number proposed. And, with the direction and reference at the end of this chapter, those that are willing, and curious herein, may make a logarithm for any natural number desired. I have not added hereto any table of logarithms at this time;

and

and what I may do hereafter, in order thereunto, I do not presume to promise. I doubt not but some may both examine some table or other, or make by this method one *de novo*, and satisfy themselves about the same; and some have told me, since my communicating this method unto them, that if the first makers of tables of these numbers had made them by such easy ways, they did not doubt but their tables might have been somewhat more exact. Howsoever, it pleased God, who is the giver of every good and perfect gift, to raise and endue such men with great ability and patience, to perform those tables, with so much difficulty and labour as their methods did require, and for common uses sufficient. And with such eagerness did that age embrace and pursue the invention of these numbers, that Ullach, a Dutchman, had exhibited a table of logarithms to 10 places for 100000, before the learned Henry Briggs's table, which he had in part done to 15 places, could be accomplished by him.—So exceeding glad were they of the invention. And the learned Henry Briggs, in his Epistle Dedicatory to our most gracious King's father, when Prince of Wales, saith, that amongst the ancients there is not found any footsteps of these numbers. Of whose construction and uses the said Henry Briggs hath written, in his *Arithmetica Logarithmica*, most learnedly and copiously. And now follows the figurative part of making the logarithm of 2 to 25 places.

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 9, in order to make the half logarithm of 2.

a	333333333333333333333333333333
a^3	370370370370370370370370370
a^5	41152263374485596707819
a^7	4572473708276177411980
a^9	508052634252908601331
a^{11}	56450292694767622370
a^{13}	6272254743863069152
a^{15}	696917193762563239
a^{17}	77435243751395915
a^{19}	8603915972377324
a^{21}	955990663597480
	106221184844164
	11802353871574
	1311372652397
	145708072489
	16189785832
	1798865092
	199873899
	22208211
	2467579
	274175
	30464
	3385
	376
	42

Differentia $\frac{1}{1}$
 Unitas $\frac{1}{1}$
 Numerus $\frac{1}{2}$
 Proposit. $\frac{1}{2}$
 Summa $\frac{1}{3}$

$\left. \begin{array}{l} \text{Differentia } \frac{1}{1} \\ \text{Unitas } \frac{1}{1} \\ \text{Numerus } \frac{1}{2} \\ \text{Proposit. } \frac{1}{2} \\ \text{Summa } \frac{1}{3} \end{array} \right\} = \frac{a}{3} \times \frac{aaa}{3} = \frac{1}{9}$

You may perceive, that if 1 be added to 2, and subtracted from it, it leaveth a result of $\frac{1}{3}$; which multiplied into itself, maketh $\frac{1}{9}$; and therefore these numbers are continually divided by 9.

These numbers are quotes from those on the opposite side, by dividing them by 1, 3, 5, 7, 9, &c, and are $\frac{2^A}{2} + \frac{2^C}{2} + \frac{2^E}{2} + \frac{2^G}{2} + \frac{2^I}{2} + \frac{2^L}{2} + \frac{2^N}{2} + \&c$, correspondent to the last preceding calculation; which added, make half the area of the compound curvilinear trapezia xnpav for the half logarithm of 2 to 25 places.

I	333333333333333333333333333333	A
III	123456790123456790123457	C
V	8230452674897119341564	E
VII	653210529753739630283	G
IX	56450292694767622370	I
XI	5131844790433420216	L
XIII	482481134143313012	N
XV	46461146250837549	P
XVII	4555014338317407	R
XIX	452837682756702	T
XXI	45523364933213	W
XXIII	4618312384529	Y
XXV	472094154863	
XXVII	48569357496	
XXIX	5024416293	
XXXI	522251156	
XXXIII	54511063	
XXXV	5710683	
XXXVII	600222	
XXXIX	63271	
XLI	6687	
XLIII	7085	
XLV	75	
XLVII	8	
XLIX		

34657359027997265470861605
 half the logarithm of 2,

6931471805599453094172321
 the logarithm of 2.

Thus

Thus have we calculated the logarithm for 2 to 25 places, after Dairy's rule, or rather James Gregory's, which method maketh far greater dispatch than that in chap V; for this calculation, though to 25 places, is sooner performed than that of 15 places, in chap. V; as, by comparing them, is very perspicuous and manifest.

And now we have exemplified the rule Dairy declared; and I am apt to believe he had studied well Gregory's said *Exercitationes*, though he was not pleased to tell any more thereof, but that others should take pains therein as well as he; and, truly, if John Collins had not acquainted me with Gregory's works, I had done the work, but not with that satisfaction I met with from James Gregory's books: and here you have it in a more familiar discourse and dialect than that of James Gregory's, being altogether analitical; and if any letter or symbol be mistaken in his, it is very great study and labour to find, and to set it to rights.

I find James Gregory had calculated the hyperbolical logarithm for 2, in his *Vera Circuli & Hyperbolæ Quadratura*, to 25 places, which agreeth with this calculation but to 17 places. I have not raised the logarithm for 2 to his doctrine in that book, but am satisfied this calculation for the logarithm of 2, in this chapter, is true, to an unit, in the 25th place, and may be, in two hours, very well examined by any one that will take the pains to do it, and they shall find it to be as herein calculated. And to this I have the concurrence of the most ingenious and laborious Mr. Abraham Sharp, who (from the occasion of my publishing formerly two sheets of the praxis hereof as a specimen) hath shewn me his calculation of the logarithm of 2, and some others, to 40 places; the like, I suppose, not hitherto heard of or seen. Without all doubt, Gregory found, that Mercator's lucky invention of squaring the hyperbola, was of far more dispatch than that of his *Vera Circuli & Hyperbolæ Quadratura*, or else he would not have wrote on Mercator: but so excellently hath Gregory illustrated Mercator, that a better way of squaring the hyperbola, I suppose, hath not, nor may be found.

We shall, for the future, keep to this last method of calculating those examples hereafter following. And now we proceed to calculate a logarithm for $1\frac{1}{4}$; for having got the logarithm of 2, we have the logarithm of 4, and all the powers thereof; so then, if we make a logarithm of $1\frac{1}{4}$, and add to it the logarithm of 4, we shall have the logarithm of 5; because that $4 \times 1\frac{1}{4}$, makes 5; that is, 4 multiplied by $1\frac{1}{4}$, makes 5: and this method I took to make the logarithm of 5; and having got the logarithm of 5, to the logarithm of 5 add the logarithm of 2, and you have the logarithm of 10. And, when I have shewn this, I shall produce Briggs's logarithm of 2, by one single division; for that all sorts of logarithms are proportional the one to the other. We therefore now hasten to make the logarithm for $1\frac{1}{4}$.

The Calculation of the Logarithm for $1\frac{1}{4}$ to 25 Places.

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 81.

These numbers are quotes from those on the opposite side, they being divided by 1, 3, 5, 7, 9, &c; which added, make half the logarithms for $1\frac{1}{4}$ to 25 places.

a	I I	I	I A
a ³	13717421124828532235940	III	4572473708276177411980 C
a ⁵	169350878084302867111	V	33870175616860573422 E
a ⁷	2090751581287689718	VII	298678797326812817 G
	25811747917131972	IX	286797199079244I I
	318663554532493 .	XI	28969414048408 L
	3934117957191	XIII	302624458245 N
	48569357496	XV	3237957166 P
	599621697	XVII	35271865 R
	7402737	XIX	389618 T
	91392	XXI	4352 W
	1128	XXIII	49 Y
	I4		I

This number is $\frac{1}{2}$ the logarithm for $1\frac{1}{4}$,

This number is the logarithm of $1\frac{1}{4}$,

III 5717756571048778831475

2231435513142097557662951

Adding 1 to $1\frac{1}{4}$, it maketh $2\frac{1}{4}$, and subtracted from it, leaveth $\frac{1}{4}$, which result maketh a fraction of $\frac{1}{9}$; for $2\frac{1}{4}$ being reduced into fourths, make $\frac{9}{4}$, so the result of $\frac{1}{4}$ and $\frac{9}{4}$ (rejecting the denominators) is $\frac{1}{9}$, as above; which squared, maketh $\frac{1}{81}$: so are these numbers, therefore, continually divided by 81, to make the infinite series. By decimals, it presently sheweth itself to be a fraction of $\frac{1}{9}$. Thus the difference, or numerator, is 25, the sum, or denominator, 2,25; which fraction $\frac{25}{225}$ is equal to $\frac{1}{9}$.

Differentia - 25

Unitas I

To divide the numbers on the other side (to make the infinite series) by 81, is easy enough; for it is but dividing twice by 9, or taking the ninth part twice, and rejecting or cancelling the first; so is it very readily done; and the whole operation hereof may very well

be performed in two hours' time: and thus have we got the logarithm for $1\frac{1}{4}$ to 25 places; and now shall proceed to make the logarithm of 10, which is, by adding together the logarithm of 2, 3 times, and that makes the logarithm of 8; and that added to the logarithm of $1\frac{1}{4}$, makes the logarithm of 10; and the logarithm of 2 subtracted from the logarithm of 10, leaveth the logarithm of 5, and is to the same effect as is before.

Differentia	-	25	
Unitas		1	
Numerus	}	1,25	
Propositus			
Summa	-	2,25	

Logarithm

Logarithm of	2.	6931471805599453094172321
Logarithm of	8.	20794415416798359282516963
Logarithm of	$1\frac{1}{4}$.	2231435513142097557662951
Logarithm of	10.	23025850929940456840179914
Logarithm of	5.	16094379124341003746007593

We have now made and exhibited the logarithms of 2, 5, and 10, and from these you may make all their composites.

And now we proceed to make the logarithm of 3 to 25 places, which we shall shew two ways; first, all at once, from a compound curvilinear trapezia, or hyperbolical space; secondly, by a composition of two logarithms, viz. of 2 and $1\frac{1}{2}$, for that $2 \times 1\frac{1}{2}$, maketh 3; and this latter we chiefly recommend. The compound curvilinear trapezia, or hyperbolical space, $N \Delta \odot AVE$, we have, in the foregoing chapter, shewn to be equal to the uncompounded $CAVFO$, and also equal to $AVF\Phi B$; we shall calculate the logarithm for 3 according to the compounded space, and by the third and fourth tables you may know the lengths of the base and perpendiculars.

The base $N\Delta$ is 10, therefore $cA = 5 = cx$. Now, forasmuch as Dairy's rule of adding to, and subtracting 1 from 3, produces the half length of the base, agreeable to the third table, we shall shew how to calculate the half of the logarithm for 3, as we did for the half logarithm of 2.

Adding 1 to 3, it makes 4, and subtracting 1 from 3, leaveth 2; which maketh a result or fraction of $\frac{2}{4} = \frac{1}{2}$.

Now, dividing 1 and 25 cyphers by $\frac{1}{2}$, the quotient is 5, for the first term of your infinite series. Now, forasmuch as $\frac{1}{2} \times \frac{1}{2}$ maketh $\frac{1}{4}$, the second term must be, therefore, $\frac{1}{4}$ of the first, and so on; as was discoursed before, in making the half logarithm of 2.

Having made the infinite series as followeth, you divide each of those numbers (which, as before taught, are proportional) by 1, 3, 5, 7, 9, &c, and these added, make half the area of $N\Delta\Phi AVE$ for the half logarithm of 3.

The

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 4, in order to make the $\frac{1}{2}$ log. of 3 to 25 places.

These numbers are quotes from those on the opposite side, those being divided by 1, 3, 5, 7, 9, &c; which added, make $\frac{1}{2}$ the log. of 3 to 25 places.

a	5000000000000000000000000000
aaa	125
$aaaaa$	3^{125}
a^7	78125
a^9	1953125
a^{11}	48828125
a^{13}	1220703125
a^{15}	30517578125
a^{17}	762939453125
a^{19}	19073486328125
a^{21}	476837178203125
	11920928955078125
	298023223876953125
	74505805969238281
	18626451492309570
	4656612873077392
	1164153218269343
	291038304567336
	72759576141834
	18189894035458
	4547473508864
	1136868377216
	284217094304
	71054273576
	17763568394
	4440892099
	1110223025
	277555756
	69333936
	17333485
	4333371
	1083343
	270836
	67709
	16927
	4232
	1058
	264
Differentia	2
Unitas	1
Numerus	3
Propositus	3
Summa	4

$$\left. \begin{array}{l} \text{Differentia} \\ \text{Unitas} \\ \text{Numerus} \\ \text{Propositus} \end{array} \right\} 3 = \frac{a}{2} \times \frac{aaa}{2} = \frac{1}{4}$$

I	5000000000000000000000000000
III	4166666666666666666666666666
V	625
VII	11160714295714285714285
IX	2170138888888888888888888888
XI	4438920454545454545454545454
XIII	93900240384615384615
XV	2034505208333333333333333333
XVII	4487879136029411765
XIX	1003867701480263158
XXI	227065313430059523
XXIII	51830125891644022
XXV	11920928855078125
XXVII	2759474295156975
XXIX	642291430769295
XXXI	150213318486367
XXXIII	32247067220283
XXXV	8315389130495
XXXVII	1966475030861
XXXIX	466407539294
XLI	110913988021
XLIII	26438799470
XLV	6315935429
XLVII	1511793055
XLIX	362521804
LI	87076316
LIII	20947604
LV	5046469
LVII	1216385
LIX	293788
LXI	71039
LXIII	17196
LXV	4167
LXVII	1010
LXIX	245
LXXI	59
LXXIII	15
LXXV	3

Half the log. of 3. 5493061443340548456976226
The log. of 3. 10986122886681096913952452

We

We shall now proceed to make the logarithm for 3 the second way, which is from the logarithms of 2 and $\frac{1}{2}$.

The Infinite Series, or Numbers continually proportional.

These numbers are continually multiplied by 4, in order to make the half of the logarithm of $1\frac{1}{2}$ to 25 places.

<i>a</i>	20000000000000000000000000000000
<i>aaa</i>	8
<i>aaaaa</i>	32
<i>a⁷</i>	128
<i>a⁹</i>	512
<i>a¹¹</i>	2048
	8192
	32768
	131072
	524288
	2097152
	8388608
	33554432
	1342177
	53687
	2147
	85

Differentia $\frac{1}{2}$
 —
 Unitas - 1
 Numerus } $1\frac{1}{2}$
 Propositus }
 —
 Summa - $2\frac{1}{2}$

$$a \quad aaa \quad 33554432$$

$$= \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$$

$$1342177$$

$$53687$$

$$2147$$

$$85$$

To make this infinite series, I should divide by 25 continually; but if you multiply by 4, and transfer it answerably, it will be the same thing: because $\frac{1}{5}$ of 1 is $\frac{2}{10}$, and that multiplied in itself, is $\frac{4}{100}$. Therefore, multiplying by 4 and dividing by 100, is the same thing as multiplying by 25. And thus this infinite series is made very speedily, in order to make the half of the logarithm for 1 and $\frac{1}{2}$.

Having now made the logarithm for $1\frac{1}{2}$, you add to it the logarithm of 2, and that makes the logarithm for 3, which will be found, as before, to be the same number.

And now we proceed to make the logarithm for 7, and then we shall have all to 11: in order thereunto, we make the logarithm for $1\frac{2}{5}$, or $1\frac{4}{10}$, and add that to the logarithm of 5, and it will produce the logarithm of 7, for that $1\frac{2}{5}$ multiplied by 5, maketh 7, or $1\frac{4}{10} \times 5 = 7$.

These numbers are quotes from those on the opposite side, they being divided by 1, 3, 5, 7, 9, 11, &c; and added, make half the logarithm for $1\frac{1}{2}$ to 25 places.

I	20000000000000000000000000000000
III	26666666666666666666666666666666
V	64
VII	18285714285714285714
IX	56888888888888888888
XI	1861818181818181818
XIII	630153846153846
XV	21845333333333334
XVII	771011764706
XIX	27594105204
XXI	998643809
XXIII	36472209
XXV	1342177
XXVII	49711
XXIX	1851
XXXI	69
XXXIII	3

2027325540540821909890065
 half the logarithm of $1\frac{1}{2}$,
 4054651081081643819780131
 the logarithm of $1\frac{1}{2}$,
 6931471805599453094172321
 the logarithm for 2,
 10986122886681096913952452
 the logarithm for 3,

The

The Infinite Series, or Numbers continually proportional.

These numbers are continually divided by 36, in order to make half the logarithm of $1\frac{2}{3}$, or $1\frac{4}{6}$.

<i>a</i>	16666666666666666666666666666666
.	27777777777777777777777777777777
<i>aaa</i>	46296296296296296296296296296296
.	77160493827160493827160493827160
<i>aaaaa</i>	1286008230452674897119214334705075445816186
.	357224508459076360315953741807651272672
<i>a⁷</i>	992290301275212112165381716879202019
.	275636194798670034593936579977864
<i>a⁹</i>	765656096662972127609349443829
.	212682249073043544704151217
<i>a¹¹</i>	5907840252039846400420016410667366
.	273511122845585187175975312
Differentia	4 } 12662552
Unitas	1 } 2110425
Numerus	1 } 351737
Proposit.	4 } 58623
Summa	- 2 4 } 9737
	1623
	270

This series is made by dividing twice by 6, which is all one as if you divided at once by 36; and so every other number is the proper number of the series to be divided by 1, 3, 5, 7, 9, 11, &c, as in the opposite column, to make the half logarithm for $1\frac{4}{6}$.

N. B. The logarithm of 5, in page 69, being put in the room of this in the second column, will produce 1945910149055313305105353 for the true logarithm of 7; those two last numbers being part mistaken.

These numbers are quotes from those on the opposite side, those being divided by 1, 3, 5, 7, 9, 11, &c; and added, make half the logarithm of $1\frac{2}{3}$, or $1\frac{4}{6}$.

I	16666666666666666666666666666666
III	15432098765432098765432
V	257201646090534979424
VII	5103207263701090863
IX	110254477819468014
XI	2505783589078819
XIII	58896622820229
XV	1417881660487
XVII	34752001483
XIX	863719335
XXI	21707237
XXIII	550546
XXV	14069
XXVII	360
XXIX	9

1682361183106064652522972
half the logarithm of $1\frac{4}{6}$,
3364722366212129305045944
the logarithm of $1\frac{2}{3}$, or $1\frac{4}{6}$,
16094379124341003168895254
the logarithm of 5,
19459101490553132473941198
the logarithm of 7.

Having

Having by this calculation made the half logarithm of $1\frac{4}{10}$, if we double it, and to that add the logarithm of 5, that addition will produce the logarithm of 7, as was required. And now we have all the logarithms to 11, and to make the logarithms from 10 to 100, it will not be much difficulty to proceed after the foregoing methods; as to make the logarithm of 11, you have for the first term a , the result or fraction $\frac{5}{6}$, and for aa , it will be $\frac{25}{36}$, which is very easy to work; and for the logarithm of 13, you make it of 12 multiplied by $1\frac{1}{12}$; and so it is for the first term a , the result or fraction $\frac{1}{12}$; and for the second aa , it is $\frac{1}{612}$, which 625 is $\approx \frac{5}{8}$; and so may you make many easy compendiums for the prime numbers between 10 and 100, and also, not with great difficulty, from 1000 to 10000; and when you have made some logarithms, you will perceive how the differences arise; and having for composites, logarithms in a readiness, greater and lesser than the prime or incomposite very near, it will be, by the difference, no great difficulty to make a logarithm for such a prime very readily and easily. And they that are curious herein, may have compendiums hereof in James Gregory's aforesaid *Vera Circuli & Hyperbolæ Quadratura*, to make logarithms for prime or incomposite numbers, to which I shall refer him. And here I shall content myself to have exemplified James Gregory's method in his *Exercitationes Geometricæ* to so many examples of logarithms as I have herein calculated to 25 places, and shall, in the next chapter, shew how to produce from these geometrical logarithms Briggs's logarithms.

C H A P. VIII.

HAVING, in the preceding chapter, made the logarithms for 2, 3, 4, 5, 6, 7, 8, 9, and 10, according to the geometrical figure, or hyperbola, I require the logarithm of 2, according to Briggs's table. Forasmuch as all logarithms are proportional, it is as the hyperbolical logarithm of 10, is to its logarithm of 2 :: so is Briggs's logarithm of 10 to this logarithm of 2. The operation followeth.

By this division it doth appear, that this quotient doth agree with Briggs's table of logarithms for his logarithm number of 2, whereby it is apparent he did produce the logarithm for 2 to 15 places very true; though I have been told it was eight persons' work for a year's time after his method, which was by large and many extractions of the square root; and if it was so to 15 places, it would have been very tedious, if not impossible, for them to have produced the logarithm of 2 to 25 places, as before herein is shewn, and done by us; and both the hyperbolical and Briggs's logarithm to 25 places, may very well be calculated and done, according to the foregoing method, in half a day's time; by which method herein beforegoing one may make a table of logarithms, in a short space, to what Pardie (a French author), in his *Elements of Geometry*, hath declared; for he saith, he knew more than 20 persons engaged for 20 years, with indefatigable assiduity, to calculate the logarithms. He doth not say to how many places; but the greatest radius that I have seen of any

French author is but 11 places, which, I suppose, must be the same as Vlacq's. And the logarithm for 2, 3, 4, or 5, &c to 11 places, according to the method in this book, may be very well done and performed in less than two hours' time.

This divisor is half the logarithm of 10, according to the hyperbola.

This quotient is the logarithm of 2, according to Briggs's table.

This dividend is compounded of half the hyperbolical logarithm of 2, and Briggs's logarithm of 10.

Divisor.	Quotient.
.	
115129254649702281	(301029995663981190

Dividend.
.
34657359027997264,000000000000000000
1185820330865797.
3453378436877419.
115079334388337338.
114630052036052851
110137228513207981
65208993284759281
76443659599081405
73561068092600364
45835153027789954
112963766328792697
93474371440605431
13709677208436262
21967517434660339
104445919096901109
8295905121690561

The reader may now see, that logarithms derived from this figure, or the hyperbola, are not only more perceptible and intelligible, but with far more certainty and expedition produced, than what was known in former times.

The divisor in the foregoing work differs 2 units in the 18th place from the half logarithm of 10 before herein calculated; and the reason is, that I took Gregory's logarithm of 10, in his *Vera Circuli & Hyperbolæ Quadratura de bene esse*; and having calculated the half logarithm of 2, as before, I was very desirous to see if we could produce Briggs's logarithm of 2 to 15 places, as by the division is manifest; and this I did long before I met with Gregory's other book of his *Exercitationes Geometricæ*; for, since I got that book, I did calculate, *de novo*, the logarithm of 10 to 25 places, according to his doctrine in that book, and as before herein is done. And the calculation of the logarithm of 10, as before, doth agree with Gregory's former book but to 17 places. Howsoever, the division beforegoing is sufficient to produce Briggs's logarithm for 2 to 15 places; and if any shall be so curious as to produce
Briggs's

Briggs's logarithm for 25 places, he may rely on the foregoing examples herein, and may, in 4 hours' time, examine the foregoing calculations thereof, and in as little time produce Briggs's logarithm for them to the like number of places.

Having this division ready done, long before the publishing hereof, I have contented myself to insert it here, whereby the studious may soon perceive what to do further to gratify himself herein.

I do not add hereto any table of logarithms, that being not my design at this time, but only to shew how Briggs's, or any other logarithms, may be derived from the doctrine beforegoing; and also for the curious, at his will and pleasure, to examine, whether any logarithms formerly published be truly made or not.

As for the various uses of logarithms I add none here, but refer the reader to such authors (whereof there are plenty) who have, long before, written largely and learnedly, as the first inventor, the famous Lord Napier, Henry Briggs, Edmund Gunter, Richard Norwood, Wingate, and divers others; as also my father, John Speidell; in which the reader may meet with many excellent uses of the logarithms in all parts of the mathematicks. And I do find, my father printed several sorts of logarithms, but at last concluded, that the decimal, or Briggs's logarithms, were the best sort for a standard logarithm; and did also print the same several ways, so ordered, whereby they might be applied to arithmetical questions, and other operations for the solution thereof, with ease and readiness.

END OF SPEIDELL'S LOGARITHMOTECHNIA.

E X T R A C T S

FROM THE

PHILOSOPHICAL TRANSACTIONS,

V O L. XIX.

1. *An easy Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line, or Sum of the Secants; with various Methods for computing the same to the utmost Exactness. By Dr. E. Halley.*

IT is now near one hundred years since our worthy countryman, Mr. Edward Wright, published his *Correction of Errors in Navigation*; a book well deserving the perusal of all such as design to use the sea. Therein he considers the course of a ship on the globe, steering obliquely to the meridian; and having shewn, that the departure from the meridian is, in all cases, less than the difference of longitude, in the ratio of radius to the secant of the latitude, he concludes, that the sum of the secants of each point of the quadrant being added successively, would exhibit a line divided into spaces, such as the intervals of the parallels of latitude ought to be in a true sea chart, whereon the meridians are made parallel lines, and the rhombs, or oblique courses, represented by right lines. This is commonly known by the name of the meridian line, which, though it generally be called Mercator's, was yet undoubtedly Mr. Wright's invention, as he has made it appear in his preface. And the table thereof is to be met with in most books treating of navigation, computed with sufficient exactness for the purpose.

It was first discovered by chance, and, as far as I can learn, first published by Mr. Henry Bond, as an addition to Norwood's *Epitome of Navigation*,
about

about fifty years since, that the meridian line was analogous to a scale of logarithmick tangents of half the complements of the latitudes. The difficulty to prove the truth of this proposition, seemed such to Mr. Mercator, the author of *Logarithmotechnia*, that he proposed to wager a good sum of money, against who so would fairly undertake it, that he should not demonstrate either that it was true or false. And, about that time, Mr. John Collins, holding a correspondence with all the eminent mathematicians of the age, did excite them to this inquiry.

The first that demonstrated the said analogy, was the excellent Mr. James Gregory, in his *Exercitationes Geometricæ*, published anno 1668; which he did not without a long train of consequences, and complication of proportions, whereby the evidence of the demonstration is in a great measure lost, and the reader wearied before he attain it. Nor with less work and apparatus hath the celebrated Dr. Barrow, in his *Geometrical Lectures* (Lect. XI. app. 1.), proved, that the sum of all the secants of any arch is analogous to the logarithm of the ratio of radius + sine to rad. — sine; or, which is all one, that the meridional parts answering to any degree of latitude, are as the logarithms of the rationes of the versed sines of the distances from both the poles. Since which, the incomparable Dr. Wallis (on occasion of a paralogism committed by one Mr. Norris in this matter) has more fully and clearly handled this argument, as may be seen in number 176 of the *Transactions*. But neither Dr. Wallis nor Dr. Barrow, in their said Treatises, have any where touched upon the aforesaid relation of the meridian line to the logarithmick tangent; nor hath any one, that I know of, yet discovered the rule for computing independently the interval of the meridional parts answering to any two given latitudes.

Wherefore having attained, as I conceive, a very facile and natural demonstration of the said analogy, and having found out the rule for exhibiting the difference of meridional parts between any two parallels of latitude, without finding both the numbers whereof they are the difference, I hope I may be intitled to a share in the improvements of this useful part of geometry. And first let us demonstrate the following

PROPOSITION.

The meridian line is a scale of logarithmick tangents of the half complements of the latitudes.

For this demonstration it is requisite to premise these four lemmata.

Lemma 1. In the stereographick projection of the sphere upon the plane of the equinoctial, the distances from the center, which, in this case, is the pole, are laid down by the tangents of half those distances; that is, of half the complements of the latitudes.—This is evident from Eucl. 3, 20.

Lemma 2. In the stereographick projection, the angles under which the circles intersect each other are, in all cases, equal to the spherical angles they represent; which is, perhaps, as valuable a property of this projection as that of all

the

the circles of the sphere thereon appearing circles. But this not being vulgarly known, must not be assumed without a

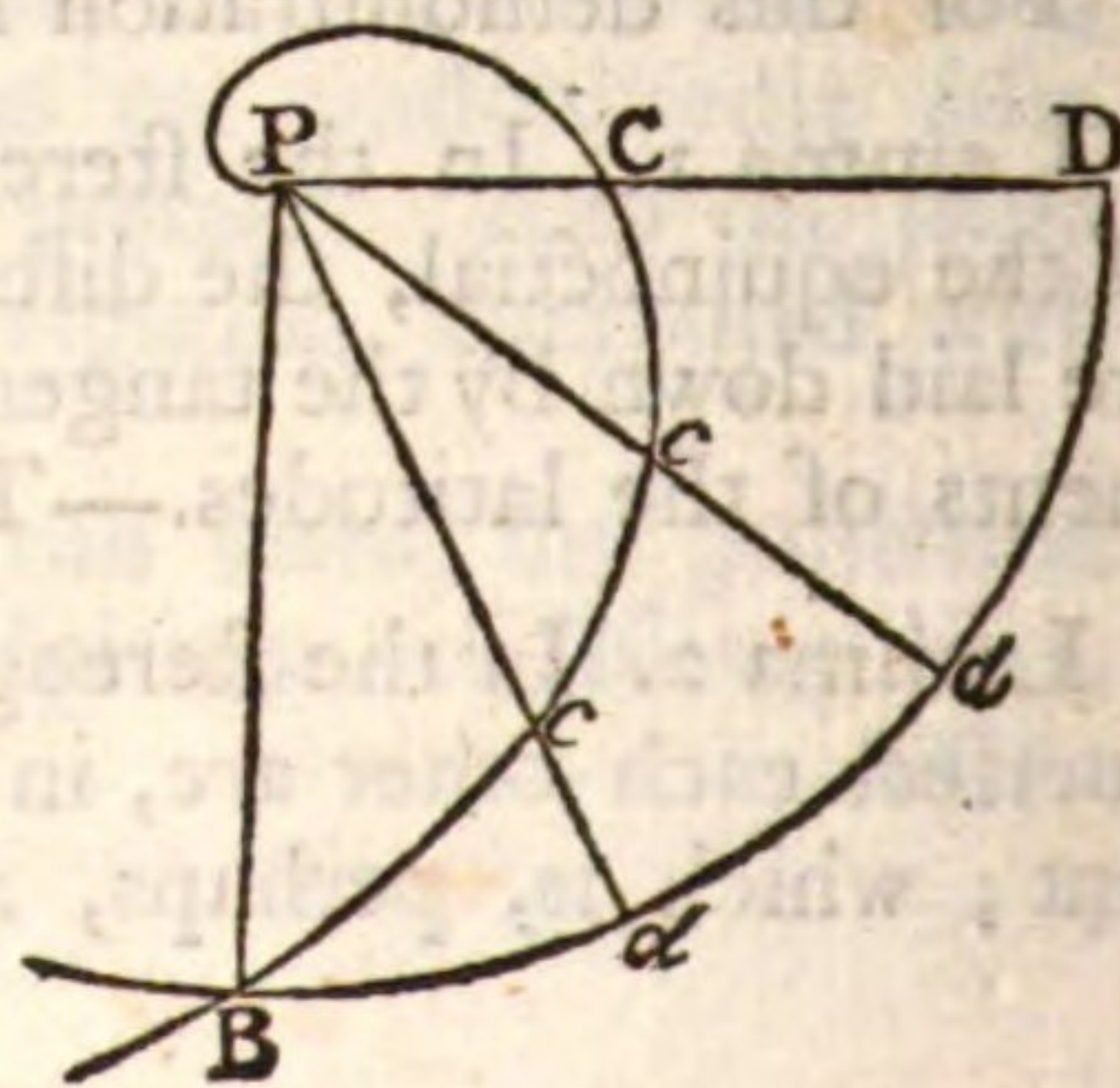
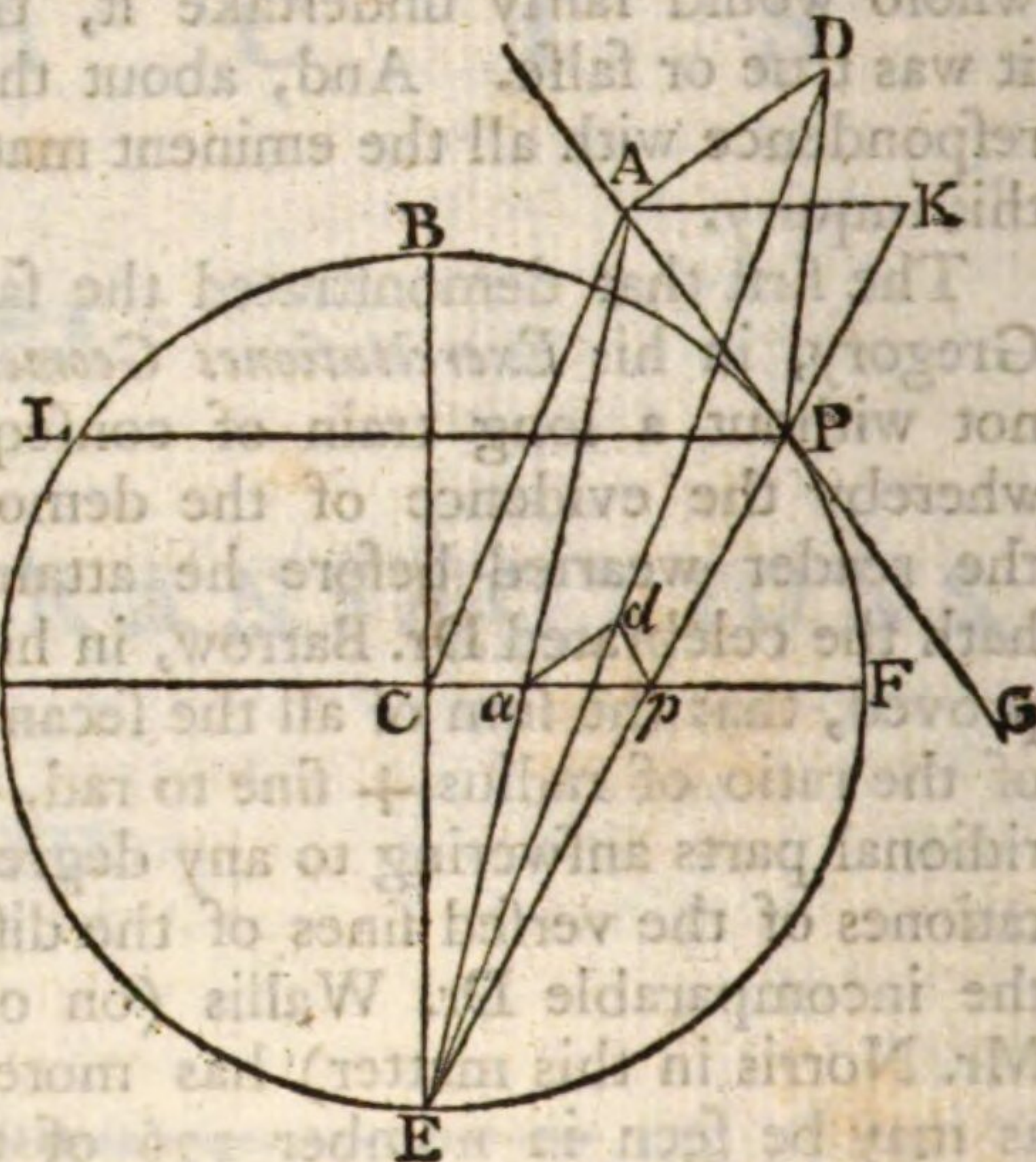
DEMONSTRATION.

Let $EBPL$ be any great circle of the sphere, E the eye placed in its circumference, c its centre, P any point thereof, and let FCO be supposed a plane erected at right angles to the circle $EBPL$, on which FCO we design the sphere to be projected. Draw EP crossing the plain FCO in p , and p shall be the point P projected. To the point P draw the tangent APG , and on any point thereof, as A , erect a perpendicular AD at right angles to the plane $EBPL$, and draw the lines PD , AC , DC , and the angle APD shall be equal to the spherical angle contained between the planes APC , DPC . Draw also AE , DE , intersecting the plain FCO in the points a and d , and join ad , pd ; I say, the triangle adp is similar to the triangle ADP , and the angle apd equal to the angle APD . Draw PL , AK , parallel to FO , and, by reason of the parallels, ap will be to ad as AK to AD . But (by Eucl. 3, 32) in the triangle AKP , the angle $AKP = LPE$, is also equal to $APK = EPG$; wherefore the sides AK , AP , are equal, and it will be as ap to ad , so AP to AD ; whence the angles DAP , dap , being right, the angle APD will be equal to the angle apd ; that is, the spherical angle is equal to that on the projection, and that in all cases; which was to be proved.

This lemma I lately received from Mr. Demoivre, though I since understand, from Dr. Hook, that he long ago produced the same thing before the society. However, the demonstration, and the rest of the discourse, is my own.

Lemma 3. On the globe the rhomb lines make equal angles with every meridian, and, by the foregoing lemma, they must likewise make equal angles with the meridians in the stereographick projection on the plane of the equator; they are therefore, in that projection, proportional spirals about the pole point.

Lemma 4. In the proportional spiral it is a known property, that the angles BPC , or the arches BD , are exponents of the rationes of BP to PC ; for if the arch BD be divided into innumerable equal parts, right lines drawn from them to the centre P , shall divide the curve $Bccc$ into an infinity of proportionals; and all the lines Pc shall be an infinity of proportionals between PB and PC , whose number is equal to all the points dd in the arch BD ; whence, and by what I have delivered in the next ensuing discourse, it follows, that as BD to Bd , or as the angle



BPC to the angle BPC, so is the logarithm of the ratio of PB to PC, to the logarithm of the ratio of PB to PC.

From these lemmata our proposition is very clearly demonstrated; for by the first, PB, PC, PC, are the tangents of half the complements of the latitudes in the stereographick projection; and by the last of them the differences of longitude, or angles at the pole between them, are logarithms of the rationes of those tangents one to the other. But the nautical meridian line is no other than a table of the longitudes, answering to each minute of latitude on the rhomb line, making an angle of 45 degrees with the meridian; wherefore the meridian line is no other than a scale of logarithmick tangents of the half complements of the latitudes; *quod erat demonstrandum*.

Coroll. 1. Because that in every point of any rhomb line the difference of latitude is to the departure, as the radius to the tangent of the angle that rhomb makes with the meridian; and those equal departures are every where to the differences of longitude, as the radius to the secant of the latitude; it follows, that the differences of longitude are, on any rhomb, logarithms of the same tangents, but of a differing species, being proportioned to one another as are the tangents of the angles made with the meridian.

Coroll. 2. Hence any scale of the logarithm tangents (as those of the vulgar tables made after Briggs's form, or those made to Napier's, or any other form whatsoever) is a table of the differences of longitude to the several latitudes, upon some determinate rhomb or other; and therefore as the tangent of the angle of such rhomb to the tangent of any other rhomb, so the difference of the logarithms of any two tangents to the difference of longitude on the proposed rhomb, intercepted between the two latitudes, of whose half complements you took the logarithm tangents.

And since we have a very complete table of logarithm tangents of Briggs's form, published by Vlacq, anno 1633, in his *Canon Magnus Triangulorum Logarithmicus*, computed to ten decimal places of the logarithm, and to every ten seconds of the quadrant (which seems to be more than sufficient for the nicest calculator) I thought fit to inquire the oblique angle with which that rhomb line crosses the meridian, whereon the said Canon of Vlacq precisely answers to the differences of longitudes, putting unity for one minute thereof, as in the common meridian line. Now the momentary argument or fluxion of the tangent line at 45 degrees is exactly double to the fluxion of the arch of the circle, as may easily be proved; and the tangent of 45 being equal to radius, the fluxion also of the logarithm tangent will be double to that of the arch, if the logarithm be of Napier's form; but for Briggs's form, it will be as the same doubled arch multiplied into 0,43429 &c, or divided by 2,30258 &c. Yet this must be understood only of the addition of an indivisible arch, for it ceases to be true if the arch have any determinate magnitude.

Hence it appears, that if any one minute be supposed unity, the length of the arch of one minute being ,000290888208665721596154 &c, in parts of the radius, the proportion will be as unity to 2,908882 &c, so radius to the tangent of $71^{\circ} 1' 42''$, whose logarithm is 10.46372611720718325204 &c; and under that angle is the meridian intersected by that rhomb line, on which the differences of Napier's logarithm tangents of the half complements of the latitudes are the true

true differences of longitude estimated in minutes and parts, taking the first four figures for integers. But for Vlacq's tables we must say,

As 2302585 &c to 2908882 &c, so radius to 1,26331143874244569212 &c, which is the tangent of $51^{\circ} 38' 9''$, and its logarithm 10,101510428507720941162 &c; wherefore in the rhomb line which makes an angle of $51^{\circ} 38' 9''$ with the meridian, Vlacq's logarithm tangents are the true differences of longitude. And this compared with our second corollary may suffice for the use of the tables already computed.

But if a table of logarithm tangents be made by extraction of the root of the infiniteth power whose index is the length of the arch you put for unity (as for minutes the ,0002908882th &c power), which we will call a ; such a scale of tangents shall be the true meridian line, or sum of all the secants, taken infinitely many. Here the reader is desired to have recourse to my little Treatise of Logarithms, in the ensuing discourse, that I may not need to repeat it. By what is there delivered, it will follow, that putting t for the excess or defect of any tangent above or under the radius or tangent of 45 , the logarithm of the ratio of radius to such tangent will be

$\frac{1}{m}$ into $t - \frac{1}{2}tt + \frac{1}{3}ttt - \frac{1}{4}tttt + \frac{1}{5}t^5$ &c when the arch is greater than 45^{gr} , or $\frac{1}{m}$ into $t + \frac{1}{2}tt + \frac{1}{3}ttt + \frac{1}{4}tttt + \frac{1}{5}t^5$ &c when it is less than 45^{gr} . And, by the same doctrine, putting τ for the tangent of any arch, and t for the difference thereof from the tangent of another arch, the logarithm of their ratio will be

$\frac{1}{m}$ into $\frac{t}{\tau} + \frac{tt}{2\tau\tau} + \frac{t^3}{3\tau\tau} + \frac{t^4}{4\tau^4} + \frac{t^5}{5\tau^5}$ &c when τ is the greater term, or $\frac{1}{m}$ into $\frac{t}{\tau} - \frac{tt}{2\tau\tau} + \frac{t^3}{3\tau^3} - \frac{t^4}{4\tau^4} + \frac{t^5}{5\tau^5}$ &c when τ is the lesser term.

And if m be supposed ,0002908882 &c = a , its reciprocal $\frac{r}{a}$ will be 3437, 7467707849392526 &c, which multiplied into the aforesaid series, shall give precisely the difference of meridional parts between the two latitudes to whose half complements the assumed tangents belong. Nor is it material from whether pole you estimate the complements, whether the elevated or depressed, the tangents being to one another in the same ratio as their complements, but inverted.

In the same discourse I also shewed, that the series might be made to converge twice as swift, all the even powers being omitted; and putting τ for the sum of the two tangents, the same logarithm would be $\frac{2}{m}$ or $\frac{2r}{a}$ into $\tau + \frac{t^3}{3\tau^3} + \frac{t^5}{5\tau^5} + \frac{t^7}{7\tau^7} + \frac{t^9}{9\tau^9}$ &c; but the ratio of τ to t , or of the sum of two tangents to their difference, is the same as that of the sine of the sum of the arches to the sine of their difference: wherefore, if s be put for the sine complement of the middle latitude, and s for the sine of half the difference of latitudes, the same series will be $\frac{2r}{a}$ into $\frac{s}{s} + \frac{s^3}{3s^3} + \frac{s^5}{5s^5} + \frac{s^7}{7s^7} + \frac{s^9}{9s^9}$ &c; wherein, as the differences of latitudes are smaller, fewer steps will suffice. And if the equator be put for the middle latitude, and consequently $s = R$, and s to the sine of

the latitude, the meridional parts reckoned from the equator will be $\frac{s}{a} + \frac{s^3}{3rra} + \frac{s^5}{5r^4a} + \frac{s^7}{7r^6a}$ &c, which is coincident with Dr. Wallis's solution, in number 176 of the Philosophical Transactions. And this same series being half the logarithm of the ratio of $R + s$ to $R - s$, that is, of the versed fines of the distances from both poles, does agree with what Dr. Barrow had shewn in his XIth lecture.

The same ratio of τ to t may be expressed also by that of the sum of the co-fines of the two latitudes to the fine of their difference, as likewise by that of the fines of the sum of the two latitudes to the difference of their co-fines, or by that of the versed fine of the sum of the co-latitudes to the difference of the fines of the latitudes, or as the same difference of the fines of the latitudes to the versed fine of the difference of the latitudes; all which are in the same ratio of the co-fine of the middle latitude, to the fine of half the difference of the latitudes. As it were easy to demonstrate, if the reader were not supposed capable to do it himself, upon a bare inspection of a scheme duly representing these lines.

This variety of expression of the same ratio I thought not fit to be omitted, because by help of the rationality of the fine of 30gr. in all cases where the sum or difference of the latitudes is 30gr. 60gr. 90gr. 120gr. or 150 degrees, some one of them will exhibit a simple series, wherein great part of the labour will be saved. And besides, I am willing to give the reader his choice, which of these equippolent methods to make use of; but for his exercise shall leave the prosecution of them, and the compendia arising therefrom, to his own industry; contenting myself to consider only the former, which, for all uses, seems the most convenient, whether we design to make the whole meridian line or any part thereof, viz. $\frac{2r}{a}$ into $\frac{s}{s} + \frac{s^3}{3s^3} + \frac{s^5}{5s^5} + \frac{s^7}{7s^7} + \frac{s^9}{9s^9}$ &c; wherein a is the length of any arch which you design shall be the integer or unity in your meridional parts (whether it be a minute, league, or degree, or any other), s the co-fine of the middle latitude, and s the fine of half the difference of latitudes; but the secants being the reciprocals of the co-fines, $\frac{s}{s}$ will be equal to $\frac{fs}{rr}$ putting f for the secant of the middle latitude; and $\frac{2r}{a}$ into $\frac{s}{s}$ will be $= \frac{2fs}{ar}$. This multiplied by $\frac{ss}{3ss}$, that is, by $\frac{ffss}{3rrrr}$ will give the second step; and that again by $\frac{3ffss}{5rrrr}$, the third step, and so forward till you have completed as many places as you desire. But the squares of the fines being in the same ratio with the versed fines of the double arches, we may, instead of $\frac{ss}{3ss}$, assume for our multiplicator $\frac{v}{3v}$, or the versed fine of the difference of the latitudes, divided by thrice the versed fine of the sum of the co-latitudes, &c, which is the utmost compendium I can think of for this purpose, and the same series will become $\frac{2sr}{as}$ into $1 + \frac{v}{3v} + \frac{v^2}{5v^2} + \frac{v^3}{7v^3} + \frac{v^4}{9v^4}$ &c.

Hereby we are enabled to estimate the default of the method of making the meridian line by the continued addition of the secants of æquidifferent arches, which, as the difference of those arches are smaller, does still nearer and nearer approach the truth. If we assume, as Mr. Wright did, the arch of one minute to be unity, and one minute to be the common difference of a rank of arches, it will be, in all cases, as the arch of one minute to its chord :: so the secant of the middle latitude to the first step of our series. This, by reason of the near equality between a and $2s$, which are to one another in the ratio of unity to $1 - 0,00000000352566457713$ &c, will not differ from the secant f but in the ninth figure, being less than it in that proportion. The next step being $+\frac{2f^3s^3}{3ar^5}$ will be equal to the cube of the secant of the middle latitude multiplied into $\frac{2ss}{3arr} = 0,00000000705132908715$; which therefore, unless the secant exceed ten times radius, can never amount to 1 in the fifth place. These two steps suffice to make the meridian line, or logarithm tangent to far more places than any tables of natural secants yet extant are computed to; but if the third step be required, it will be found to be $+f^5$ into $\frac{2s^5}{5ar^4} = 0,00000000000000000089498$: by all which it appears, that Mr. Wright's table does no where exceed the true meridian parts by fully half a minute; which small difference arises by his having added continually the secants of $1', 2', 3',$ &c, instead of $0\frac{1}{2}', 1\frac{1}{2}', 2\frac{1}{2}', 3\frac{1}{2}',$ &c; but, as it is, it is abundantly sufficient for nautical uses. That in Sir Jonas Moor's New System of the Mathematics is much nearer the truth, but the difference from Wright is scarce sensible till you exceed those latitudes where navigation ceases to be practicable; the one exceeding the truth by about half a minute, the other being a very small matter deficient therefrom.

For an example easy to be imitated by whosoever pleases, I have added the true meridional parts to the first and last minutes of the quadrant; not so much that there is any occasion for such accuracy, as to shew that I have obtained, and laid down herein, the full doctrine of these spiral rhombs which are of so great concern in the art of navigation.

The first minute is 1.00000001410265862178.

The second - 2,00000005641063806707.

The last, or $89^\circ 59'$, is 30374,9634311414228643.

And not 32348,5279, as Mr. Wright has it, by adding the secants of every whole minute; nor 30249,8, as Mr. Oughtred's rule makes it, by adding the secants of every other half minute; nor 30364,3, as Sir Jonas Moor had concluded it, by I know not what method, though in the rest of his table he follows Oughtred.

And this may suffice to shew how to derive the true meridian line from the sines, tangents, or secants, supposed ready made; but we are not destitute of a method for deducing the same independently from the arch itself. If the latitude from the equator be estimated by the length of its arch A , radius being unity, and the arch put for an integer be a , as before, the meridional parts answering to that latitude will be $\frac{1}{a}$ into $A + \frac{1}{6}A^3 + \frac{1}{24}A^5 + \frac{61}{84}A^7$ or $\frac{61}{5040}A^7$

+ $\frac{11}{1386} A^9$ or $\frac{1385}{362880} A^9$ &c; which converges much swifter than any of the former series, and besides has the advantage of A increasing in arithmetical progression, which would be of great ease, if any should undertake, *de novo*, to make the logarithm tangents, or the meridian line, to many more places than now we have them. The logarithm tangent to the arch of $45 + \frac{1}{2}A$ being no other than the aforesaid series $A + \frac{1}{6}A^3 + \frac{1}{24}A^5$ &c, in Napier's form, or the same multiplied into 0,43429 &c, for Briggs's.

But because all these series toward the latter end of the quadrant do converge exceeding slowly, so as to render this method almost useless, or, at least, very tedious, it will be convenient to apply some other arts, by assuming the secants of some intermediate latitudes; and you may for s , or the sine of α , the arch of half the difference of latitudes, substitute $\alpha - \frac{1}{8}\alpha^3 + \frac{1}{180}\alpha^5 - \frac{1}{15040}\alpha^7 + \frac{1}{562880}\alpha^9$ &c, according to Mr. Newton's rule for giving the sine from the arch; and if α be no more than a degree, a very few steps will suffice for all the accuracy that can be desired.

And if α be commensurable to a ; that is, if it be a certain number of those arches with which you make your integer, then will $\frac{\alpha}{a}$ be that number; which if we call n , the parts of the meridional line will be found to be

$$\frac{f_n}{r} \text{ into } \left\{ \begin{array}{l} 1 + \frac{f^2 \alpha \alpha}{3r^4} + \frac{f^4 \alpha^4}{5r^8} + \frac{f^6 \alpha^6}{7r^{12}} \text{ \&c.} \\ - \frac{\alpha \alpha}{6rr} - \frac{f^2 \alpha^4}{6r^6} - \frac{f^4 \alpha^6}{6r^{10}} \text{ \&c.} \\ + \frac{1 \alpha^4}{120r^4} + \frac{13f^2 \alpha^6}{360r^8} \text{ \&c.} \\ - \frac{1 \alpha^6}{5040r^6} \text{ \&c.} \end{array} \right.$$

In this the first two steps are generally sufficient for nautical uses, especially when neither of the latitudes exceed 60 degrees, and the difference of latitudes doth not pass 30 degrees.

But I am sensible I have already said too much for the learned, though too little for the learner: to such I can recommend no better treatise than Dr. Wallis's precedent Discourse, wherein he has, with his usual brevity and that perspicuity peculiar to himself, handled this subject from the first principles, which here, for the most part, we suppose known.

I need not shew how, by regressive work, to find the latitudes from the meridional parts, the method being sufficiently obvious: I shall only conclude with the proposal of a problem which remains to make this doctrine complete, and that is this:

A ship sails from a given latitude, and, having run a certain number of leagues, has altered her longitude by a given angle; it is required to find the course steered. The solution hereof would be very acceptable, if not to the public, at least to the author of this tract; being likely to open some further light into the mysteries of geometry.

To conclude, I shall only add, that unity being radius; the co-sine of the arch A , according to the same rules of Mr. Newton, will be $1 - \frac{1}{2}A^2 + \frac{1}{24}A^4 - \frac{1}{720}A^6 + \frac{1}{40320}A^8 - \frac{1}{3628800}A^{10}$ &c; from which, and the former series exhibiting the sine by the arch, by division, it is easy to conclude, that the natural tangent of the arch A is $A + \frac{1}{3}A^3 + \frac{2}{15}A^5 + \frac{17}{315}A^7 + \frac{62}{2835}A^9$, &c, and the natural secant to the same arch $1 + \frac{1}{2}A^2 + \frac{5}{24}A^4 + \frac{61}{720}A^6 + \frac{277}{8064}A^8$ &c; and from the arithmetick of infinites, the number of these secants being the arch A , it follows, that the sum total of all the infinite secants on that arch is $A + \frac{1}{6}A^3 + \frac{1}{24}A^5 + \frac{61}{3040}A^7 + \frac{277}{72376}A^9$ &c; the which, by what foregoes, is the logarithm tangent of Napier's form, for the arch of $45\text{gr.} + \frac{1}{2}A$, as before.

And collecting the infinite sum of all the natural tangents on the said arch A , there will arise $\frac{1}{2}AA + \frac{1}{12}A^4 + \frac{1}{48}A^6 + \frac{17}{2520}A^8 + \frac{31}{4175}A^{10}$ &c, which will be found to be the logarithm of the secant of the same arch A .

2. *A most compendious and facile Method of constructing the Logarithms, exemplified and demonstrated from the Nature of Numbers, without any Regard to the Hyperbola; with a speedy Method for finding the Number from the Logarithm given.*
By Dr. E. Halley.

THE invention of the logarithms is justly esteemed one of the most useful discoveries in the art of numbers, and accordingly has had an universal reception and applause; and the great Geometricians of this age have not been wanting to cultivate this subject with all the accuracy and subtilty a matter of that consequence doth require; and they have demonstrated several very admirable properties of these artificial numbers which have rendered their construction much more facile than by those operose methods at first used by their truly noble inventor, the Lord Napier, and our worthy countryman, Mr. Briggs.

But, notwithstanding all their endeavours, I find very few of those who make constant use of logarithms, to have attained an adequate notion of them, to know how to make or examine them, or to understand the extent of the use of them; contenting themselves with the tables of them, as they find them, without daring to question them, or caring to know how to rectify them, should they be found amiss; being, I suppose, under the apprehension of some great difficulty therein. For the sake of such, the following tract is principally intended; but not without hopes, however, to produce something that may be acceptable to the most knowing in these matters.

But first it may be requisite to premise a definition of logarithms, in order to render the ensuing Discourse more clear, the rather, because the old one, *Numerorum Proportionalium æqui Differentes comites*, seems too scanty to define them fully. They may more properly be said to be *Numeri Rationum Exponentes*; wherein we consider ratio as a *quantitas sui generis*, beginning from the ratio of equality, or 1 to $1 = 0$; being affirmative when the ratio is increasing, as of unity to a greater number, but negative when decreasing; and these rationes

we suppose to be measured by the number of ratiunculæ contained in each. Now these ratiunculæ are so to be understood as in a continued scale of proportionals infinite in number, between the two terms of the ratio, which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculæ between any two terms, as the logarithm of one ratio is to the logarithm of the other. Thus if there be supposed between 1 and 10 an infinite scale of mean proportionals, whose number is 100000 &c, *in infinitum*, between 1 and 2 there shall be 30102 &c, of such proportionals, and between 1 and 3 there will be 47712 &c, of them; which numbers therefore are the logarithms of the rationes of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the logarithms of 10, 2, and 3.

But if, instead of supposing the logarithms composed of a number of equal ratiunculæ proportional to each ratio, we shall take the ratio of unity to any number, to consist always of the same infinite number of ratiunculæ, their magnitude in this case will be as their number in the former; wherefore if between unity and any number proposed there be taken any infinity of mean proportionals, the infinitely little augment or decrement of the first of those means from unity, will be a ratiuncula; that is, the momentum or fluxion of the ratio of unity to the said number. And seeing that in these continual proportionals all the ratiunculæ are equal, their sum, or the whole ratio, will be as the said momentum is directly; that is, the logarithm of each ratio will be as the fluxion thereof. Wherefore if the root of any infinite power be extracted out of any number, the differentiola of the said root from unity shall be as the logarithm of that number. So that logarithms thus produced may be of as many forms as you please to assume infinite indices of the power whose root you seek; as if the index be supposed 100000 &c, infinitely, the roots shall be the logarithms invented by the Lord Napier; but if the said index were 2302585 &c, Mr. Briggs's logarithms would immediately be produced. And if you please to stop at any number of figures, and not to continue them on, it will suffice to assume an index of a figure or two more than your intended logarithm is to have, as Mr. Briggs did; who, to have his logarithms true to 14 places, by continual extraction of the square root, at last came to have the root of the 140737488355328th power; but how operose that extraction was will be easily judged by who so shall undertake to examine his Calculus.

Now, though the notion of an infinite power may seem very strange, and, to those that know the difficulty of the extraction of the roots of high powers, perhaps, impracticable; yet, by the help of that admirable invention of Mr. Newton, whereby he determines the uncixæ or numbers prefixed to the members composing powers (on which chiefly depends the doctrine of series), the infinity of the index contributes to render the expression much more easy; for if the the infinite power to be resolved be put (after Mr. Newton's method)

$$p + pq, p + pq \Big|^{1/m} \text{ or } 1 + q \Big|^{1/m}, \text{ instead of } 1 + \frac{1}{m}q + \frac{1-m}{2mm}qq + \frac{1-3m+2mm}{6m^3}q^3 \\ + \frac{1-6m+11mm-6m^2}{24m^4}q^4 \text{ \&c (which is the root when } m \text{ is finite), becomes } 1 + \\ \frac{1}{m}q - \frac{1}{2m}qq + \frac{1}{3m}q^3 - \frac{1}{4m}q^4 + \frac{1}{5m}q^5 \text{ \&c, } mm \text{ being infinite infinite; and conse-} \\ \text{quently}$$

quently whatever is divided thereby vanishing. Hence it follows, that $\frac{1}{m}$ multiplied into $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5$ &c, is the augment of the first of our mean proportionals between unity and $1 + q$, and is therefore the logarithm of the ratio of 1 to $1 + q$; and whereas the infinite index m may be taken at pleasure, the several scales of logarithms to such indices will be as $\frac{1}{m}$, or reciprocally as the indices. And if the index be taken 10000 &c, as in the case of Napier's logarithms, they will be simply $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.

Again, if the logarithm of a decreasing ratio be sought, the infinite root of $1 - q$, or $\sqrt[m]{1 - q}$, is $1 - \frac{1}{m}q - \frac{1}{2m}q^2 - \frac{1}{3m}q^3 - \frac{1}{4m}q^4 - \frac{1}{5m}q^5 - \frac{1}{6m}q^6$ &c; whence the decrement of the first of our infinite number of proportionals will be $\frac{1}{m}$ into $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$, &c; which therefore will be as the logarithm of the ratio of unity to $1 - q$. But if m be put 10000 &c, then the said logarithm will be $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c.

Hence the terms of any ratio being a and b , q becomes $\frac{b-a}{a}$, or the difference divided by the lesser term when it is an increasing ratio; or $\frac{b-a}{b}$ when it is decreasing, or as b to a . Whence the logarithm of the same ratio may be doubly expressed; for putting x for the difference of the terms a and b , it will be either

$$\begin{aligned} \frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} \text{ \&c,} \\ \text{or } \frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} \text{ \&c.} \end{aligned}$$

But if the ratio of a to b be supposed divided into two parts, viz. into the ratio of a to the arithmetical mean between the terms, and the ratio of the said arithmetical mean to the other term b , then will the sum of the logarithms of those two ratios be the logarithm of the ratio of a to b ; and substituting $\frac{1}{2}x$ instead of $\frac{1}{2}a + \frac{1}{2}b$, the said arithmetical mean, the logarithms of those ratios will be, by the foregoing rule,

$$\begin{aligned} \frac{1}{m} \text{ in } \frac{x}{z} + \frac{xx}{2zz} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} \text{ \&c,} \\ \text{and } \frac{1}{m} \text{ in } \frac{x}{z} - \frac{xx}{2zz} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} \text{ \&c,} \\ \text{the sum whereof } \frac{1}{m} \text{ in } \frac{2x}{z} + * + \frac{2x^3}{3z^3} + * + \frac{2x^5}{5z^5} + * + \frac{2x^7}{7z^7} \text{ \&c,} \end{aligned}$$

will be the logarithm of the ratio of a to b , whose difference is x and sum z . And this series converges twice as swift as the former, and therefore is more proper for the practice of making logarithms; which it performs with that expedition, that where x the difference is but the hundredth part of the sum, the first

first step $\frac{2x}{z}$ suffices to seven places of the logarithm, and the second step to twelve. But if Briggs's first twenty chiliads of logarithms be supposed made, as he has very carefully computed them to fourteen places, the first step alone is capable to give the logarithm of any intermediate number true to all the places of those tables.

After the same manner may the difference of the said two logarithms be very fitly applied to find the logarithms of prime numbers, having the logarithms of the two next numbers above and below them: for the difference of the ratio of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b , is the ratio of ab to $\frac{1}{4}zz$, and the half of that ratio is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical. And consequently the logarithm thereof will be the half difference of the logarithms of those rationes, viz. $\frac{1}{m}$ into $\frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8}$, &c; which is a theorem of good dispatch to find the logarithm of $\frac{1}{2}z$. But the same is yet much more advantageously performed by a rule derived from the foregoing, and beyond which, in my opinion, nothing better can be hoped. For the ratio of ab to $\frac{1}{4}zz$, or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, has the difference of its terms $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$, or the square of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$, which, in the present case of finding the logarithms of prime numbers, is always unity, and calling the sum of the terms $\frac{1}{4}zz + ab = yy$, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$, or $\frac{1}{2}z$ will be found $\frac{1}{m}$ in $\frac{1}{yy} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{1}{9y^9}$, &c, which converges very much faster than any theorem hitherto published for this purpose.

Here note $\frac{1}{m}$ is all along applied to adapt these rules to all sorts of logarithms. If m be 10000 &c, it may be neglected, and you will have Napier's logarithms, as we hinted before; but if you desire Briggs's logarithms, which are now generally received, you must divide your series by

2,302585092994045684017991454684364207601101488628772976033328,
or multiply it by the reciprocal thereof, viz.

0,434294481903251827651128918916605082294397005803666566114454.

But to save so operose a multiplication (which is more than all the rest of the work), it is expedient to divide this multiplicator by the powers of z or y continually, according to the direction of the theorem, especially where x is small and integer, reserving the proper quotes to be added together, when you have produced your logarithm to as many figures as you desire; of which method I will give a specimen.

If the curiosity of any gentleman that has leisure would prompt him to undertake to do the logarithms of all prime numbers under 100000 to 25 or 30 figures, I dare assure him, that the facility of this method will invite him thereto; nor can any thing more easy be desired. And, to encourage him, I here give the logarithms of the first prime numbers under 20 to 60 places, computed by the accurate pen of Mr. Abraham Sharp (from whose industry and capacity the world may in time expect great performances), as they were communicated to me by our common friend, Mr. Euclid Speidell.

N^o.

N^o.

Logarithm.

2	0,301029995663981195213738894724493026768189881462108541310427
3	0,477121254719662437295027903255115309200128864190695864829866
7	0,845098040014256830712216258592636193483572396323965406503835
11	1,041392685158225040750199971243024241706702190466453094596539
13	1,113943352306837769206541895026246254561189005053673288598083
17	1,230448921378273028540169894328337030007567378425046397380368
19	1,278753600952828961536333475756929317951129337394497598906819

The next prime number is 23, which I will take for an example of the foregoing doctrine; and, by the first rules, the logarithm of the ratio of 22 to 23 will be found to be either

$$\frac{1}{22} - \frac{1}{968} + \frac{1}{31944} - \frac{1}{937024} + \frac{1}{25768160} \&c,$$

$$\text{or } \frac{1}{23} + \frac{1}{1058} + \frac{1}{36501} + \frac{1}{1119364} + \frac{1}{32181715} \&c;$$

as likewise that of the ratio of 23 to 24 by a like process,

$$\frac{1}{23} - \frac{1}{1058} + \frac{1}{36501} - \frac{1}{1119364} + \frac{1}{32181715} \&c,$$

$$\text{or } \frac{1}{24} + \frac{1}{1152} + \frac{1}{41472} + \frac{1}{1327104} + \frac{1}{39813120} \&c.$$

And this is the result of the doctrine of Mercator, as improved by the learned Dr. Wallis. But by the second theorem, viz. $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} \&c$, the same logarithms are obtained by fewer steps; to wit,

$$\frac{2}{45} + \frac{2}{273375} + \frac{2}{922640625} + \frac{2}{2615686171875} \&c,$$

$$\text{and } \frac{2}{47} + \frac{2}{311469} + \frac{2}{1146725035} + \frac{2}{3546361843241} \&c;$$

which was invented and demonstrated in the hyperbolick spaces analogous to the logarithms, by the excellent Mr. James Gregory, in his *Exercitationes Geometricæ*, and since further prosecuted by the aforesaid Mr. Speidell, in a late Treatise, in English, by him published on this subject. But the demonstration, as I conceive, was never till now perfected without the consideration of the hyperbola, which, in a matter purely arithmetical, as this is, cannot be so properly applied. But what follows, I think, I may more justly claim as my own, viz. that the logarithm of the ratio of the geometrical mean to the arithmetical between 22 and 24, or of $\sqrt{528}$ to 23, will be found to be either

$$\frac{1}{1058} + \frac{1}{1119364} + \frac{1}{888215334} + \frac{1}{626487882248} \&c,$$

$$\text{or } \frac{1}{1057} + \frac{1}{3542796579} + \frac{1}{659676558485285} \&c.$$

All these series being to be multiplied into 0,4342944819 &c, if you design to make the logarithm of Briggs. But, with great advantage in respect of the work,

work, the said 4342944819 &c, is divided by 1057, and the quotient thereof again divided by three times the square of 1057, and that quotient again by $\frac{2}{3}$ of that square, and that quotient by $\frac{7}{3}$ thereof, and so forth, till you have as many figures of your logarithm as you desire; as, for example, the logarithm of the geometrical mean between 22 and 24 is found by the logarithms of 2, 3, and 11, to be

		1.361,316,961,266,906,129,450,091,726,698,05
1057	) 43429 &c	(;...410,874,628,101,468,143,473,158,863,68
3 in 1117249	) 41087 &c	(;...122,585,215,441,818,294,600,74
$\frac{2}{3}$ in 1117249	) 12258 &c	(;...65,832,351,843,761,75
$\frac{7}{3}$ in 1117249	) 65832 &c	(;...42,088,297,65
$\frac{2}{7}$ in 1117249	) 42088 &c	(;...29,30
Summa		1.361,727,836,017,592,878,867,777,112,251,17

Which is the logarithm of 23 to 32 places, and obtained by five divisions with very small divisors; all which is much less work than simply multiplying the series into the said multiplicator 43429, &c.

Before I pass on to the converse of this problem, or to shew how to find the number appertaining to a logarithm assigned, it will be requisite to advertise the reader, that there is a small mistake in the aforesaid Mr. James Gregory's *Vera Quadratura Circuli & Hyperbolæ*, published at Padua, anno 1667; wherein he applies his quadrature of the hyperbola to the making the logarithms. In page 48, he gives the computation of the Lord Napier's logarithm of 10 to 25 places, and finds it 2,302,585,092,994,045,624,017,870, instead of 2,302,585,092,994,045,684,017,991, erring in the eighteenth figure; as I was assured upon my own examination of the number I here give you, and by comparison thereof with the same wrought by another hand, agreeing therewith to 57 of the 60 places. Being desirous to be satisfied how this difference arose, I took the no small trouble of examining Mr. Gregory's work, and at length found, that in the inscribed polygon of 512 sides, in the eighteenth figure, was a 0, instead of 9; which being rectified, and the subsequent work corrected therefrom, the result did agree to an unit with our number. And this I propose, not to cavil at an easy mistake in managing of so vast numbers, especially by a hand that has so well deserved of the mathematical sciences, but to shew the exact co-incidence of two so very differing methods to make logarithms, which might otherwise have been questioned.

From the logarithm given to find what ratio it expresses, is a problem that has not been so much considered as the former, but which is solved with the like ease, and demonstrated by a like process, from the same general theorem of Mr. Newton. For, as the logarithm of the ratio of 1 to $1 + q$ was proved to be

$\overline{1 + q}^{\frac{1}{m}} - 1$, and that of the ratio of 1 to $1 - q$ to be $1 - \overline{1 - q}^{\frac{1}{m}}$; so the logarithm (which we will from henceforth call L) being given, $1 + L$ will be equal to $\overline{1 + q}^{\frac{1}{m}}$ in the one case, and $1 - L$ will be equal to $\overline{1 - q}^{\frac{1}{m}}$ in the other;

consequently $\overline{1 + L}^m$ will be equal to $1 + q$, and $\overline{1 - L}^m$ to $1 - q$; that is,

according to Mr. Newton's said rule, $1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 + \frac{1}{120}m^5L^5$ &c, will be $= 1 + q$, and $1 - mL + \frac{1}{2}m^2L^2 - \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 - \frac{1}{120}m^5L^5$ &c, will be equal to $1 - q$, m being any infinite index whatsoever, which is a full and general proposition from the logarithm given to find the number, be the species of logarithm what it will. But if Napier's logarithm be given, the multiplication by m is saved (which multiplication is indeed no other than the reducing the other species to his), and the series will be more simple, viz. $1 + L + \frac{1}{2}LL + \frac{1}{6}L^3 + \frac{1}{24}L^4 + \frac{1}{120}L^5$ &c, or $1 - L + \frac{1}{2}LL - \frac{1}{6}L^3 + \frac{1}{24}L^4 - \frac{1}{120}L^5$ &c. This series, especially in great numbers, converges so slowly, that it were to be wished it could be contracted.

If one term of the ratio, whereof L is the logarithm, be given, the other term will be easily had by the same rule; for if L were Napier's logarithm of the ratio of a the lesser to b the greater term, b would be the product of a into $1 + L + \frac{1}{2}LL + \frac{1}{6}LLL$ &c, $= a + aL + \frac{1}{2}aLL + \frac{1}{6}aL^3$ &c. But if b were given, a would be $= b - bL + \frac{1}{2}bLL - \frac{1}{6}bL^3$ &c. Whence, by the help of the chiliads, the number appertaining to any logarithm will be exactly had to the utmost extent of the tables. If you seek the nearest next logarithm, whether greater or lesser, and call its number a , if lesser, or b , if greater, than the given L , and the difference thereof from the said nearest logarithm you call l ; it will follow, that the number answering to the logarithm L will be either a into $1 + l + \frac{1}{2}ll + \frac{1}{6}lll + \frac{1}{24}l^4 + \frac{1}{120}l^5$ &c, or else b into $1 - l + \frac{1}{2}ll - \frac{1}{6}lll + \frac{1}{24}l^4 - \frac{1}{120}l^5$ &c; wherein, as l is less, the series will converge the swifter. And if the first 20000 logarithms be given to fourteen places, there is rarely occasion for the three first steps of this series to find the number to as many places. But for Vlacq's great canon of 100000 logarithms, which is made but to ten places, there is scarce ever need for more than the first step $a + al$ or $a + mal$, in one case, or else $b - bl$ or $b - mbl$, in the other, to have the number true to as many figures as those logarithms consist of.

If future industry shall ever produce logarithmick tables to many more places than now we have them, the aforesaid theorems will be of more use to reduce the correspondent natural numbers to all the places thereof. In order to make the first chiliad serve all uses, I was desirous to contract this series, wherein all the powers of l are present, into one, wherein each alternate power might be wanting; but found it neither so simple nor uniform as the other. Yet the first step thereof is, I conceive, most commodious for practice, and withal exact enough for numbers not exceeding fourteen places, such as are Mr. Briggs's large table of logarithms; and therefore I recommend it to common use.—It is thus: $a + \frac{al}{1 - \frac{1}{2}l}$ or $b - \frac{bl}{1 + \frac{1}{2}l}$ will be the number answering to the logarithm given, differing from the truth by but one half of the third step of the former series. But that which renders it yet more eligible, is, that, with equal facility, it serves for Briggs's, or any other sort of logarithm, with the only variation of

writing $\frac{1}{m}$ instead of 1; that is, $a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}$ and $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$ or $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$

and

and $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$; which are easily resolved into analogies, viz. As 43429 &c
 $-\frac{1}{2}l$ to 43429 $+\frac{1}{2}l ::$ so is a to the number sought; or, as 43429 &c $+\frac{1}{2}l$ to
 43429 $-\frac{1}{2}l ::$ so is b to the said number sought.

If more steps of this series be desired, it will be found as follows, $a + \frac{al}{1 - \frac{1}{2}l}$
 $-\frac{\frac{1}{2}al^3}{1 - l} + \frac{\frac{1}{3}al^5}{1 - 2l}$ &c, as may easily be demonstrated by working out the di-
 visions in each step, and collecting the quotes, whose sum will be found to agree
 with our former series.

Thus, I hope, I have cleared up the doctrine of logarithms, and shewn their
 construction and use, independent from the hyperbola, whose affections have hi-
 therto been made use of for this purpose, though this be a matter purely arith-
 metical, nor properly demonstrable from the principles of geometry. Nor have
 I been obliged to have recourse to the method of indivisibles, or the arithmetick
 of infinites, the whole being no other than an easy corollary to Mr. Newton's
 General Theorem for forming Roots and Powers.

END OF DR. HALLEY'S DISCOURSES.

N O T E S

O N

SOME OF THE MORE DIFFICULT PASSAGES

O F T H E

FOREGOING DISCOURSE

O F

D R. E D M U N D H A L L E Y.

B Y *FRANCIS MASERES*, E S Q.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

THE foregoing discourse of Dr. Halley on the construction of logarithms has always been considered, even by persons of great skill in the mathematics, as a very obscure and difficult tract: I have therefore thought it might be useful to draw up the following explanatory notes upon many of the more difficult passages of it, by which, I hope, I have rendered them sufficiently intelligible. The reader will, however, observe, that I have made no notes on some of the most important parts of this discourse, to wit, those parts in which the author gives us the investigation of the two logarithmick serieses $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, *ad infinitum*, and $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, *ad infinitum*, and the two anti-logarithmick serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$, *ad infinitum*, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$, *ad infinitum*. The reason of my omitting to make notes on these important passages, is, that I never have been able perfectly to understand them. I have, however, given full investigations of the two former of these infinite serieses, in the "Remarks on the said Serieses" (which were first published by Mr. Mercator and Dr. Wallis), contained in the former volume of this Collection of Tracts, from page 235 to page

page 344; and I have given the like investigations of the two latter of these serieses, or the two anti-logarithmick serieses $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c, ad\ infinitum$, and $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c, ad\ infinitum$, in the subsequent discourse, intituled, "An Appendix to the foregoing Remarks." And it seems probable, that these investigations of these several serieses are either the same with those given, or intended to be given, by Dr. Halley, in the preceeding discourse, or very analogous to them, and founded on the same principles of arithmetick and abstract reasoning on the nature of ratios, without having recourse to the hyperbola, or any other geometrical figure. I therefore refer the reader to those two discourses, in the former volume of those Tracts, for the investigations of these four serieses; and I flatter myself, that, by the attentive perusal of those two discourses, together with the following notes on those passages of Dr. Halley's preceeding tract which I have been able to understand, and which seemed to stand in need of explanation, he will be able to understand, and perceive the truth of, all the conclusions contained in the foregoing tract of Dr. Halley, notwithstanding its obscurity.

NOTE I.

IN page 86, line 16, &c.—Hence the terms of any ratio being a and b , q becomes $\frac{b-a}{a}$, or the difference divided by the lesser term, when it is an increasing ratio, or $\frac{b-a}{b}$, when it is decreasing, or as b to a .

In this passage, Dr. Halley supposes b to be greater than a , and he calls a ratio of minority, as that of a to b , an increasing ratio, because it proceeds from a lesser term to a greater; and he calls a ratio of majority, as that of b to a , a decreasing ratio, because it proceeds from a greater term to a lesser. This seems to be an odd kind of language: but definitions and the meaning of words are arbitrary, or what we please to make them; and this is evidently the sense in which Dr. Halley here uses the expressions of an increasing and a decreasing ratio.

The propositions therefore that are affirmed by Dr. Halley in this passage of his discourse, are these two, to wit: first, that if a be to b as 1 is to $1 + q$, q will be $= \frac{b-a}{a}$; and, secondly, that if b be to a as 1 is to $1 - q$, q will be $= \frac{b-a}{b}$. Now these propositions may be proved in the manner following.

In

In the first place, if a is to b as 1 is to $1 + q$, we shall have, *invertendo*, $b : a :: 1 + q : 1$, and, *dividendo*, $b - a : a :: 1 + q - 1 : 1$, or $b - a : a :: q : 1$; and consequently $q \times a = \overline{b - a} \times 1$, and $q = \frac{\overline{b - a} \times 1}{a} = \frac{b - a}{a}$. Q. E. D.

And secondly, if b is to a as 1 is to $1 - q$, we shall have, *dividendo*, $b - a : a :: 1 - q : 1$, or $b - a : a :: q : 1 - q$. But, by the supposition, $a : b :: 1 - q : 1$; therefore, *ex æquo*, we shall have $b - a : b :: q : 1$, and consequently $q \times b = \overline{b - a} \times 1$, and $q = \frac{\overline{b - a} \times 1}{b} = \frac{b - a}{b}$. Q. E. D.

N O T E II.

IN page 86, line 18, &c.—*Whence the logarithm of the same ratio may be doubly expressed: for, putting x for the difference of the terms a and b , it will be either $\frac{1}{m} \times$ the infinite series $\frac{x}{b} + \frac{xx}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + \&c$, or $\frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + \&c$.*

For if 1 be to $1 + q$ as a is to b , the logarithm of the ratio of a to b will be equal to the logarithm of the ratio of 1 to $1 + q$, that is, to $\frac{1}{m} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$. But q is in this case $= \frac{b - a}{a} = \frac{x}{a}$. Therefore $\frac{1}{m} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$ is $= \frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + \&c$. Consequently $\frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + \&c$, is the logarithm of the ratio of a to b . Q. E. D.

Secondly, if 1 is to $1 - q$ as b is to a , the logarithm of the ratio of b to a will be equal to the logarithm of the ratio of 1 to $1 - q$, that is, to $\frac{1}{m} \times$ the infinite series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$. But q is in this case $= \frac{b - a}{b} = \frac{x}{b}$. Therefore $\frac{1}{m} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$ is $= \frac{1}{m} \times$ the series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + \&c$. Consequently $\frac{1}{m} \times$ the series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + \&c$ is the logarithm of the ratio of b to a . But the logarithm of the ratio of b to a is equal to the logarithm of the ratio of a to b ; because those two ratios are equal,

equal, though contrary, to each other. Therefore $\frac{1}{m} \times$ the infinite series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + \&c$ is equal to $\frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + \&c$, which is the logarithm of the ratio of a to b .

Therefore the logarithm of the ratio of a to b is equal either to $\frac{1}{m} \times$ the infinite series $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} + \&c$, or to $\frac{1}{m} \times$ the infinite series $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} + \&c$, agreeably to Dr. Halley's assertion in the text.

Q. E. D.

NOTE III.

IN page 86, line 23, &c.—But if the ratio of a to b be supposed to be divided into two parts, viz. into the ratio of a to the arithmetical mean between the terms, and the ratio of the said arithmetical mean to the other term b , then will the sum of the logarithms of those two rationes be the logarithm of the ratio of a to b ; and, substituting $\frac{1}{2}z$ instead of $\frac{1}{2}a + \frac{1}{2}b$, the said arithmetical mean, the logarithms of those rationes will be, by the foregoing rule,

$\frac{1}{m}$ into the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$,
and $\frac{1}{m}$ into the series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$,
and the sum of these serieses, or

$\frac{1}{m}$ into the series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \&c$,

will be the logarithm of the ratio of a to b , whose difference is x , and sum z .

This passage requires some explanation. Dr. Halley affirms in it, that if a and b are two given quantities, of which b is the greater, and x be put for their difference $b - a$, and z for their sum $b + a$, the logarithm of the ratio of a to $\frac{z}{2}$, or to $\frac{b+a}{2}$, will be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$ ad infinitum, and that the logarithm of the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to b will

b will be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$ *ad infinitum*. Now this may be shewn in the manner following.

Let the ratio of a to $\frac{z}{2}$, or to $\frac{b+a}{2}$ (which is evidently a ratio of minority, because b is greater than a , and consequently $\frac{b+a}{2}$ is greater than $\frac{a+a}{2}$, or than a) be equal to the ratio of $1 - q$ to 1 ; then will $2a$ be to z in the same proportion of $1 - q$ to 1 . Therefore, *dividendo*, $z - 2a$ will be to z as $1 - (1 - q)$, or q , is to 1 ; and consequently q will be $= \frac{z - 2a}{z} \times 1 = \frac{z - 2a}{z} = \frac{b + a - 2a}{z} = \frac{b - a}{z} = \frac{x}{z}$. Therefore $\frac{1}{m} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, *ad infinitum*, will be $= \frac{1}{m} \times$ the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$ *ad infinitum*. But $\frac{1}{m} \times$ the series $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c$, *ad infinitum*, is the logarithm of the ratio of $1 - q$ to 1 ; therefore $\frac{1}{m} \times$ the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$, *ad infinitum*, will be the logarithm of the same ratio of $1 - q$ to 1 , and consequently of the ratio of a to $\frac{z}{2}$, or $\frac{b+a}{2}$, which is equal to it. Q. E. D.

Secondly, let the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to b (which is evidently a ratio of minority, because b is greater than a , and consequently $2b$ is greater than $b + a$, and $\frac{2b}{2}$, or b , than $\frac{b+a}{2}$) be equal to the ratio of 1 to $1 + q$.

Then will z be to $2b$ in the same proportion of 1 to $1 + q$, and consequently, *dividendo*, $2b - z$ will be to z as $1 + q - 1$, or q , is to 1 . Therefore q will be $= \frac{2b - z}{z} \times 1 = \frac{2b - z}{z} = \frac{2b - (b + a)}{z} = \frac{2b - b - a}{z} = \frac{b - a}{z} = \frac{x}{z}$. Therefore $\frac{1}{m} \times$ the infinite series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, *ad infinitum*, will be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$ *ad infinitum*. But $\frac{1}{m} \times$ the series $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c$, *ad infinitum*, is the logarithm of the ratio of 1 to $1 + q$; therefore $\frac{1}{m} \times$ the series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$, *ad infinitum*, is also the logarithm of the ratio of 1 to $1 + q$, and consequently of the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to b , which is equal to it. Q. E. D.

And hence it follows that the logarithm of the ratio of a to b (which is equal to the sum of the two ratios of a to $\frac{b+a}{2}$ and of $\frac{b+a}{2}$ to b) will be equal to the sum

sum of $\frac{1}{m} \times$ the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$ *ad infinitum*, and $\frac{1}{m} \times$ the series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$ *ad infinitum*, or to $\frac{1}{m} \times$ the series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \frac{2x^9}{9z^9} + \frac{2x^{11}}{11z^{11}} + \&c$ *ad infinitum*; that is, the logarithm of the ratio of a to b , whereof b is the greater, and of which the difference $b - a$ is denoted by x , and the sum $b + a$ is denoted by z , is equal to $\frac{1}{m} \times$ the infinite series $\frac{2x}{z} + \frac{2x^3}{z^3} + \frac{2x^5}{z^5} + \frac{2x^7}{z^7} + \frac{2x^9}{z^9} + \frac{2x^{11}}{z^{11}} + \&c$, agreeably to Dr. Halley's last assertion in the passage here explained.

Q. E. D.

Dr. Halley observes that this series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \&c$ converges twice as swift as either of the two former serieses $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$ and $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$; and says, that, on that account, it is more proper than those other serieses for the practice of making logarithms. But this circumstance is not of any great importance, because there is almost as much trouble in computing four terms, or any other given number of terms, of the series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \&c$, as in computing twice the same number of terms of the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \frac{x^7}{7z^7} + \frac{x^8}{8z^8} + \&c$, or the series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \frac{x^7}{7z^7} - \frac{x^8}{8z^8} + \&c$. For in both cases we must compute the same number of powers of the fraction $\frac{x}{z}$; in which the principal labour of the business consists; for when once we have computed the first eight powers of $\frac{x}{z}$, to wit, $\frac{x}{z}$, $\frac{x^2}{z^2}$, $\frac{x^3}{z^3}$, $\frac{x^4}{z^4}$, $\frac{x^5}{z^5}$, $\frac{x^6}{z^6}$, $\frac{x^7}{z^7}$, and $\frac{x^8}{z^8}$, or any other number of those powers, the dividing them by their indexes 2, 3, 4, 5, 6, 7, 8, &c, which are very small and easy numbers, is a matter of very little difficulty; and the avoiding half of these operations of division, by making use of the third series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \&c$, is but a small saving of trouble.

NOTE IV.

IN page 87, line 8, &c.—For the difference of the ratios of a to $\frac{1}{2}z$, and of $\frac{1}{2}z$ to b , is the ratio of ab to $\frac{1}{4}zz$; and the half of that ratio is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the geometrical mean to the arithmetical; and consequently the logarithm thereof will be the half difference of the logarithms of those rationes, viz. $\frac{1}{m}$ into the series $\frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \&c$ ad infinitum; which is a theorem of good dispatch to find the logarithm of $\frac{1}{2}z$.

The propositions contained in this passage may be proved in the manner following.

The ratio of a to $\frac{1}{2}z$ is equal to the ratio of $a \times b$ to $\frac{1}{2}z \times b$, or of ab to $\frac{1}{2}bz$; and the ratio of $\frac{1}{2}z$ to b is equal to the ratio of $\frac{1}{2}z \times \frac{1}{2}z$ to $b \times \frac{1}{2}z$; that is, to the ratio of $\frac{zz}{4}$ to $\frac{1}{2}bz$. Therefore the excess of the ratio of a to $\frac{1}{2}z$ above the ratio of $\frac{1}{2}z$ to b will be equal to the excess of the ratio of ab to $\frac{1}{2}bz$ above the ratio of $\frac{zz}{4}$ to $\frac{1}{2}bz$. But the excess of the ratio of ab to $\frac{1}{2}bz$ above the ratio of $\frac{zz}{4}$ to $\frac{1}{2}bz$ is evidently the ratio of ab to $\frac{zz}{4}$. Therefore the excess of the ratio of a to $\frac{1}{2}z$ above the ratio of $\frac{1}{2}z$ to b is equal to the ratio of ab to $\frac{zz}{4}$; which is the first proposition affirmed in the foregoing passage. Q. E. D.

Secondly, the ratio of $\sqrt{ab}$ to $\frac{1}{2}z$ is equal to half the ratio of ab to $\frac{zz}{4}$; that is, the ratio of the geometrical mean proportional between the two quantities a and b to the arithmetical mean between the same quantities is equal to half the ratio of ab to $\frac{zz}{4}$. But since the ratio of ab to $\frac{zz}{4}$ has been shewn to be equal to the difference of the ratios of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b , it follows that the ratio of $\sqrt{ab}$ to $\frac{z}{2}$ will be equal to half the difference of the said ratios; that is, the ratio of the geometrical mean proportional between the two quantities a and b to the arithmetical mean between the same quantities will be equal to half the difference between the ratios of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b . Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the said geometrical mean to the said arithmetical mean, will be equal to half the difference of the logarithms of the ratios of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b . But the logarithm of the ratio of a to $\frac{1}{2}z$ has been shewn, in Note III. to be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c$ ad infinitum; and the logarithm of the ratio of $\frac{1}{2}z$ to b has been shewn, in the same note, to be $= \frac{1}{m} \times$ the infinite series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c$ ad infinitum. Therefore the excess of the
logarithm

logarithm of the ratio of a to $\frac{1}{2}z$ above the logarithm of the ratio of $\frac{1}{2}z$ to b will be $= \frac{1}{m} \times$ the series $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \&c - \frac{1}{m} \times$ the series $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} + \&c = \frac{1}{m} \times$ the series $\frac{x}{z} - \frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^5}{5z^5} + \frac{x^6}{6z^6} + \frac{x^6}{6z^6} + \&c = \frac{1}{m} \times$ the series $\frac{2x^2}{2z^2} + \frac{2x^4}{4z^4} + \frac{2x^6}{6z^6} + \&c$; and half the said excess will be $= \frac{1}{m} \times$ the series $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \&c$. Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}z$, or to $\frac{b+a}{2}$, or of the geometrical mean proportional between the quantities a and b to the arithmetical mean between the same quantities, will be $= \frac{1}{m} \times$ the infinite series $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \frac{x^{10}}{10z^{10}} + \frac{x^{12}}{12z^{12}} + \&c$ *ad infinitum*.

Q. E. D.

Thirdly, by this series we may derive the logarithm of the ratio of $\frac{z}{2}$, or $\frac{b+a}{2}$, to 1, from the logarithms of the ratios of a to 1, and of b to 1. For when these two logarithms are known, we need only add them together, and we shall thereby obtain the logarithm of the ratio of ab to 1, the half of which will be the logarithm of the ratio of $\sqrt{ab}$, or the geometrical mean between a and b , to 1. And if we add to this logarithm the logarithm of the ratio of $\frac{z}{2}$ to $\sqrt{ab}$, or of the arithmetical mean between a and b to the said geometrical mean, which may be computed by means of the expression $\frac{1}{m} \times$ the series $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \frac{x^{10}}{10z^{10}} + \frac{x^{12}}{12z^{12}} + \&c$, *ad infinitum*, the sum will be the logarithm of the ratio of $\frac{z}{2}$ to 1, or of $\frac{b+a}{2}$ to 1, or of the said arithmetical mean between a and b to 1.

Q. E. D.

If the number b exceeds the number a by 2, we shall have $x (= b - a) = 2$, and $z (= b + a = a + 2 + a) = 2a + 2$, and consequently $\frac{x}{z} = \frac{2}{2a+2} = \frac{1}{a+1}$. Therefore the series $\frac{x^2}{2z^2} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} + \frac{x^{10}}{10z^{10}} + \frac{x^{12}}{12z^{12}} + \&c$, *ad infinitum*, will, in this case, be $= \frac{1}{2 \times (a+1)^2} + \frac{1}{4 \times (a+1)^4} + \frac{1}{6 \times (a+1)^6} + \frac{1}{8 \times (a+1)^8} + \frac{1}{10 \times (a+1)^{10}} + \frac{1}{12 \times (a+1)^{12}} + \&c$ *ad infinitum*; and consequently the logarithm of the ratio of $\frac{z}{2}$, or $\frac{2a+2}{2}$, or $a+1$, to $\sqrt{ab}$, or $\sqrt{a \times a+2}$, or $\sqrt{aa+2a}$, or of the arithmetical mean between a and b , or between a and $a+2$, to the geometrical mean between them, will be $= \frac{1}{m} \times$ the infinite series $\frac{1}{2 \times (a+1)^2} + \frac{1}{4 \times (a+1)^4} + \frac{1}{6 \times (a+1)^6} + \frac{1}{8 \times (a+1)^8} + \frac{1}{10 \times (a+1)^{10}} + \frac{1}{12 \times (a+1)^{12}} + \&c$ *ad infinitum*.

Thus, for example, if a was $= 22$, and $b = 24$, and we had already computed the logarithms of the ratios of all the prime numbers under 23 to 1, we might make use of the last-mentioned series to find the logarithm of the ratio of the prime number 23 to 1.

For since we had already computed the logarithms of the ratios of the prime numbers 2 and 3 and 11 to 1, we might from thence derive the logarithms of the ratios of 22 and 24 to 1, by mere addition; because the logarithm of the ratio of 22 to 1 is $= L. \frac{22}{11} + L. \frac{11}{1} = L. \frac{2}{1} + L. \frac{11}{1}$, and the logarithm of the ratio of 24 to 1 is $= L. \frac{24}{8} + L. \frac{8}{1} = L. \frac{3}{1} + L. \frac{8}{1} = L. \frac{3}{1} + L. \frac{2^3}{1} = L. \frac{3}{1} + 3 \times L. \frac{2}{1}$. And having thus found the logarithms of the ratios of 22 and 24 to 1, we should have the logarithm of the ratio of 22×4 , or 528, to 1, $= L. \frac{528}{24} + L. \frac{24}{1} = L. \frac{22}{1} = L. \frac{24}{1}$; or the sum of the logarithms of the ratios of 22 to 1 and 24 to 1, would be that of the ratio of 528 to 1. Therefore half the sum of those logarithms would be the logarithm of the ratio of the square-root of 528, or of the geometrical mean proportional between 22 and 24, to 1. And, lastly, if to this logarithm of the ratio of $\sqrt{528}$ to 1 we should add the logarithm of the ratio of 23, or the arithmetical mean between 22 and 24, to $\sqrt{528}$, which may be computed by means of the expression $\frac{1}{m} \times$ the infinite series $\frac{1}{2 \times \sqrt{a+1}^2} + \frac{1}{4 \times \sqrt{a+1}^4} + \frac{1}{6 \times \sqrt{a+1}^6} + \frac{1}{8 \times \sqrt{a+1}^8} + \frac{1}{10 \times \sqrt{a+1}^{10}} + \frac{1}{12 \times \sqrt{a+1}^{12}} + \&c, ad\ infinitum$, or $\frac{1}{m} \times$ the series $\frac{1}{2 \times 23^2} + \frac{1}{4 \times 23^4} + \frac{1}{6 \times 23^6} + \frac{1}{8 \times 23^8} + \frac{1}{10 \times 23^{10}} + \frac{1}{12 \times 23^{12}} + \&c, ad\ infinitum$, or $\frac{1}{m} \times$ the series $\frac{1}{2 \times 529} + \frac{1}{4 \times 529^2} + \frac{1}{6 \times 529^3} + \frac{1}{8 \times 529^4} + \frac{1}{10 \times 529^5} + \frac{1}{12 \times 529^6} + \&c$, the sum will be the logarithm of the ratio of 23 to 1.

Q. E. I.

This method of computing the logarithm of the ratio of 23 to 1 is the same in substance with that by which we computed the same logarithm in the 9th Example of the foregoing Remarks on Mr. Mercator's and Dr. Wallis's Serieses, Art. 32, 80, and 95, pages 264, 304, and 328.

NOTE

NOTE V.

IN page 87, line 13.—But the same is yet much more advantageously performed by a rule derived from the foregoing, and beyond which, in my opinion, nothing better can be hoped; for the ratio of ab to $\frac{1}{4}zz$, or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, has the difference of its terms $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$, or the square of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$; which, in the present case of finding the logarithms of prime numbers, is always unity; and, calling the sum of the terms $\frac{1}{4}zz + ab = yy$, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$, or $\frac{1}{2}z$, will be found $= \frac{1}{m} \times$ the series $\frac{1}{yy} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{1}{9y^9} + \&c$, ad infinitum; which converges much faster than any theorem hitherto published for this purpose.

Since z is $= b + a$, zz will be $= bb + 2ba + aa$. Now bb , ba , and aa , are three quantities in continued geometrical proportion, bb being to ba as b is to a , and ba being to aa also as b is to a . Therefore, by El. 5, 25, the sum of the two extreme terms will be greater than twice the middle term; that is, $bb + aa$ will be greater than $2ab$. Therefore $bb + aa + 2ab$, or $bb + 2ab + aa$, will be greater than $2ab + 2ab$, or $4ab$. Therefore zz (which is $= bb + 2ab + aa$) will be greater than $4ab$, and consequently $\frac{zz}{4}$ will be greater than ab .

Now let ab be put $= A$, and $\frac{zz}{4} = B$; then will B be greater than A . Let $B - A$, or the difference of B and A , or $\frac{zz}{4}$ and ab , be called D , and $B + A$, or the sum of B and A , or $\frac{zz}{4}$ and ab , be called s .

It is shewn above, in Note III. that, if a and b be any two unequal quantities, of which b is the greater, and x be put equal to their difference $b - a$, and z to their sum $b + a$, the logarithm of the ratio of a to b will be equal to $\frac{1}{m} \times$ the infinite series $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5} + \frac{2x^7}{7z^7} + \frac{2x^9}{9z^9} + \frac{2x^{11}}{11z^{11}} + \&c$. It follows therefore, that, in the present case, in which there are two unequal quantities A and B , of which B is the greater, and the difference of the said quantities is denoted by D , and their sum by s , the logarithm of the ratio of A to B will be $= \frac{1}{m} \times$ the infinite series $\frac{2D}{s} + \frac{2D^3}{3s^3} + \frac{2D^5}{5s^5} + \frac{2D^7}{7s^7} + \frac{2D^9}{9s^9} + \frac{2D^{11}}{11s^{11}} + \&c$. But the ratio of A to B is equal to the ratio of ab to $\frac{zz}{4}$, because A is $= ab$, and B is $= \frac{zz}{4}$. Therefore the logarithm of the ratio of ab to $\frac{zz}{4}$ will be $= \frac{1}{m} \times$ the infinite series $\frac{2D}{s} + \frac{2D^3}{3s^3} + \frac{2D^5}{5s^5} + \frac{2D^7}{7s^7} + \frac{2D^9}{9s^9} + \frac{2D^{11}}{11s^{11}} + \&c$. Therefore the logarithm of half the ratio of ab to $\frac{zz}{4}$, or of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, will be $= \frac{1}{m} \times$ the infinite series $\frac{D}{s} + \frac{D^3}{3s^3} + \frac{D^5}{5s^5} + \frac{D^7}{7s^7} + \frac{D^9}{9s^9} + \frac{D^{11}}{11s^{11}} + \&c$; that is, the logarithm

logarithm of the ratio of $\sqrt{ab}$, the geometrical mean between the numbers a and b , to $\frac{z}{2}$, or $\frac{b+a}{2}$, the arithmetical mean between the same numbers, is $= \frac{1}{m} \times$ the series $\frac{D}{s} + \frac{D^3}{3s^3} + \frac{D^5}{5s^5} + \frac{D^7}{7s^7} + \frac{D^9}{9s^9} + \frac{D^{11}}{11s^{11}} + \&c$, *ad infinitum*.

Now let the difference of the numbers a and b be 2, or let b be $= a + 2$.

Then will ab be $(= a \times \overline{a+2}) = aa + 2a$; and $\frac{zz}{4}$ will be $(= \frac{a+b}{4})^2 = \frac{a+a+2}{4}^2 = \frac{2a+2}{4}^2 = \frac{4a^2+8a+4}{4} = a^2 + 2a + 1$; and consequently

$\frac{zz}{4} - ab$ will be $(= a^2 + 2a + 1 - \overline{a^2 + 2a}) = 1$. But $\frac{zz}{4} - ab$ is $= D$.

Therefore, in this case, or when b is $= a + 2$, D will be $= 1$; and consequently $D^3, D^5, D^7, D^9, D^{11}$, and all the following powers of D , will also be equal to 1. Therefore the logarithm of the ratio of $\sqrt{ab}$, or the geometrical mean between the numbers a and b , to $\frac{z}{2}$, or $\frac{b+a}{2}$, the arithmetical mean between the same numbers, will, in this case, or when b is $= a + 2$, become $= \frac{1}{m} \times$ the infinite series $\frac{1}{s} + \frac{1}{3s^3} + \frac{1}{5s^5} + \frac{1}{7s^7} + \frac{1}{9s^9} + \frac{1}{11s^{11}} + \&c$ *ad infinitum*. Therefore, if, instead of s , we substitute yy in the terms of this series, or suppose yy to be equal to $\frac{zz}{4} + ab$, or the sum of $\frac{zz}{4}$ and ab , we shall have $\frac{1}{m} \times$ the series $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} + \&c$, *ad infinitum*, for the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of the geometrical mean between the two numbers a and b , or a and $a + 2$, to the arithmetical mean $(\frac{b+a}{2},$ or $\frac{a+2+a}{2},$ or $\frac{2a+2}{2},$ or) $a + 1$, between the same numbers.

Q. E. I.

An Example of the Computation of a Logarithm by Means of the foregoing Series.

Let it be required to find, by means of this last series, the logarithm of the ratio of the geometrical mean between the numbers 22 and 24 to the arithmetical mean between the same numbers; that is, to the number 23.

Here we have $a = 22$, $b = 24$, $ab (= 22 \times 24) = 528$, and $\sqrt{ab} = \sqrt{528}$, and $z (= b + a = 24 + 22) = 46$, and consequently $\frac{z}{2} = 23$, and $\frac{zz}{4} (= 23^2) = 529$, and $yy (= \frac{zz}{4} + ab = 529 + 528) = 1057$, and $y^4 (= 1057^2) = 1,117,249$. Therefore the series $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} + \&c$ will be $= \frac{1}{1057} + \frac{1}{3 \times 1057^3} + \frac{1}{5 \times 1057^5} + \frac{1}{7 \times 1057^7} + \frac{1}{9 \times 1057^9} + \frac{1}{11 \times 1057^{11}} + \&c$, and $\frac{1}{m} \times$ the series $\frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} + \frac{1}{11y^{22}} + \&c$

&c will be $= \frac{1}{m} \times$ the series $\frac{1}{1057} + \frac{1}{3 \times 1057^3} + \frac{1}{5 \times 1057^5} + \frac{1}{7 \times 1057^7} + \frac{1}{9 \times 1057^9} + \frac{1}{11 \times 1057^{11}} + \&c$; that is, the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of $\sqrt{528}$ to 23, will be $= \frac{1}{m} \times$ the series $\frac{1}{1057} + \frac{1}{3 \times 1057^3} + \frac{1}{5 \times 1057^5} + \frac{1}{7 \times 1057^7} + \frac{1}{9 \times 1057^9} + \frac{1}{11 \times 1057^{11}} + \&c$ *ad infinitum*.

Now $\frac{1}{m} \times$ the series $\frac{1}{1057} + \frac{1}{3 \times 1057^3} + \frac{1}{5 \times 1057^5} + \frac{1}{7 \times 1057^7} + \frac{1}{9 \times 1057^9} + \frac{1}{11 \times 1057^{11}} + \&c$ is $=$ the series $\frac{1}{m \times 1057} + \frac{1}{3^m \times 1057^3} + \frac{1}{5^m \times 1057^5} + \frac{1}{7^m \times 1057^7} + \frac{1}{9^m \times 1057^9} + \frac{1}{11^m \times 1057^{11}} + \&c$; which, if we put c for the first term $\frac{1}{m \times 1057}$, and D for the second term $\frac{1}{3^m \times 1057^3}$, and E for the third term $\frac{1}{5^m \times 1057^5}$, and F, G, H, &c, for the fourth, fifth, sixth, and other following terms of the series, will be $=$ the series $\frac{1}{m \times 1057} + \frac{c}{3 \times 1057^2} + \frac{3^D}{5 \times 1057^2} + \frac{5^E}{7 \times 1057^2} + \frac{7^F}{9 \times 1057^2} + \frac{9^G}{11 \times 1057^2} + \&c = \frac{1}{m \times 1057} + \frac{c}{3 \times 1,117,249} + \frac{3^D}{5 \times 1,117,249} + \frac{5^E}{7 \times 1,117,249} + \frac{7^F}{9 \times 1,117,249} + \frac{9^G}{11 \times 1,117,249} + \&c$. Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of $\sqrt{528}$ to 23, is $=$ the series $\frac{1}{m \times 1057} + \frac{c}{3 \times 1,117,249} + \frac{3^D}{5 \times 1,117,249} + \frac{5^E}{7 \times 1,117,249} + \frac{7^F}{9 \times 1,117,249} + \frac{9^G}{11 \times 1,117,249} + \&c$ *ad infinitum*.

But in Briggs's system of logarithms $\frac{1}{m}$ is $=$
 $\frac{1}{2.302,585,092,994,045,684,017,991,454,684,364,207, \&c} = 0.434,294,481,903,251,827,651,128,918,916,605,082, \&c$. Therefore $\frac{1}{m \times 1057}$ is $(= \frac{1}{m} \times \frac{1}{1057}) = 0.434,294,481,903,251,827,651,128,918,916,605,082, \&c \times \frac{1}{1057} = 0.434,294,481,903,251,827,651,128,918,916,605,082, \&c \div 1057 = 0.000,410,874,628,101,468,143,473,158,863,68$; and the series $\frac{1}{m \times 1057} + \frac{c}{3 \times 1,117,249} + \frac{3^D}{5 \times 1,117,249} + \frac{5^E}{7 \times 1,117,249} + \frac{7^F}{9 \times 1,117,249} + \frac{9^G}{11 \times 1,117,249} + \&c$ is $=$ the series $0.000,410,874,628,101,468,143,473,158,863,68 + \frac{c}{3 \times 1,117,249} + \frac{3^D}{5 \times 1,117,249} + \frac{5^E}{7 \times 1,117,249} + \frac{7^F}{9 \times 1,117,249} + \frac{9^G}{11 \times 1,117,249} + \&c =$

$$\begin{aligned}
& 0;000,410,874,628,101,468,143,473,158,863,68 \\
& + 0; \dots, \dots, \dots, 122,585,215,441,818,294,600,74 \\
& + 0; \dots, \dots, \dots, \dots, \dots, 065,832,351,843,761,75 \\
& + 0; \dots, \dots, \dots, \dots, \dots, \dots, 042,088,297,65 \\
& + 0; \dots, \dots, \dots, \dots, \dots, \dots, \dots, 029,30 \\
& + \&c \\
& = 0;000,410,874,750,686,749,417,685,385,553,12 \&c.
\end{aligned}$$

Therefore the logarithm of the ratio of $\sqrt{ab}$ to $\frac{z}{2}$, or $\frac{b+a}{2}$, or of $\sqrt{528}$ to 23 is $= 0.000,410,874,750,686,749,417,685,385,553,12, \&c.$ Q. E. I.

The logarithm of the ratio of 1 to 23 is equal to the sum of the logarithms of the ratios of 1 to $\sqrt{528}$ and of $\sqrt{528}$ to 23. The logarithm of the ratio of 1 to $\sqrt{528}$ is equal to half the logarithm of the ratio of 1 to 528, or to half the sum of the logarithms of the ratios of 1 to 22 and of 22 to 528, or of 1 to 22 and of 1 to 24, or to half the sum of the logarithms of the ratios of 1 to 2, and of 2 to 22, and of 1 to 3, and of 3 to 24, or to half the sum of the logarithms of the ratios of 1 to 2, and of 1 to 11, and of 1 to 3, and of 1 to 8, or to $\frac{1}{2} L. \frac{1}{2} + \frac{1}{2} L. \frac{1}{11} + \frac{1}{2} L. \frac{1}{3} + \frac{1}{2} L. \frac{1}{8}$, and therefore may be derived from the logarithms of those lesser ratios by addition. And, if it be so derived, it will be found to be $= 1.361,316,961,266,906,129,450,091,726,698,05, \&c.$ If therefore we add to this logarithm the logarithm of the ratio of $\sqrt{528}$ to 23, which has just now been found to be $= 0.000,410,874,750,686,749,417,685,385,553,12, \&c.$ the sum, to wit, $1.361,727,836,017,592,878,867,777,112,251,17, \&c.$ will be the logarithm of the ratio of 1 to 23, or of 23 to 1, in Briggs's system, or, according to the common way of expressing ourselves on this subject, the logarithm of the number 23.

NOTE VI.

IN page 90, line 11.—If one term of the ratio, whereof L is the logarithm, be given, the other term will easily be had by the same rule: for, if L was Napier's logarithm of the ratio of a , the lesser, to b , the greater, term, b would be the product of a into the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c.$ ad infinitum, $=$ the series $a + aL + \frac{aL^2}{2} + \frac{aL^3}{2.3} + \frac{aL^4}{2.3.4} + \frac{aL^5}{2.3.4.5} + \frac{aL^6}{2.3.4.5.6} + \&c.$ ad infinitum. But if b was given, a would be $=$ the series $b - bL + \frac{bL^2}{2} - \frac{bL^3}{2.3} + \frac{bL^4}{2.3.4} - \frac{bL^5}{2.3.4.5} + \frac{bL^6}{2.3.4.5.6} - \&c.$ ad infinitum.

The

The first proposition asserted in this passage is, "that, if any number, or quantity, called a , be given, of the ratio of which to another quantity called b , which is greater than it, L is the logarithm in Napier's System, the said greater quantity b , may be derived from a and L , by computing the value of the series $a + aL + \frac{aL^2}{2} + \frac{aL^3}{2.3} + \frac{aL^4}{2.3.4} + \frac{aL^5}{2.3.4.5} + \frac{aL^6}{2.3.4.5.6} + \&c$, *ad infinitum*, to which it will be equal." Now this may be proved in the manner following.

Suppose k to be of such a magnitude, that $1 + k$ shall be to 1 as b is to a . Then will L (which is the logarithm of the ratio of b to a) be also the logarithm of the ratio of $1 + k$ to 1 . Therefore $1 + k$ will be $=$ the series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$, *ad infinitum*, and consequently that series will be to 1 as b is to a . Therefore $b \times 1$, or b , will be $= a \times$ the said series $1 + L + \frac{L^2}{2} + \frac{L^3}{2.3} + \frac{L^4}{2.3.4} + \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} + \&c$, *ad infinitum*, $= a + aL + \frac{aL^2}{2} + \frac{aL^3}{2.3} + \frac{aL^4}{2.3.4} + \frac{aL^5}{2.3.4.5} + \frac{aL^6}{2.3.4.5.6} + \&c$, *ad infinitum*.

Q. E. D.

The Second Proposition asserted in the foregoing passage is, "That, if any number, or quantity, called b , be given, of the ratio of which to a lesser quantity, called a , the logarithm, in Napier's system, is known and denoted by L , the said lesser quantity a may be derived from the said greater quantity b , and the logarithm L , by computing the value of the series $b - bL + \frac{bL^2}{2} - \frac{bL^3}{2.3} + \frac{bL^4}{2.3.4} - \frac{bL^5}{2.3.4.5} + \frac{bL^6}{2.3.4.5.6} - \&c$, *ad infinitum*, to which it will be equal." This may be proved in the manner following.

Suppose b to be of such a magnitude that 1 shall be to $1 - k$ in the same proportion as b to a . Then will L be the logarithm of the ratio of 1 to $1 - k$, as well as of the ratio of b to a . Therefore $1 - k$ will be $=$ the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$, *ad infinitum*. Therefore 1 will be to this series in the same proportion as b is to a ; and consequently $a \times 1$, or a , will be $= b \times$ the series $1 - L + \frac{L^2}{2} - \frac{L^3}{2.3} + \frac{L^4}{2.3.4} - \frac{L^5}{2.3.4.5} + \frac{L^6}{2.3.4.5.6} - \&c$, *ad infinitum* $=$ the series $b - bL + \frac{bL^2}{2} - \frac{bL^3}{2.3} + \frac{bL^4}{2.3.4} - \frac{bL^5}{2.3.4.5} + \frac{bL^6}{2.3.4.5.6} - \&c$, *ad infinitum*.

Q. E. D.

NOTE VII.

IN page 90, line 15th.—Whence, by the help of the Chiliads, the number appertaining to any logarithm will be exactly had to the utmost extent of the Tables. If you seek the nearest next logarithm, whether greater or lesser; and call its number a , if lesser, or b , if greater, than the given L , and the difference thereof from the said nearest logarithm you call l ; it will follow that the number answering to the logarithm L will be either $a \times$ the series $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} + \&c$, ad infinitum, or else $b \times$ the series $1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} - \&c$, ad infinitum.

The first proposition contained in this passage is as follows; to wit, “That, if any logarithm in Napier’s system of Logarithms be given, and the same be denoted by L , and we seek in a Table of Napier’s Logarithms the logarithm which is nearest to L of all those which are less than L , and denote the said logarithm by the Greek letter Λ and the number corresponding to it (or of the ratio of which to unity it is the logarithm) by a , and the excess of L above Λ by the small letter l , the number corresponding to the given logarithm L , (or of the ratio of which to unity the said given logarithm L is the logarithm,) will be $= a \times$ the series $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} + \&c$. *ad infinitum*.” This may be shewn in the manner following.

Let b be put for the number hitherto unknown, which corresponds to the given logarithm L , or of the ratio of which to unity L is the logarithm. We shall then be to prove that b is $= a \times$ the series $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} + \&c$ *ad infinitum*, or $=$ the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$ *ad infinitum*.

Now, since L is the logarithm of the ratio of b to 1, and Λ is the logarithm of the ratio of a to 1, it follows that l , or $L - \Lambda$, will be the logarithm of the ratio which is equal to the excess of the ratio of b to 1 above the ratio of a to 1; that is, l will be the logarithm of the ratio of b to a . But it has been shewn above in Note 6th, that if l is the logarithm of the ratio of b to a , (the quantity b being greater than a ,) b will be equal to the series $a + al + \frac{al^2}{2} + \frac{al^3}{2.3} + \frac{al^4}{2.3.4} + \frac{al^5}{2.3.4.5} + \frac{al^6}{2.3.4.5.6} + \&c$, *ad infinitum*, or the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$, *ad infinitum*. Therefore the ratio of b to 1, or the ratio corresponding to the given logarithm L , is that of the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$, *ad infinitum* to 1. Q. E. D.

The second proposition contained in the foregoing passage is as follows, to wit, “That if any logarithm in Napier’s System of logarithms be given, and the same be denoted by the letter L , and you seek in a table of logarithms the logarithm

logarithm which is nearest to L of all those that are greater than L , and denote the said logarithm by the Greek letter Λ , and the number corresponding to it (or of the ratio of which to unity it is the logarithm,) by b , and the excess of Λ above L by the small letter l , the number corresponding to the given logarithm L , (or of the ratio of which to unity L is the logarithm,) will be equal to $b \times$ the series $1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} - \&c$, *ad infinitum*, or to the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum*, or the ratio corresponding to the given logarithm L , is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum* to 1." This may be shewn in the manner following.

Let a be put for the number, hitherto unknown, which corresponds to the given logarithm L , or of the ratio of which to unity L is the logarithm. We shall then be to prove that a is equal to $b \times$ the series $1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} - \&c$, *ad infinitum*, or to the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum*.

Now, since L is the logarithm of the ratio of a to 1, and Λ is the logarithm of the ratio of b to 1, it follows that l , or $\Lambda - L$, will be the logarithm of the ratio which is equal to the excess of the ratio of b to 1 above the ratio of a to 1, that is, of the ratio of b to a . But it has been shewn above in Note 6th that, if l is the logarithm of the ratio of b to a , (the quantity b being greater than a ,) a will be equal to the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{2.3} + \frac{bl^4}{2.3.4} - \frac{bl^5}{2.3.4.5} + \frac{bl^6}{2.3.4.5.6} - \&c$, *ad infinitum*, or the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum*. Therefore the ratio of a to 1, or the ratio corresponding to the given logarithm L , is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$ *ad infinitum* to 1. Q. E. D.

In the foregoing passage, and the remarks made to explain it, the logarithms L , Λ , and l , have been supposed to be Napier's Logarithms of the ratios to which they respectively belong. But the tables of logarithms that are most in use at present, give us only the logarithms of Briggs's system, in which the logarithm of any given ratio is less than Napier's logarithm of the same ratio in the proportion of 1 to 2.302,585,092,994,045,684, &c, or of 0.434,294,481, 903,251,827, &c to 1. If therefore we are required to find the ratio corresponding to any given logarithm of Briggs's system, we must proceed as follows.

Let the given logarithm of Briggs's system, whereof we are required to find the corresponding ratio, be called B , and Napier's logarithm of the same ratio be called L . Also let the next lesser logarithm to B that is set down in the Tables be denoted by the small Greek letter λ , and Napier's logarithm of the same ratio be denoted by the capital Greek letter Λ . And let the difference be-

tween the logarithms B and λ of Briggs's system be called d , and the difference between the corresponding logarithms L and Λ in Napier's System be called l . Then will L be $= 2.302,585,092,994,045,684, \&c \times B$, and Λ will be $= 2.302,585,092,994,045,684, \&c \times \lambda$, and $L - \Lambda$ will be $= 2.302,585, \&c \times B - 2.302,585, \&c \times \lambda = 2.302,585, \&c \times B - \lambda$; that is, l will be $= 2.302,585, \&c \times d$; or, (if, for brevity's sake, we put $m = 2.302,585,092,994,045,684, \&c$,) l will be $= m d$, and consequently l^2 will be $= m^2 d^2$, and $l^3 = m^3 d^3$, and $l^4 = m^4 d^4$, and $l^5, l^6, \&c = m^5 d^5, m^6 d^6, \&c$.

Further, let a be put for the number found in the tables corresponding to the logarithm λ , which is the next less logarithm to the given logarithm B, that is set down in the tables. Then will a be the number corresponding likewise to the logarithm Λ in Napier's system. But it has been shewn in the first part of this note, that, upon these suppositions, the ratio corresponding to L, or of which L is the logarithm in Napier's system, is that of the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$ *ad infinitum* to 1, which is equal to that of the series $a + am d + \frac{am^2 d^2}{2} + \frac{am^3 d^3}{6} + \frac{am^4 d^4}{24} + \frac{am^5 d^5}{120} + \frac{am^6 d^6}{720} + \&c$ *ad infinitum* to 1. Therefore the ratio corresponding to the given logarithm B, in Briggs's System, is that of the series $a + am d + \frac{am^2 d^2}{2} + \frac{am^3 d^3}{6} + \frac{am^4 d^4}{24} + \frac{am^5 d^5}{120} + \frac{am^6 d^6}{720} + \&c$ to 1. Q. E. I.

Secondly, Let the given logarithm of Briggs's system, whereof we are required to find the corresponding ratio, be denoted, as before, by the letter B, and the logarithm of the same ratio in Napier's system be likewise, as before, denoted by L. But let the small Greek letter λ be put for the nearest logarithm of Briggs's system, to the given logarithm B of all those that are greater than B, that is set down in the table, instead of being put for the next lesser logarithm, as in the former case. And let the capital Greek letter Λ be put for Napier's logarithm of the same ratio of which λ is the logarithm in Briggs's System. Also let d be put for $\lambda - B$, or the difference of the logarithms λ and B in Briggs's system, and l be put for $\Lambda - L$, or the difference of the logarithms Λ and L in Napier's system. Then will L be $= 2.302,585,092,994,045,684, \&c \times B$, and $\Lambda = 2.302,585,092,994,045,684, \&c \times \lambda$, and consequently $\Lambda - L$ will be $= 2.302,585, \&c \times \lambda - 2.302,585, \&c \times B = 2.302,585, \&c \times \lambda - B = 2.302,585, \&c \times d$, that is, l will be $= 2.302,585,092,994,045,684, \&c \times d$; or, (if we put $m = 2.302,585,092,994,045,684, \&c$,) l will be $= m d$; and consequently l^2 will be $= m^2 d^2$, and $l^3 = m^3 d^3$, and $l^4 = m^4 d^4$, and $l^5, l^6, \&c = m^5 d^5, m^6 d^6, \&c$.

Further, let b be put for the number found in the tables corresponding to the logarithm λ , which is the next greater logarithm to the given logarithm B, which is set down in the tables. Then will b be the number corresponding likewise to the logarithm Λ in Napier's system. But it has been shewn in the second part of this note, that, upon these suppositions, the ratio corresponding to L, or of which L is the logarithm in Napier's system, is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum* to 1. Therefore the ratio corresponding

sponding to B, or of which B is the logarithm in Briggs's system, (which is the same ratio of which L is the logarithm in Napier's system,) is that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum* to 1, or (because l is $= md$) that of $b - bmd + \frac{bm^2d^2}{2} - \frac{bm^3d^3}{6} + \frac{bm^4d^4}{24} - \frac{bm^5d^5}{120} + \frac{bm^6d^6}{720} - \&c$, *ad infinitum* to 1. Q. E. I.

In making the foregoing deductions I have constantly used the letters L, Λ , and l , to denote the logarithms of their respective ratios in Napier's system, and have employed the letters B, λ , and d , to denote the logarithms of the same ratios respectively in Briggs's system. This I have done, in order to preserve the reasonings as clear and distinct as possible. But Dr. Halley, in page 90, line 24th, uses the letter l to denote the logarithms of the same ratio both in Napier's and Briggs's systems. His words are these: "But, for Vlacq's great canon of 100,000 logarithms, which is made but to ten places, there is scarce ever need for more than the first step $a + al$, or $a + mal$, in one case, or else $b - bl$, or $b - mbl$, in the other, to have the number true to as many figures as those logarithms consist of." In this passage the letter l signifies a logarithm of Napier's system in the expressions $a + al$ and $b - bl$; and it signifies a logarithm of Briggs's system, and consequently ml signifies the corresponding logarithm in Napier's system, in the expressions $a + mal$ and $b - mbl$. These last expressions $a + mal$ and $b - mbl$ correspond to the two expressions $a + amd$ and $b - bmd$, in the foregoing deductions, or to the two first terms of the two infinite serieses $a + amd + \frac{am^2d^2}{2} + \frac{am^3d^3}{6} + \frac{am^4d^4}{24} + \frac{am^5d^5}{120} + \frac{am^6d^6}{720} + \&c$, *ad infinitum*, and $b - bmd + \frac{bm^2d^2}{2} - \frac{bm^3d^3}{6} + \frac{bm^4d^4}{24} - \frac{bm^5d^5}{120} + \frac{bm^6d^6}{720} - \&c$, *ad infinitum*, l signifying the same quantity in Dr. Halley's notation as d signifies in ours.

NOTE VIII.

IN page 90, line 29.—If future industry shall ever produce logarithmick tables to many more places than now we have them, the aforesaid theorems will be of more use to deduce the correspondent numbers to all the places thereof. In order to make the first Chiliad serve all uses, I was desirous to contract the series, wherein all the powers of 1 are present, into one wherein each alternate power might be wanting; but found it neither so simple nor so uniform as the other. Yet the first step thereof is, I conceive, most commodious for practice, and withal exact enough for numbers

not exceeding fourteen places, such as are those of Mr. Briggs's large table of logarithms; and therefore I recommend it to common use. It is thus; $a + \frac{al}{1 - \frac{1}{2}l}$, or $b - \frac{bl}{1 + \frac{1}{2}l}$, will be the number answering to the logarithm given, differing from the truth, but by one half of the third step of the former series.

This passage contains two propositions, to wit, 1st, "That the two terms $a + \frac{al}{1 - \frac{1}{2}l}$, or $a + \frac{1l}{1 - \frac{1}{2}l}$, are very nearly equal to the whole series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$, *ad infinitum*;" and 2dly, "That the two terms $b - \frac{bl}{1 + \frac{1}{2}l}$, or $b - \frac{bl}{1 + l}$, are very nearly equal to the whole series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum*."

The first of these propositions may be proved in the manner following.

The fraction $\frac{al}{1 - \frac{1}{2}l}$ is equal to the geometrical progression $al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} + \&c$, *ad infinitum*. Therefore the two terms $a + \frac{al}{1 - \frac{1}{2}l}$ are equal to the series $a + al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} + \&c$, *ad infinitum*, which agrees with the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$, *ad infinitum* in the three first terms, and exceeds it only by the series $(\frac{al^3}{4} - \frac{al^3}{6} + \frac{al^4}{8} - \frac{al^4}{24} + \frac{al^5}{16} - \frac{al^5}{120} + \frac{al^6}{32} - \frac{al^6}{720} + \&c$, *ad infinitum*, or $\frac{3al^3}{12} - \frac{2al^3}{12} + \frac{3al^4}{24} - \frac{al^4}{24} + \frac{30al^5}{480} - \frac{4al^5}{480} + \frac{45al^6}{32 \times 45} - \frac{2al^6}{2 \times 720} + \&c$, *ad infinitum*, or $\frac{al^3}{12} + \frac{2al^4}{24} + \frac{26al^5}{480} + \frac{45al^6}{1440} - \frac{2al^6}{1440} + \&c$, *ad infinitum*, or) $\frac{al^3}{12} + \frac{al^4}{12} + \frac{13al^5}{240} + \frac{43al^6}{1440} + \&c$, *ad infinitum*, of which the first term $\frac{al^3}{12}$ is equal to only half of $\frac{al^3}{6}$, which is the fourth term of the series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$, *ad infinitum*, or the third quantity that is to be computed and added to the first term a , or (according to Dr. Halley's expression) the third step of the said series.

Q. E. D.

The second proposition contained in the aforesaid passage may be proved in the manner following.

The fraction $\frac{bl}{1 + \frac{1}{2}l}$ is equal to the series $bl - \frac{bl^2}{2} + \frac{bl^3}{4} - \frac{bl^4}{8} + \frac{bl^5}{16} - \frac{bl^6}{32} + \&c$, *ad infinitum*. Therefore the two terms $b - \frac{bl}{1 + \frac{1}{2}l}$ are equal to $b -$ the second $l - \frac{bl^2}{2} + \frac{bl^3}{4} - \frac{bl^4}{8} + \frac{bl^5}{16} - \frac{bl^6}{32} + \&c$, *ad infinitum*, that is, to the series

ries $b - bl + \frac{bl^2}{2} - \frac{bl^3}{4} + \frac{bl^4}{8} - \frac{bl^5}{16} + \frac{bl^6}{32} - \&c$, *ad infinitum*, which agrees with the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum* in the three first terms, and falls short of it by the series $\frac{bl^3}{4} - \frac{bl^3}{6} - \frac{bl^4}{8} + \frac{bl^4}{24} + \frac{bl^5}{16} - \frac{bl^5}{120} - \frac{bl^6}{32} + \frac{bl^6}{720} + \&c$, *ad infinitum*, or the series $\frac{bl^3}{12} - \frac{bl^4}{12} + \frac{13bl^5}{240} - \frac{43bl^6}{1440} + \&c$, *ad infinitum*, of which the first term $\frac{bl^3}{12}$ is equal to only half of $\frac{bl^3}{6}$ which is the fourth term of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, or the third quantity that is to be computed and added to the first term a , which is what I understand Dr. Halley to mean by the expression of the *the third step* of the said series.

Q. E. D.

NOTE IX.

IN page 90, line 3 from the bottom.—*But that which renders it more eligible, is, that with equal facility it serves for Briggs's, or any other sort of logarithms, with the only variation of writing $\frac{1}{m}$ instead of 1, that is, $a + \frac{al}{\frac{1}{m} - 1}$ and $b - \frac{bl}{\frac{1}{m} + 1}$, or $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}}$ and $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}}$, which are easily resolved into analogies, viz. as 0.434, 29 $\&c - \frac{1}{2}l$ is to 0.434, 29 $\&c + \frac{1}{2}l$, so is a to the number sought, or as 0.434, 29 $\&c + \frac{1}{2}l$, is to 0.434, 29 $\&c - \frac{1}{2}l$, so is b to the number sought.*

In this passage, (if I understand it rightly,) Dr. Halley uses the letter l for the logarithm of a ratio in some other system than Napier's, and supposes that the logarithms of any given ratios in such other system are to the logarithms of the same ratios in Napier's system in the proportion of 1 to m . This notation seems to throw some obscurity upon the passage, to remove which I shall substitute d instead of l , as above in Note 7th, and keep l to denote the logarithm of the same ratio in Napier's system. And then the foregoing passage will be found to contain the two following propositions; to wit, 1st, "That, if B be any given logarithm in Briggs's system, or in any other system of logarithms different from Napier's, and L be the logarithm of the same ratio in Napier's system: whereof B is the logarithm in the other system, and the proportion of the logarithms of any given ratios in Napier's system to the logarithms of the same ratios in the other system be that of m to 1; and λ be the next lesser logarithm to B that is found in the tables, and a be the number found in the tables

" bles

bles corresponding to λ , or of the ratio of which to unity λ is the logarithm;
 “ and Λ be Napier’s logarithm of the same ratio of a to 1; and d be $= B - \lambda$,
 “ and l be $= L - \Lambda$; the ratio corresponding to the given logarithm B will be
 “ very nearly equal to that of $a + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$ or $a + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$, to 1; and, secondly,
 “ That, if B be any given logarithm in Briggs’s system, or in any other system
 “ of logarithms different from Napier’s, and L be the logarithm of the same
 “ ratio in Napier’s system whereof B is the logarithm in the other system; and
 “ the proportion of the logarithms of any given ratios in Napier’s system,
 “ to the logarithms of the same ratios in the said other system be that of m to 1;
 “ and λ be the next greater logarithm than B that is found in the tables, and b
 “ be the number found in the tables corresponding to λ , or of the ratio of
 “ which to unity λ is the logarithm; and Λ be Napier’s logarithm of the same
 “ ratio of b to 1; and d be $= \lambda - B$, and l be $= \Lambda - L$; the ratio corre-
 “ sponding to the given logarithm B , will be very nearly equal to that of
 $b - \frac{b d}{\frac{1}{m} + \frac{1}{2} d}$, or $b - \frac{b d}{\frac{1}{m} + \frac{1}{2} d}$, to 1.”

These two propositions may be proved in the manner following.

By what is shewn in Note 7th it appears that, if L be any given logarithm of Napier’s system, and Λ the next less logarithm set down in a table of Napier’s logarithms, and a is the number corresponding to the said tabular logarithm Λ , (or the number of the ratio of which to unity Λ is the logarithm,) and l be $= L - \Lambda$, the ratio corresponding to the given logarithm L , will be that of the series $a + a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \frac{a l^6}{720} + \&c$, *ad infinitum* to 1, or, (because the series $a + a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \frac{a l^6}{720} + \&c$, *ad infinitum*, is nearly equal to $a + \frac{a l}{1 - \frac{l}{2}}$), nearly, that of $a + \frac{a l}{1 - \frac{l}{2}}$ to 1. Now the ratio corresponding to the logarithm L in Napier’s system, is the same as the ratio corresponding to B in the other system, whose logarithms are to those of the same ratios in Napier’s system as 1 to m . And l , or $L - \Lambda$, is $(= m B - m \lambda = m \times (B - \lambda) = m d$; and consequently $a l$ is $= a m d$; and $\frac{l}{2}$ is $= \frac{m d}{2}$, and $1 - \frac{l}{2}$ is $(= 1 - \frac{m d}{2}) = \frac{2 - m d}{2}$, and $\frac{a l}{1 - \frac{l}{2}}$ is $(= \frac{a m d}{\frac{2 - m d}{2}} = a m d \times \frac{2}{2 - m d} = \frac{2 a m d}{2 - m d} = \frac{a m d}{1 - \frac{m d}{2}} = \frac{a d}{\frac{1}{m} - \frac{1}{2} d})$ and therefore $a + \frac{a l}{1 - \frac{l}{2}}$ is $= a + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$. Therefore the ratio corresponding to the given logarithm B in the system of logarithms in which the logarithms of any given ratios are to Napier’s logarithms of the same ratios in the proportion of 1 to m , is, nearly, that of $a + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$ to 1. Q. E. D.

Secondly, it appears likewise by what is shewn in Note 7th, that, if L be any given logarithm in Napier’s system, and Λ be the next greater logarithm set down in a table of Napier’s logarithms, and b is the number corresponding to the said tabular logarithm Λ , (or the number, of the ratio of which to unity Λ is

is the logarithm,) and l be $= \Lambda - L$, the ratio corresponding to the given logarithm L will be that of the series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum* to 1, or (because the said series $b - bl + \frac{bl^2}{2} - \frac{bl^3}{6} + \frac{bl^4}{24} - \frac{bl^5}{120} + \frac{bl^6}{720} - \&c$, *ad infinitum* is nearly equal to $b - \frac{bl}{1 + \frac{l}{2}}$), it will be,

nearly, that of $b - \frac{bl}{1 + \frac{l}{2}}$ to 1. Now the ratio corresponding to the given

logarithm L in Napier's system is the same as the ratio corresponding to the given logarithm B in the other system, whose logarithms are to those of the same ratios in Napier's system in the proportion of 1 to m . And l , or $\Lambda - L$, is $(= m\lambda - mB = m \times \lambda - B) = md$; and consequently bl is $= bmd$, and $\frac{l}{2}$ is $= \frac{md}{2}$, and $1 + \frac{l}{2}$ is $(= 1 + \frac{md}{2}) = \frac{2 + md}{2}$, and $\frac{bl}{1 + \frac{l}{2}}$ is $(=$

$$\frac{bmd}{\frac{2 + md}{2}} = bmd \times \frac{2}{2 + md} = \frac{2bmd}{2 + md} = \frac{bmd}{1 + \frac{md}{2}} = \frac{bd}{\frac{1}{m} + \frac{1}{2}}$$

$$= b - \frac{bd}{\frac{1}{m} + \frac{1}{2}}. \text{ Therefore the ratio corresponding to the given logarithm } B$$

in the system of logarithms in which the logarithms of any given ratios are to those of the same ratios in Napier's system in the proportion of 1 to m , is, nearly, that of $b - \frac{bd}{\frac{1}{m} + \frac{1}{2}}$ to 1. Q. E. D.

Besides the two propositions which we have just demonstrated, the foregoing passage contains the two following propositions, to wit, 1st, "That $a +$

$$\frac{al}{\frac{1}{m} - \frac{1}{2}l} \text{ is equal to } \frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l},"$$

$$\text{and 2dly, "That } b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l} \text{ is equal to } \frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l};"$$

or, according to the notation used in the foregoing part of this note, (in which d answers to the logarithm denoted by l in Dr. Halley's

$$\text{expression,) "That } a + \frac{ad}{\frac{1}{m} - \frac{1}{2}d} \text{ is equal to } \frac{\frac{1}{m}a + \frac{1}{2}da}{\frac{1}{m} - \frac{1}{2}d},"$$

$$\text{and "That } b - \frac{bd}{\frac{1}{m} + \frac{1}{2}d} \text{ is equal to } \frac{\frac{1}{m}b - \frac{1}{2}db}{\frac{1}{m} + \frac{1}{2}d}."$$

Now this may be easily shewn in the manner following.

The former quantity $a + \frac{a d}{\frac{1}{m} - \frac{1}{2} d}$ is $(= \frac{a \times \sqrt{\frac{1}{m} - \frac{1}{2} d}}{\frac{1}{m} - \frac{1}{2} d} + \frac{a d}{\frac{1}{m} - \frac{1}{2} d})$

$$= \frac{\frac{1}{m} a - \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d} + \frac{a d}{\frac{1}{m} - \frac{1}{2} d} = \frac{\frac{1}{m} a - \frac{1}{2} a d + a d}{\frac{1}{m} - \frac{1}{2} d} = \frac{\frac{1}{m} a + \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d}; \text{ and}$$

the latter quantity $b - \frac{b d}{\frac{1}{m} + \frac{1}{2} d}$ is $(= \frac{b \times \sqrt{\frac{1}{m} + \frac{1}{2} d}}{\frac{1}{m} + \frac{1}{2} d} - \frac{b d}{\frac{1}{m} + \frac{1}{2} d})$

$$= \frac{\frac{1}{m} b + \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d} - \frac{b d}{\frac{1}{m} + \frac{1}{2} d} = \frac{\frac{1}{m} b + \frac{1}{2} b d - b d}{\frac{1}{m} + \frac{1}{2} d} = \frac{\frac{1}{m} b - \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d}.$$

Q. E. D.

Lastly, the expression $\frac{\frac{1}{m} a + \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d}$, which is equal to b , affords us the fol-

lowing analogy for determining the value of b ; to wit, as $\frac{1}{m} - \frac{1}{2} d$ is to $\frac{1}{m}$

+ $\frac{1}{2} d$, so is a to $(\frac{\frac{1}{m} a + \frac{1}{2} a d}{\frac{1}{m} - \frac{1}{2} d}, \text{ or}) b$, or the number sought; or, if d de-

note a logarithm in Briggs's system, and consequently m be $= 2.302,585,092, 994,045,684, \&c$, and $\frac{1}{m}$ be $\frac{1}{2.302,585,092,994,045,684, \&c.}$ or, $0.434,294,481, 903,251,827, \&c$, we shall have $0.434,294, \&c$, $-\frac{1}{2} d$ to $0.434,294, \&c$, $+\frac{1}{2} d$ as a to b , or the number sought.

And the expression $\frac{\frac{1}{m} b - \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d}$, which is equal to a , will afford us the

following analogy for determining the value of a ; to wit, as $\frac{1}{m} + \frac{1}{2} d$ is to

$\frac{1}{m} - \frac{1}{2} d$, so is b to $(\frac{\frac{1}{m} b - \frac{1}{2} b d}{\frac{1}{m} + \frac{1}{2} d}, \text{ or}) a$, or the number sought; or, if d

be a logarithm in Briggs's system, and consequently $\frac{1}{m}$ be $= 0.434,294,481, 903,251,827, \&c$, we shall have $0.434,294, \&c$, $+\frac{1}{2} d$ to $0.434,294, \&c$, $-\frac{1}{2} d$ as b is to a , or the number sought.

NOTE

NOTE X.

IN page 91, line 4th.—If more steps of this series be desired, it will be found to be as follows, to wit, $a + \frac{a l}{1 - \frac{1}{2} l} - \frac{\frac{1}{2} a l^3}{1 - l} + \frac{\frac{1}{3} a l^5}{1 - 2 l} \&c$; as may easily be demonstrated by working out the divisions in each step, and collecting the quotes, whose sum will be found to agree with our former series.

Dr. Halley here affirms that the series $a + \frac{a l}{1 - \frac{1}{2} l} - \frac{\frac{1}{2} a l^3}{1 - l} + \frac{\frac{1}{3} a l^5}{1 - 2 l} \&c$, is equal (as far as these four terms go, or as far as to the term that involves the fifth power of l), to the series $a + a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \frac{a l^6}{720} + \&c$. This assertion is true as far as the fourth power of l , and nearly, but not accurately, so with respect to the term $\frac{a l^5}{120}$, which involves the fifth power of l .

This will appear by converting the three fractions $\frac{a l}{1 - \frac{1}{2} l}$, $\frac{\frac{1}{2} a l^3}{1 - l}$, and $\frac{\frac{1}{3} a l^5}{1 - 2 l}$ into serieses by dividing their numerators by their denominators, and then adding together the first and third quotients, or serieses so obtained, and subtracting the second quotient from their sum. For, if this be done, we shall have $\frac{a l}{1 - \frac{1}{2} l}$, or $\frac{a l}{l}$, $= a l + \frac{a l^2}{2} + \frac{a l^3}{4} + \frac{a l^4}{8} + \frac{a l^5}{16} + \frac{a l^6}{32} + \&c$. and

$\frac{\frac{1}{2} a l^3}{1 - l} = \frac{1}{2} a l^3 + \frac{1}{2} a l^4 + \frac{1}{2} a l^5 + \frac{1}{2} a l^6 + \&c$, and $\frac{\frac{1}{3} a l^5}{1 - 2 l} = \frac{1}{3} a l^5 + \frac{2}{3} a l^6 + \&c$. Therefore $\frac{a l}{1 - \frac{1}{2} l} - \frac{\frac{1}{2} a l^3}{1 - l} + \frac{\frac{1}{3} a l^5}{1 - 2 l}$ will be $=$

$$a l + \frac{a l^2}{2} + \frac{a l^3}{4} + \frac{a l^4}{8} + \frac{a l^5}{16} + \frac{a l^6}{32} + \&c. \\ - \frac{1}{2} a l^3 - \frac{1}{2} a l^4 - \frac{1}{2} a l^5 - \frac{1}{2} a l^6 - \&c. \\ + \frac{1}{3} a l^5 + \frac{2}{3} a l^6 + \&c.$$

$$= a l + \frac{a l^2}{2} + \frac{3 a l^3}{12} + \frac{3 a l^4}{24} + \frac{a l^5}{16} + \frac{a l^6}{32} + \&c.$$

$$- \frac{a l^3}{12} - \frac{2 a l^4}{24} - \frac{a l^5}{12} - \frac{a l^6}{12} - \&c.$$

$$+ \frac{a l^5}{35} + \frac{2 a l^6}{35} + \&c.$$

$$= a l + \frac{a l^2}{2} + \frac{2 a l^3}{12} + \frac{a l^4}{24} + \frac{a l^5}{16} + \frac{a l^6}{32} + \&c.$$

$$- \frac{a l^5}{12} - \frac{a l^6}{12} - \&c.$$

$$+ \frac{a l^5}{35} + \frac{2 a l^6}{35} + \&c.$$

$$= a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{3 a l^5}{48} + \frac{3 a l^6}{96} + \&c.$$

$$- \frac{4 a l^5}{48} - \frac{8 a l^6}{96} - \&c.$$

$$+ \frac{a l^5}{35} + \frac{2 a l^6}{35} + \&c.$$

$$= al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} - \frac{al^5}{48} - \frac{5al^6}{96} + \&c.$$

$$+ \frac{al^5}{35} + \frac{2al^6}{35} + \&c.$$

$= al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{13al^5}{1680} + \frac{17al^6}{3360} + \&c$; and consequently the series $a + \frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{2}al^3}{1-l} + \frac{\frac{1}{3}al^5}{1-2l}$ will be $= a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{13al^5}{1680} + \frac{17al^6}{3360}$, which agrees with the original series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$ *ad infinitum* in the five first terms, but differs from it in the sixth and seventh terms $\frac{13al^5}{1680}$ and $\frac{17al^6}{3360}$, of which the former is a little less than $\frac{al^5}{120}$, and the latter is considerably greater than $\frac{al^6}{720}$. For $\frac{al^5}{120}$ is $= \frac{14al^5}{14 \times 120} = \frac{14al^5}{1680}$, which is greater than $\frac{13al^5}{1680}$ by the difference $\frac{al^5}{1680}$; and $\frac{al^6}{720}$ is $= \frac{14al^6}{14 \times 720} = \frac{14al^6}{10,080}$, which is less than $\frac{17al^6}{10,080}$, or $\frac{3 \times 17al^6}{3 \times 3360}$, or $\frac{17al^6}{3360}$, by $\frac{37al^6}{10,080}$, or, nearly, $\frac{al^6}{272}$, which is more than double of $\frac{al^6}{720}$. Q. E. D.

In order to make the series $a + \frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{2}al^3}{1-l} + \frac{\frac{1}{3}al^5}{1-2l} - \&c$, which is asserted by Dr. Halley to be equal to the original series $a + al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{al^5}{120} + \frac{al^6}{720} + \&c$ *ad infinitum*, agree with the said original series in its sixth term $\frac{al^5}{120}$, we must increase the fourth term $\frac{\frac{1}{3}al^5}{1-2l}$ from $\frac{\frac{1}{3}al^5}{1-2l}$ or $\frac{\frac{7}{24}al^5}{1-2l}$ to $\frac{\frac{7}{24}al^5}{1-2l}$. For then we shall have

$$\frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{2}al^3}{1-l} + \frac{\frac{7}{24}al^5}{1-2l} =$$

$$al + \frac{al^2}{2} + \frac{al^3}{4} + \frac{al^4}{8} + \frac{al^5}{16} + \frac{al^6}{32} + \&c,$$

$$- \frac{al^2}{12} - \frac{al^4}{12} - \frac{al^5}{12} - \frac{al^6}{12} - \&c,$$

$$+ \frac{7al^5}{240} + \frac{7al^6}{240} + \&c.$$

$$= al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} + \frac{3al^5}{48} + \frac{3al^6}{96} + \&c,$$

$$- \frac{4al^5}{48} - \frac{8al^6}{96} - \&c,$$

$$+ \frac{7al^5}{240} + \frac{7al^6}{120} + \&c.$$

$$= al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} - \frac{al^5}{48} - \frac{5al^6}{96} - \&c,$$

$$+ \frac{7al^5}{240} + \frac{7al^6}{120} + \&c.$$

$$= al + \frac{al^2}{2} + \frac{al^3}{6} + \frac{al^4}{24} - \frac{5al^5}{240} - \frac{50al^6}{960} - \&c,$$

$$+ \frac{7al^5}{240} + \frac{56al^6}{960} + \&c.$$

$= a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{2 a l^5}{240} + \frac{6 a l^6}{960} + \&c = a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24}$
 $+ \frac{a l^5}{120} + \frac{a l^6}{160} + \&c$, and consequently $a + \frac{a l}{1 - \frac{1}{2} l} - \frac{\frac{1}{2} a l^3}{1 - l} + \frac{\frac{7}{240} a l^5}{1 - 2 l} = a$
 $+ a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \frac{a l^6}{160} + \&c$, which agrees with the original
series $a + a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \frac{a l^6}{720} + \&c$ *ad infinitum* in the six
first terms, or as far as the sixth term $\frac{a l^6}{720}$ inclusively. Q. E. D.

Dr. Halley seems to think that there is much less trouble in computing the
four terms $a + \frac{a l}{1 - \frac{1}{2} l} - \frac{\frac{1}{2} a l^3}{1 - l} + \frac{\frac{7}{240} a l^5}{1 - 2 l}$ than in computing the first six terms
of the original series $a + a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \&c$, to which six
terms the foregoing four terms are equal. But I much doubt whether this opi-
nion is well founded; because the divisors $1 - \frac{1}{2} l$, $1 - l$, and $1 - 2 l$, of the
second, third, and fourth terms of the said new series will be much longer
numbers, and consequently more troublesome to use as divisors, than the num-
bers 2, 6, 24, and 120, which are the divisors of the third, fourth, fifth, and
sixth terms of the said original series: and in both serieses it will be necessary
to raise l to the fifth power, in order to obtain the terms $\frac{\frac{7}{240} a l^5}{1 - 2 l}$ and $\frac{a l^5}{120}$, in
finding which power of l a considerable part of the difficulty of the computa-
tion will consist. I should therefore, in a computation of this kind, incline to
make use of the original series $a + a l + \frac{a l^2}{2} + \frac{a l^3}{6} + \frac{a l^4}{24} + \frac{a l^5}{120} + \frac{a l^6}{720} +$
 $\&c$ *ad infinitum*, rather than of Dr. Halley's said new series.

NOTE XI.

IN page 91, line 8th.—Thus, I hope, I have cleared up the doctrine of loga-
rithms, and shewn their construction and use independent of the hyperbola, whose
affections have hitherto been made use of for this purpose, though this be a matter
purely arithmetical, nor properly demonstrable from the principles of geometry. Nor
have I been obliged to have recourse to the method of Indivisibles, or the arithmetick
of Infinites; the whole being no other than an easy corollary to Mr. Newton's
general theorem for forming roots and powers.

ARTICLE I.—Dr. Halley here seems to assert, that he was the first person
who had shewn the construction of logarithms without the assistance, or inter-
vention,

vention, of the hyperbola. But this assertion is not true in its full extent. For both Lord Napier, the first inventor of logarithms, and Mr. Henry Briggs, who invented, or first computed, those of the system now in use, which is called by his name, had constructed logarithms, and explained their manner of constructing them to the world, without any mention of the hyperbola. And so did the learned and sagacious John Kepler afterwards in his two tracts, first published in the years 1624 and 1625, and now republished at the beginning of this collection. And so likewise did Mr. Nicholas Mercator in the former and larger part of his tract, intitled, *Logarithmotechnia*, which was first published in the year 1668, and has now been republished in this collection. Dr. Halley must therefore be supposed to mean only that the two logarithmick serieses $q - \frac{q^2}{2} + \frac{q^3}{3} - \frac{q^4}{4} + \frac{q^5}{5} - \frac{q^6}{6} + \&c, ad\ infinitum$, and $q + \frac{q^2}{2} + \frac{q^3}{3} + \frac{q^4}{4} + \frac{q^5}{5} + \frac{q^6}{6} + \&c, ad\ infinitum$, (which had been published by Mr. Mercator and Dr. Wallis for the purpose of constructing, or computing, logarithms more conveniently and expeditiously than by the methods used by Napier and Briggs, and the other old computers of logarithms,) had been derived by Mercator and Wallis, and other writers, (as, for instance, James Gregory,) in the tracts they had published on the subject, solely from the hyperbola. And this, I suppose, was true.

II. Dr. Halley says further, “*That the doctrine of logarithms, being a matter merely arithmetical, is not properly demonstrable from the principles of geometry.*” But this is by no means a just conclusion; since numbers may as well be represented by geometrical lines, or areas, (such as the asymptotick spaces, or areas, of an hyperbola,) as by any other quantities whatsoever, and the inferences obtained by means of such a representation of numbers to the eyes, or senses, are *at least* as clear and satisfactory as those obtained by considering them merely in the abstract. Indeed, when we speak of *abstract numbers*, we can mean only *commensurable quantities in general*, or *commensurable quantities of any kind*, without confining our ideas to any one particular sort of quantities. Thus, for example, when we say, “that 3 times the number 7 is equal to the number 21,” the only meaning of the proposition is, “that, if we take 3 parcels of men, or of horses, or of houses, or of cannon-balls, or of lines a mile long, or of any thing else, and each parcel consists of 7 individuals, or units, of the things under consideration, that is, 7 men, or 7 horses, or 7 houses, or 7 cannon-balls, or 7 miles, and we add the said 3 parcels together, the sum thereby arising will be 21 of the said individuals, or units, to wit, men, horses, houses, cannon-balls, or miles; and that this will be equally true, whether we speak of one of these sorts of things, or of any other of them.” Now certainly our ideas upon this subject will not be the more clear, and our conclusions the more certain, for our not confining ourselves to the consideration of only one of these subjects, but extending our thoughts to all of them at once, as we do when we consider them in the abstract: but they will, on the contrary, be rather less so, it being a plainer and easier proposition to understand, “That 3 times 7 miles make 21 miles,” than “That 3 times 7 things of any kind make 21 things of the same kind.” When therefore it is shewn,

that the asymptotick spaces in an hyperbola that are intercepted between ordinates that are in the same ratio to each other are equal, and consequently that the asymptotick areas that are intercepted between ordinates whose ratios to each other are unequal, are proportional to, or measures of, the ratios of such ordinates, we have a clearer idea of these measures of the said ratios, or, in other words, of these logarithms of the said ratios, than when we endeavour to form a conception of the measures, or logarithms, of the ratios of mere abstract numbers to each other. And the conclusions that are drawn concerning these asymptotick spaces; or visible logarithms, are likely to be more clear and satisfactory than those that might be drawn concerning abstract logarithms, and at the same time must evidently be full as certain. It is not therefore true, as Dr. Halley has asserted, "That the doctrine of logarithms, being a matter purely arithmetical, is not properly demonstrable from the principles of geometry;" but it is true, on the contrary, "that the said doctrine may be very clearly and satisfactorily demonstrated in that manner."

III. In addition to what has been here advanced concerning Dr. Halley's assertion, we may observe, that Euclid himself, the most clear and accurate writer on the mathematicks that ever lived, has, in the fifth book of his Elements, (which is of the most abstract nature possible, and treats, not of lines and angles, and the areas of right-lined figures, as the former books do, but of quantities in general, and their proportions,) thought fit to represent abstract quantities by right lines, in order to assist his readers to conceive and understand what he says of them; thereby calling in the imaginations of his readers in aid, and to the ease and relief, of their understandings. And with respect to the subject now under consideration, to wit, the nature and construction of logarithms, I have always found right lines to convey to my mind the clearest conception both of logarithms, or the measures of the ratios of quantities to each other, and of the numbers, or quantities, themselves, of whose ratios to each other they are measures; and therefore I am of opinion that this doctrine cannot be better explained than by the assistance of the logarithmick curve, in which the ordinates increase on one side, and decrease on the other, *ad infinitum*, and consequently bear all possible ratios to each other, and therefore are capable of representing all sorts of numbers, however great or small, and the abscisses of the axis, or asymptote, belonging to any two different pairs of ordinates, are proportional to, or measures of, the ratios corresponding to the said ordinates. And further, I consider this curve as being somewhat preferable even to the hyperbola for the purpose of illustrating the doctrine of logarithms, because right lines, (such as the abscisses of the axis, or asymptote, of this curve,) are still more easily conceived in the imagination than the mixtilinear asymptotick areas of an hyperbola.

IV. However, though the illustrations and demonstrations of the properties of logarithms that have been given by Mr. Mercator and Dr. Wallis, and Mr. James Gregory, by means of the hyperbola, appear to me (for the reasons that have been mentioned,) to be very legitimate and satisfactory, yet I agree with Dr. Halley in thinking, that so useful and important a doctrine as this is, ought likewise to be explained, if possible, without the help of the hyperbola, and upon

upon the abstract and pure principles of arithmetick, to the end that it may be seen and understood in every possible light and mode of considering it: and therefore I have, in a preceding tract of Vol. I, intituled, "*Remarks on the two foregoing infinite Serieses of Mr. Mercator and Dr. Wallis,*" endeavoured to deduce those two serieses from principles of pure arithmetick and the abstract nature of proportion, and in a manner that I conjecture to be similar to that in which Dr. Halley has deduced them in the foregoing discourse which is the subject of these notes, though I have not been able, after the most earnest endeavours, perfectly to comprehend the Doctor's own investigation of them: and I have likewise, in the next following discourse, in Vol. I, intituled, "*An Appendix to the foregoing Remarks,*" endeavoured to give a similar investigation of the two anti-logarithmick serieses contained in the foregoing discourse of Dr. Halley, that is grounded likewise on the principles of pure arithmetick, and which I conjecture to be, if not the very same with, at least very similar to, the investigation of those two serieses given by Dr. Halley in the foregoing discourse, which, however, from the extreme brevity with which it is expressed, I have not been able to understand. The other parts of the foregoing discourse of Dr. Halley, (that is, all but the investigations of Mr. Mercator's and Dr. Wallis's two logarithmick serieses and the two anti-logarithmick serieses derived from them,) I think I have been able to understand, (though not without great labour and difficulty,) and I hope I have explained them sufficiently to the reader in the course of the ten foregoing notes; for the extraordinary length of which, the uncommon degree of obscurity which runs through the text, must be my apology.

V. Dr. Halley tells us in the latter part of the passage cited in the beginning of this note, "*That in the foregoing discourse he has not been obliged to have recourse to the method of indivisibles, or to the arithmetick of infinites;*" for which he seems much to applaud himself. But surely this applause is not deserved. For what can be a recourse to the arithmetick of infinites, if the following passages are not so? First, in page 85, the Doctor's words are as follows: "*And these rationes we suppose to be measured by the number of ratiunculæ contained in each. Now these ratiunculæ are so to be understood as in a continued scale of proportionals, infinite in number, between the two terms of the ratio; which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculæ between any other two terms as the logarithm of the one ratio is to the logarithm of the other. Thus, if there be supposed between 1 and 10 an infinite scale of mean proportionals, whose number is 100,000, &c. in infinitum; between 1 and 2 there shall be 30,102, &c. of such proportionals; and between 1 and 3 there will be 47,712, &c. of them; which numbers therefore are the logarithms of the rationes of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the logarithms of 10, 2, and 3.*"

"But, if, instead of supposing the logarithms composed of a number of equal ratiunculæ proportional to each ratio, we shall take the ratio of unity to any number to consist always of the same infinite number of ratiunculæ, their magnitudes in this case will be as their numbers in the former. Wherefore, if between unity and any number proposed, there be taken any infinity of mean proportionals, the infinitely little augment, or decrement, of the first of those means from unity
"will

“ will be a ratiuncula, that is, the momentum, or fluxion, of the ratio of unity to
 “ the said number. And, seeing that in these continual proportionals all the ratiun-
 “ culæ are equal, their sum, or the whole ratio, will be as the said momentum is
 “ directly; that is, the logarithm of each ratio will be as the fluxion thereof.
 “ Wherefore, if the root of any infinite power be extracted out of any number, the
 “ differentiola of the said root from unity shall be as the logarithm of that number.
 “ So that logarithms thus produced may be of as many forms as you please to assume
 “ infinite indices of the power whose root you seek; as, if the index be supposed
 “ 100000 &c, infinitely, the roots shall be the logarithms invented by the Lord
 “ Napier; but, if the said index were 2302585 &c, Mr. Briggs’s logarithms
 “ would immediately be produced. And, if you please to stop at any number of
 “ figures, and not to continue them on, it will suffice to assume an index of a figure
 “ or two more than your intended logarithm is to have, as Mr. Briggs did; who, to
 “ have his logarithms true to 14 places, by continual extraction of the square root,
 “ at last came to have the root of the 140,737,488,355,328th power; but how
 “ operose that extraction was will be easily judged by who so shall undertake to
 “ examine his Calculus.

“ Now, though the notion of an infinite power may seem very strange, and, to
 “ those that know the difficulty of the extraction of the roots of high powers,
 “ perhaps, impracticable: yet, by the help of that admirable invention of Mr.
 “ Newton, whereby he determines the unciæ, or numbers prefixed to the members
 “ composing powers (on which the doctrine of series chiefly depends), the infinity of
 “ the index contributes to render the expression much more easy; for if the infinite
 “ power to be resolved be put (after Mr. Newton’s method) $p + pq, \sqrt[p]{p + pq}$ or
 “ $1 + q^{\frac{1}{m}}$, instead of $1 + \frac{1}{m}q + \frac{1-m}{2mm}qq + \frac{1-3m+2mm}{6m^3}q^3 + \frac{1-6m+11mm-6m^2}{24m^4}q^4$
 “ q^4 &c (which is the root when m is finite), becomes $1 + \frac{1}{m}q - \frac{1}{2m}qq + \frac{1}{3m}$
 “ $q^3 - \frac{1}{4m}q^4 + \frac{1}{5m}q^5$ &c, mm being infinite; and consequently whatsoever
 “ is divided thereby vanishing. Hence it follows, that $\frac{1}{m}$ multiplied into $q - \frac{1}{2}qq$
 “ $+ \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5$ &c, is the augment of the first of our mean proportionals
 “ between unity and $1 + q$, and is therefore the logarithm of the ratio of 1 to
 “ $1 + q$. And, whereas the infinite index m may be taken at pleasure, the several
 “ scales of logarithms to such indices will be as $\frac{1}{m}$, or reciprocally as the indices:
 “ And, if the index be taken 10000 &c, as in the case of Napier’s logarithms, they
 “ will be simply $q - \frac{1}{2}qq + \frac{1}{3}q^3 - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.

“ Again, if the logarithm of a decreasing ratio be sought, the infinite root of
 “ $1 - q$, or $\sqrt[1]{1 - q}$, is $1 - \frac{1}{m}q - \frac{1}{2m}q^2 - \frac{1}{3m}q^3 - \frac{1}{4m}q^4 - \frac{1}{5m}q^5 - \frac{1}{6m}q^6$
 “ &c; whence the decrement of the first of our infinite number of proportionals will
 “ be $\frac{1}{m}$ into $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$, &c; which therefore will be
 “ as the logarithm of the ratio of unity to $1 - q$. But, if m be put 10000 &c,
 “ then the said logarithm will be $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c.”

In this passage we have frequent mention made of *infinite numbers, infinite indices of powers, different infinite numbers bearing proportions to each other, fluxions, momenta, and differentiolæ*, and a quantity, *mm*, that is *infinite* infinite. Surely these expressions partake both of *the method of indivisibles* and of *the arithmetick of infinites*, notwithstanding the author's boast of the contrary.

And, secondly, the following passage makes mention of infinite indexes of powers, and supposes that there may be as many different indexes of this kind as we please; which certainly is a branch of the arithmetick of infinites.

"From the logarithm given to find what ratio it expresses, is a problem that has not been so much considered as the former, but which is solved with the like ease, and demonstrated by a like process, from the same general theorem of Mr. Newton. For, as the logarithm of the ratio of 1 to $1 + q$ was proved to be $\overline{1 + q}^{\frac{1}{m}} - 1$, and that of the ratio of 1 to $1 - q$ to be $1 - \overline{1 - q}^{\frac{1}{m}}$; so the logarithm (which we will from henceforth call L) being given, $1 + L$ will be equal to $\overline{1 + q}^{\frac{1}{m}}$ in the one case, and $1 - L$ will be equal to $\overline{1 - q}^{\frac{1}{m}}$ in the other; consequently $\overline{1 + L}^m$ will be equal to $1 + q$, and $\overline{1 - L}^m$ to $1 - q$; that is, according to Mr. Newton's said rule, $1 + mL + \frac{1}{2}m^2L^2 + \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 + \frac{1}{120}m^5L^5$ &c, will be $= 1 + q$, and $1 - mL + \frac{1}{2}m^2L^2 - \frac{1}{6}m^3L^3 + \frac{1}{24}m^4L^4 - \frac{1}{120}m^5L^5$ &c, will be equal to $1 - q$, m being any infinite index whatsoever: which is a full and general proposition, from the logarithm given to find the number, be the species of logarithm what it will. But, if Napier's logarithm be given, the multiplication by m is saved (which multiplication is indeed no other than reducing the other species to his), and the series will be more simple, viz. $1 + L + \frac{1}{2}LL + \frac{1}{6}L^3 + \frac{1}{24}L^4 + \frac{1}{120}L^5$ &c, or $1 - L + \frac{1}{2}LL - \frac{1}{6}L^3 + \frac{1}{24}L^4 - \frac{1}{120}L^5$ &c."

After reading these two passages of this discourse of Dr. Halley, (which are those which, with the greatest attention I could bestow upon them, I have not been able perfectly to understand) I presume, it will be readily allowed that the Doctor is not intitled to the merit he seems to claim, of having explained the doctrine of logarithms without having had recourse to the method of indivisibles, or to the arithmetick of infinites.

VI. It is, however, possible by another method of applying Sir Isaac Newton's excellent binomial theorem, to compute a table of logarithms either of Briggs's system, or any other that may be thought fit, in the manner which Dr. Halley recommends, or without the intervention of the hyperbola or logarithmick curve, or any other geometrical figure, and likewise without having recourse to the method of indivisibles or to the arithmetick of infinites, and by the help of the principles of pure and finite arithmetick only, and the contemplation of the nature and properties of ratios, which are the quantities of which logarithms are the measures. The method of doing this I shall now proceed to explain in the following discourse; which, as it has taken its rise from the reflections and observations of Dr. Halley contained in the foregoing tract on logarithms, I shall intitle an Appendix to the said tract.

A N
A P P E N D I X
TO THE
F O R E G O I N G T R A C T
O F
D R. E D M U N D H A L L E Y
U P O N
L O G A R I T H M S;
B E I N G

A direct Method of computing the Logarithms of Ratios either in Briggs's System, or any other that may be proposed, by the help of Sir Isaac Newton's Binomial Theorem, without the intervention of the Hyperbola, or the Logarithmick Curve, or any other Geometrical Figure, and likewise without having recourse to the method of Indivisibles or the Arithmetick of Infinites.

By *FRANCIS MASERES*, Esq.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

A R T I C L E I.

THE logarithms of ratios in Briggs's System are the numbers that express the magnitudes of the said ratios, or their proportions to each other, upon a supposition that an unit is taken for the representative, or logarithm, of the ratio of 10 to 1. And the logarithms of ratios in any other system are the numbers that represent the magnitudes of the said ratios, or their proportions to each other, upon a supposition that an unit is taken for the representative, or logarithm, of some other ratio than that of 10 to 1. If therefore we can find the proportion of any other ratio to the ratio of 10 to 1, we shall thereby be

enabled to discover Briggs's logarithm of such other ratio. For, if the number found for the representative of the ratio of 10 to 1 is 1, (which is Briggs's logarithm of that ratio,) the number found for the representative of the other ratio will be Briggs's logarithm of the said other ratio: and, if the number found for the representative of the ratio of 10 to 1 is not 1, but some other number, which we may call x , either greater or less than 1, we need only diminish or increase the number found for the representative of the said other ratio in the proportion of x to 1, and the number thereby obtained will be Briggs's logarithm of the said other ratio. Thus, for example, if we should seek the proportion between the ratio of 11 to 10, and the ratio of 10 to 1, and should find it to be that of 1 to 24.158,857,928,096,6, (as we shall find it to be in the course of the following pages,) of which two numbers the lesser, to wit, 1, is the representative of the smaller ratio of 11 to 10, and the greater number 24.158,857,928,096,6, is the representative of the greater ratio of 10 to 1, we need only reduce the smaller number 1, (which is the representative of the ratio of 11 to 10,) in the proportion of 24.158,857,928,096,6, (the first representative of the ratio of 10 to 1,) to 1, (which is its second representative, or its representative in Briggs's system of logarithms,) and the number thereby obtained, to wit, $\frac{1}{24.158,857,928,096,6}$, or 0.041,392,685,158,228,29 &c, will be the new representative of the ratio of 11 to 10, or the logarithm of the said ratio in Briggs's system. And the like observation may be extended to the logarithms of any other system, in which 1 is not the logarithm of the ratio of 10 to 1, but of some other ratio. Our business therefore on the present occasion is to find a method of investigating the proportion between any given ratio (as, for example, the ratio of 11 to 10,) and the ratio of 10 to 1, which is the fundamental, or standard, ratio in Briggs's system of logarithms, or that to which all other ratios are, in that system, compared and referred. Or, if any other system of logarithms were to be computed, it would be necessary to find a method of investigating the proportion between any given ratio and the ratio of which 1 was the logarithm in that system. But, as no system of logarithms besides that of Briggs is now in use, it is needless to think of any other on this occasion.

2. Now in order to find the proportion between the standard ratio of 10 to 1 and any other ratio by the method which we are now going to explain, it will be most convenient (though not absolutely necessary) to suppose the said other ratio to be less than the ratio of 2 to 1, and indeed, for the most part, to be a great deal less than that ratio, and not to exceed the ratio of 11 to 10 or that of 10 to 9. But this will be sufficient for the purpose of computing the logarithms of all sorts of ratios, how great soever, because the greatest ratios are compounded of, and may be divided into, a number of other lesser ratios, of which some shall be exact multiples of the ratio of 10 to 1, and others shall be small ratios of the magnitudes that have been just described. Thus, for example, the ratio of 1788 to 1 is compounded of the ratio of 1788 to 1780, that of 1780 to 178, that of 178 to 89, that of 89 to 80, that of 80 to 8, and that of 8 to 1; of which the first is a very small ratio, the second is equal to the ratio of 10 to 1, the third is equal to the ratio of 2 to 1, the fourth is a
small

small ratio, very little greater than that of 88 to 80, or 11 to 10, the fifth is equal to that of 10 to 1, and the sixth is equal to three times the ratio of 2 to 1. Therefore the logarithm of the ratio of 1788 to 1 is equal to the sum of the logarithms of the ratios of 1788 to 1780, or of 447 to 445, and of 10 to 1, and of 2 to 1, and of 89 to 80, and of 10 to 1, and of three times the ratio of 2 to 1; or $\text{Log. } \frac{1788}{1}$ will be $= \text{Log. } \frac{447}{445} + \text{Log. } \frac{10}{1} + \text{Log. } \frac{2}{1} + \text{Log. } \frac{89}{80} + \text{Log. } \frac{10}{1} + 3 \text{ Log. } \frac{2}{1} = \text{Log. } \frac{447}{445} + \text{Log. } \frac{89}{80} + 2 \text{ Log. } \frac{10}{1} + 4 \text{ Log. } \frac{2}{1} = \text{Log. } \frac{447}{445} + \text{Log. } \frac{89}{80} + 2 \times 1 + 4 \text{ Log. } \frac{2}{1} = \text{Log. } \frac{447}{445} + \text{Log. } \frac{89}{80} + 2 + 4 \text{ Log. } \frac{2}{1}$. And consequently, when the logarithm of the ratio of 2 to 1, and those of the small ratios of 447 to 445, and of 89 to 80, have been computed, that of the ratio of 1788 to 1 may be derived from them by mere addition. And, in like manner, the logarithm of any other great ratio may be obtained by adding together the logarithms of the smaller ratios of which it is compounded. It is not therefore necessary, in the present investigation, to take into our consideration any other ratios but such small ones as have been mentioned, to wit, those of 10 to 9, and 11 to 10, or others of a still smaller magnitude.

3. Now, as the reasonings by which we may find the proportion of any one small ratio, (such as those of 10 to 9 and of 11 to 10,) to the standard ratio of 10 to 1, are exactly the same as those by which we may find the proportion of any other such small ratio to the same ratio of 10 to 1;—and, as particular examples are simpler and easier to understand than general problems, I shall confine myself, in the course of the following pages, to the investigation of the proportion which a single particular small ratio, namely, the ratio of 11 to 10, bears to the said standard ratio. This therefore will be the subject of the following Problem.

P R O B L E M.

4. To find in numbers the proportion between the ratio of 11 to 10 and the greater ratio of 10 to 1.

S O L U T I O N.

The ratio of 11 to 10 is equal to the ratio of $\frac{11}{10}$ to $\frac{10}{10}$, or of $\frac{10+1}{10}$ to $\frac{10}{10}$, or of $1 + \frac{1}{10}$ to 1. We are therefore to find in numbers the proportion between the ratio of $1 + \frac{1}{10}$ to 1 and the ratio of 10 to 1.

Let this proportion be that of 1 to the unknown quantity x .

Then, since the ratio of 10 to 1 is to the ratio of $1 + \frac{1}{10}$ to 1 as x is to 1, and, from the nature of the powers of quantities, (which are only continual proportionals to unity and the quantities themselves, of which they are said to be

be powers,) the ratio of $1 + \frac{1}{10}^x$ to 1 is to the ratio of $1 + \frac{1}{10}$ to 1 in the same proportion of x to 1, it follows that the ratio of $1 + \frac{1}{10}^x$ to 1 will be equal to the ratio of 10 to 1; and consequently $1 + \frac{1}{10}^x$ will be = 10. We must therefore endeavour to find the value of the index x in this equation $1 + \frac{1}{10}^x = 10$.

5. Now, by Sir Isaac Newton's binomial theorem, $1 + \frac{1}{10}^x$ is = the series $1 + x \times \frac{1}{10} + x \times \frac{x-1}{2} \times \frac{1}{100} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{1}{1000} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{1}{10,000} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{x-4}{5} \times \frac{1}{100,000} + x \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{x-4}{5} \times \frac{x-5}{6} \times \frac{1}{1,000,000} + \&c = 1 + \frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000} + \&c$. Therefore this last series $1 + \frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000} + \&c$ will be = 10; and consequently (subtracting 1 from both sides) the series $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000} + \&c$ will be = 9. This equation we must now endeavour to resolve.

6. Now the true value of x in this equation, so far as it can be expressed by eighteen decimal figures, is 24.158,857,928,096,805,5; as may be collected from Mr. Abraham Sharp's computation of Briggs's logarithm of 11 to 61 places of figures. For the first twenty-two figures of that logarithm are 1.041,392,685,158,225,040,750; that is, the first twenty-two figures of the logarithm of the ratio of 11 to 1 in Briggs's system are 1.041,392,685,158,225,040,750; and consequently the logarithm of the ratio of 11 to 10 will be equal to the excess of 1.041,392,685,158,225,040,750 above the logarithm of the ratio of 10 to 1, that is, above 1, or will be = 0.041,392,685,158,225,040,750. Therefore the proportion of the ratio of 11 to 10 to the ratio of 10 to 1, is that of 0.041,392,685,158,225,040,750 to 1, and consequently is equal to that of 1 to $\frac{1,000,000,000,000,000,000,000,000}{0.041,392,685,158,225,040,750}$, or 24.158,857,928,096,805,5 &c. This therefore is the true value of x in this problem; which I have here set down before-hand, to the end that we may see, in the course of the following investigation of it, to what degree of exactness every new step in our gradual approaches towards it, will exhibit it.

7. To find the value of x in this equation $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000} + \frac{x^6-15x^5+85x^4-225x^3+274xx-120x}{720,000,000} + \&c$

+ &c = 9, we must take a few of the first terms of the series which forms the left-hand side of the equation, and suppose them to be equal to the whole series, and consequently to the absolute term 9, and then resolve the finite equations resulting from such suppositions. And it is evident that, in this way of proceeding, the more terms of the series we retain, the nearer will the value of x thereby obtained approach to its true value; but that it must always be somewhat greater than the said true value, on account of the subsequent terms of the series which have not been retained. The values of x arising from the retention of the first term only, the two first terms, the three first terms, and the four first terms, will be as follows.

8. If we suppose the first term $\frac{x}{10}$ alone to be equal to the whole series, and consequently to 9, we shall have $x = 10 \times 9 = 90$. This value of x is more than triple of its true value, which is 24.158,857, &c.

9. If we suppose the two first terms $\frac{x}{10} + \frac{xx-x}{200}$ to be equal to the whole series, and consequently to 9, we shall have $\frac{20x}{200} + \frac{xx-x}{200} (= \frac{x}{10} + \frac{xx-x}{200}) = 9$, or $\frac{19x+xx}{200} = 9$, and consequently $19x + xx (= 200 \times 9) = 1800$. Therefore $\frac{361}{4} + 19x + xx$ will be $(= \frac{361}{4} + 1800 = \frac{361}{4} + \frac{7200}{4}) = \frac{7561}{4}$; and consequently $\frac{19}{2} + x$ will be $(= \frac{\sqrt{7561}}{2}) = \frac{86.95}{2}$, and x will be $(= \frac{86.95}{2} - \frac{19}{2} = \frac{67.95}{2}) = 33.97$. This second value of x is much nearer to its true value than the former was, but is still a good deal too large.

10. Let us then, in the third place, suppose the three terms $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000}$ to be equal to the whole series, and consequently to the absolute term 9, and investigate the value of x resulting from this supposition.

Now $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000}$ are $= \frac{600x}{6000} + \frac{30xx-30x}{6000} + \frac{x^3-3xx+2x}{6000} = \frac{572x+27xx+x^3}{6000}$. Therefore $\frac{572x+27xx+x^3}{6000}$ will, upon this supposition, be = 9, and consequently $572x + 27xx + x^3$ will be $(= 6000 \times 9) = 54,000$. We must therefore endeavour to resolve this cubick equation.

Now, if we suppose x to be equal to 30, and substitute 30 instead of x in the compound quantity $572x + 27xx + x^3$, (which forms the left-hand side of the equation $572x + 27xx + x^3 = 54,000$,) we shall have $xx = 900$, and $x^3 = 27,000$, and consequently $27xx (= 27 \times 900) = 24,300$, and $572x (= 572 \times 30) = 17160$, and $572x + 27xx + x^3 (= 17160 + 24,300 + 27,000) = 68,460$; which is greater than the absolute term, 54,000, of the cubick equation $572x + 27xx + x^3 = 54,000$. Therefore 30 must be greater than the true value of x in that equation.

Let us therefore suppose the said true value to be $30 - z$; and let $30 - z$ be substituted instead of x in the terms of the said equation, but with an omission of all the quantities that would involve either zz or z^3 . And we shall then have

$$xx (= \overline{30 - z})^2 = \overline{30}^2 - 2 \times 30 \times z + \&c) \\ = 900 - 60 \times z + \&c,$$

$$\text{and } x^3 (= \overline{30 - z})^3 = \overline{30}^3 - 3 \times \overline{30}^2 \times z + \&c \\ = \overline{30}^3 - 3 \times 900 \times z + \&c) \\ = 27000 - 2700 \times z + \&c,$$

$$\text{and consequently } 27xx (= 27 \times 900 - 60 \times z + \&c \\ = 27 \times 900 - 27 \times 60 \times z + \&c) \\ = 24,300 - 1620 \times z + \&c,$$

$$\text{and } 572x (= 572 \times 30 - z = \\ 572 \times 30 - 572 \times z) \\ = 17160 - 572 \times z,$$

$$\text{and } 572x + 27xx + x^3 = \\ 17,160 - 572 \times z \\ + 24,300 - 1620 \times z + \&c \\ + 27,000 - 2700 \times z + \&c =$$

$68,460 - 4892 \times z + \&c$. Therefore this last quantity $68,460 - 4892 \times z + \&c$, will be $= 54,000$; and consequently, (adding $4892 \times z$ to both sides) we shall have $68,460 = 54,000 + 4892 \times z$, and (subtracting $54,000$ from both sides,) $4892 \times z = 14,460$, and, lastly, (dividing both sides by 4892) $z = 2.95$. Therefore x , or $30 - z$, will be $(= 30 - 2.95) = 27.05$; that is, the root of the cubick equation $572x + 27xx + x^3 = 54,000$ will be $= 27.05$, or (neglecting the decimal fraction $.05$) $= 27$. Therefore 27 is the third approximation to the value of x in the original equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \&c = 9$; and it is considerably nearer to the truth than the foregoing, or second, approximation, 33.95 , which was derived from only the two first terms of the series.

11. And if, in the fourth place, we suppose the four first terms of the series, to wit, $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000}$, to be equal to the whole series, and consequently to the absolute term 9 , we shall have another approximation to the value of x , which will come within less than a 25th part of its true value. For we shall then have $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} = 9$; and consequently $\frac{40 \times 572x + 40 \times 27xx + 40x^3}{240,000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} = \frac{22880x + 1080xx + 40x^3}{240,000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} = \frac{22,874x + 1091xx + 34x^3 + x^4}{240,000}$; and consequently $\frac{22,874x + 1091xx + 34x^3 + x^4}{240,000}$ will be $= 9$, and $22,874x + 1091xx + 34x^3 + x^4$ will be $(= 240,000 \times 9) = 2,160,000$. We must therefore now endeavour to resolve the biquadratick equation $22,874x + 1091xx + 34x^3 + x^4 = 2,160,000$.

12. Now the root of this equation must be less than 27 . For, if we suppose x to be equal to 27 , we shall have $xx = 729$, and $x^3 = 19,683$, and $x^4 = 531,441$,

531,441, and consequently $22874x (= 22874 \times 27) = 617,598$, and $1091xx (= 1091 \times 729) = 795,539$, and $34x^3 (= 34 \times 19683) = 669,222$, and $22,874x + 1091xx + 34x^3 + x^4 (= 617,598 + 795,539 + 669,222 + 531,441) = 2,613,800$; which is greater than 2,160,000, or the absolute term of the equation $22874x + 1091xx + 34x^3 + x^4 = 2,160,000$; and consequently 27 is greater than the true value of x in that equation.

13. Let us therefore suppose the true value of x in this equation to be $27 - z$. And let $27 - z$ be substituted in its terms instead of x , but with an omission of all the quantities that shall involve either zz , z^3 , or z^4 .

And we shall then have

$$\begin{aligned} xx & (= \overline{27 - z}^2 = \overline{27}^2 - 2 \times 27 \times z + \&c) = \\ & \quad 729 - 54 \times z + \&c, \\ \text{and } x^3 & (= \overline{27 - z}^3 = \overline{27}^3 - 3 \times \overline{27}^2 \times z + \&c) \\ & \quad = \overline{27}^3 - 3 \times 729 \times z + \&c = \overline{27}^3 - 2187 \times z + \&c) \\ & \quad = 19683 - 2187 \times z + \&c, \\ \text{and } x^4 & (= \overline{27 - z}^4 = \overline{27}^4 - 4 \times \overline{27}^3 \times z + \&c) \\ & \quad = \overline{27}^4 - 4 \times 19683 \times z + \&c = \overline{27}^4 - 78,732 \times z + \&c) \\ & \quad = 531,441 - 78,732 \times z + \&c, \\ \text{and } 22874x & (= 22874 \times \overline{27 - z} = 22,874 \times \overline{27} - 22,874 \times z) \\ & \quad = 617,598 - 22,874z, \\ \text{and } 1091xx & (= 1091 \times \overline{729 - 54z} = 1091 \times \overline{729} - 1091 \times 54 \times z + \&c) \\ & \quad = 795,539 - 58,914 \times z + \&c, \\ \text{and } 34x^3 & (= 34 \times \overline{19683 - 2187z} = 34 \times \overline{19683} - 34 \times 2187 \times z + \&c) \\ & \quad = 669,222 - 74,358 \times z + \&c, \end{aligned}$$

$$\begin{aligned} & \text{and consequently } 22874x + 1091xx + 34x^3 + x^4 \\ & \quad = 617,598 - 22,874 \times z \\ & \quad + 795,539 - 58,914 \times z + \&c \\ & \quad + 669,222 - 74,358 \times z + \&c \\ & \quad + 531,441 - 78,732 \times z + \&c \end{aligned} \quad \left. \vphantom{\begin{aligned} & \text{and consequently } 22874x + 1091xx + 34x^3 + x^4 \\ & \quad = 617,598 - 22,874 \times z \\ & \quad + 795,539 - 58,914 \times z + \&c \\ & \quad + 669,222 - 74,358 \times z + \&c \\ & \quad + 531,441 - 78,732 \times z + \&c \end{aligned}} \right\} = 2,613,800 - 234,878 \times z + \&c.$$

Therefore this last quantity $2,613,800 - 234,878 \times z + \&c$, will be equal to the absolute term 2,160,000; and consequently (adding $234,878 \times z$ to both sides,) $2,613,800$ will be $= 2,160,000 + 234,878 \times z$, and (subtracting 2,160,000 from both sides,) $234,878 \times z$ will be $= 453,800$, and consequently

z will be $= \frac{453,800}{234,878} = 1.93$. Therefore x , or $27 - z$, will be $(= 27 - 1.93)$

$= 25.07$, that is, the root of the biquadratic equation $22,874x + 1091xx + 34x^3 + x^4 = 2,160,000$ is $= 25.07$; which is therefore a fourth approximation to the true value of x in the original equation

$$\begin{aligned} & \frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} \\ & + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} + \&c = 9. \end{aligned}$$

14. As this fourth approximation, 25.07, to the value of x in the original equation agrees with the third approximation to it, 27.05, in the first, or highest, figure 2, we might conclude (if we did not already know the true value of x ,) that the first figure of the true value of x in the said original equation must be 2, though, perhaps, the second figure of it may be less than 5. And, further, as the difference between the third and fourth approximations, 27.05 and 25.07, is less than 2, whereas the difference between the second and third approximations, 33.97 and 27.05, is more than 6, we have reason to suppose, (without taking the pains to try it,) that, if we were to take in another term of the series $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \&c$, and suppose the five first terms of it, to wit, $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \frac{x^5-10x^4+35x^3-50xx+24x}{12,000,000}$, to be equal to the whole series, and consequently to 9, and were to resolve the equation of the fifth order thence resulting, the difference between the value of x that would be thereby obtained, and the last, or fourth value of it, 25.07, would be less than 1, and consequently that the said fifth approximation to the value of x would be greater than 24. We might therefore conclude, without resolving any more equations for the purpose, that the true value of x in the said original equation $\frac{x}{10} + \frac{xx-x}{200} + \&c = 9$, or of the index x in the equation $1 + \frac{1}{10}^x = 10$, is greater than 24, but less than 25; and this conclusion we shall now proceed to verify by raising the said binomial quantity $1 + \frac{1}{10}$ to the 24th and 25th powers.

15. Now $1 + \frac{1}{10}$ is $= 1.1$; of which the square is 1.21, and the fourth power is $(= 1.21^2) = 1.4641$, and the eighth power is $(= 1.4641^2) = 2.143,588,81$, and the sixteenth power is $(= 2.143,588,81^2) = 4.594,972,986,357,216,1$. Therefore 1.1^{24} is $(= 1.1^{16} \times 1.1^8 = 4.594,972,986,357,216,1 \times 2.143,588,81) = 9.849,732,675,807,611,094,711,841$, and 1.1^{25} is $(= 1.1^{24} \times 1.1 = 9.849,732,675,807,611,094,711,841 \times 1.1) = 10.834,705,943,388,372,204,183,025,1$. It appears therefore that 1.1^x , or $1 + \frac{1}{10}^x$ (which is supposed to be $= 10$,) is greater than 1.1^{24} , or $1 + \frac{1}{10}^{24}$, but less than 1.1^{25} , or $1 + \frac{1}{10}^{25}$, and nearer to the former than to the latter. Therefore the index x is greater than 24, but less than 25, as we had concluded it to be in consequence of the foregoing investigations. We therefore now know for certain that the two first, or highest, figures of the true value of the index x in the equation $1 + \frac{1}{10}^x = 10$ are 24.

16. We might now proceed to investigate the value of x to a greater degree of exactness by taking in five, or six, or seven, or more, terms of the series $\frac{x}{10} + \frac{xx-x}{200}$.

+ $\frac{x^3 - 3xx + 2x}{6000}$ + &c, and supposing them to be equal to the whole series, and consequently to 9, and resolving the equations resulting from such suppositions. But the computations necessary to these resolutions would be found excessively laborious. We shall therefore lay aside the equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000}$ + &c = 9, and shall proceed to obtain, in its stead, another equation, of which the root, or unknown quantity, shall be a much smaller quantity than x , and shall even be less than 1, and in which the powers of the unknown quantity shall consequently form a decreasing progression, and therefore a few of the first terms of the series will approach much nearer to an equality with the whole series than in the former equation, and consequently our approaches to the true value of the said unknown quantity by retaining first one term, then two terms, then three terms, and, lastly, four terms, of the series, and supposing them to be equal to the whole series, and resolving the equations resulting from these suppositions, will be much swifter than our approaches to the value of x in the foregoing operations. This may be done in the manner following.

17. Since x is greater than 24, let us suppose it to be = $24 + z$.

Then will $1 + \frac{1}{10}^{24+z}$ be = $1 + \frac{1}{10}^x$, and consequently = 10.

But $1 + \frac{1}{10}^{24+z}$ is = $1 + \frac{1}{10}^{24} \times 1 + \frac{1}{10}^z$.

Therefore $1 + \frac{1}{10}^{24} \times 1 + \frac{1}{10}^z$ will be = 10.

But we have seen that $1 + \frac{1}{10}^{24}$ is = 9.849,732,675,807,611,094,711,841.

Therefore 9.849,732,675,807,611,094,711,841 $\times 1 + \frac{1}{10}^z$ is = 10; and consequently $1 + \frac{1}{10}^z$ will be (= $\frac{10.000,000,000,000,000,000,000,000,000,000}{9.849,732,675,807,611,094,711,841}$) = 1.015,255,979,947,706,347,941, &c. We must therefore endeavour to find the value of the index z in this new equation $1 + \frac{1}{10}^z = 1.015,255,979,947,706,347,941, \&c.$

18. Now, by the binomial theorem, $1 + \frac{1}{10}^z$ is = the series $1 + z \times \frac{1}{10} + z \times \frac{z-1}{2} \times \frac{1}{100} + z \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{1}{1000} + z \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{1}{10000} + \&c = 1 + \frac{z}{10} + \frac{zz-z}{200} + \frac{z^3-3zz+2z}{6000} + \frac{z^4-6z^3+11zz-6z}{240,000} + \&c$, or (to express the terms more correctly, because z is, in this case, less than 1, and consequently zz is less than z , and z^4 than z^3 ,) $1 + \frac{z}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000} - \frac{6z-11zz+6z^3-z^4}{240,000} + \&c$. Therefore this series $1 + \frac{z}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000} - \frac{6z-11zz+6z^3-z^4}{240,000} + \&c$ will be = 1.015,255,979,947,706,347,941, &c; and consequently (subtracting 1 from both sides,) the

the series $\frac{z}{10} - \frac{\sqrt{z - zz}}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{\sqrt{6z - 11zz + 6z^3 - z^4}}{240,000} + \&c$, will be $= 0.015,255,979,947,706,347,941, \&c$. This equation we must now endeavour to resolve.

19. Now here, as in the former equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} + \&c = 9$, we may make continual approaches to the true value of the root by taking in more and more terms of the series, and supposing them to be equal to the whole series, and consequently to the absolute term $0.015,255,979,947,706,347,941, \&c$, and then resolving the equations that will result from these suppositions. This may be done in the manner following.

20. If we suppose the first term, $\frac{z}{10}$, of the series to be alone equal to the whole series, and consequently to $0.015,255,979,9, \&c$, we shall have $z = 10 \times 0.015,255,979,9, \&c = 0.152,559,799, \&c$; of which the two first figures .15 are exact. For, if z be $= 0.152,559,799, \&c$, we shall have $x (= 24 + z) = 24.152,559,799, \&c$, which agrees with the more exact value of x , to wit, $24.158,857,928,096,805,5$, in the four highest figures. This is no inconsiderable approximation to the true value of z , and is obtained with hardly any trouble.

21. This first approximation $0.152, \&c$, to the value of z in the equation $\frac{z}{10} - \frac{\sqrt{z - zz}}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{\sqrt{6z - 11zz + 6z^3 - z^4}}{240,000} + \&c = 0.015,255,979,9, \&c$, is less than its true value, because the first term of the series, $\frac{z}{10}$, is not less than the whole series, (as was the case with $\frac{x}{10}$ the first term of the former series $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} + \&c$), but is greater than the whole series, on account of the diminution of its value by the subtraction of the following term $\frac{z - zz}{200}$. And for a like reason the value of z derived from the two first terms of the series will be greater than the truth, and the value of it derived from the three first terms will be less than the truth, and the value of it derived from the four first terms will be greater than the truth, and so on alternately, because any odd number of the terms will be greater than the whole series, and any even number of them will be less: all which is the consequence of z 's being less than unity.

22. In the second place, let us suppose the two first terms of the series, to wit, $\frac{z}{10} - \frac{\sqrt{z - zz}}{200}$, to be equal to the whole series, and consequently to $0.015,255,979,9, \&c$. We shall then have $\frac{20z}{200} - \frac{\sqrt{z - zz}}{200} (= \frac{z}{10} - \frac{\sqrt{z - zz}}{200}) = 0.015,255,979,9, \&c$, or $\frac{19z + zz}{200} = 0.015,255,979,9, \&c$, and consequently $19z + zz (= 200 \times 0.015,255,979,9, \&c) = 3.051,195,800, \&c$, and $\frac{361}{4} + 19z + zz$

+ zz ($= \frac{361}{4} + 3.051,195,800 = \frac{361}{4} + \frac{4 \times 3.051,195,800 \mathcal{E}c}{4} = \frac{361}{4} + \frac{12.204,783,200}{4}$) $= \frac{373.204,783,200}{4}$, and consequently $\frac{19}{2} + z$ ($= \frac{\sqrt{373.204,783,200, \mathcal{E}c}}{2}$) $= \frac{19.3185}{2}$, and z ($= \frac{19.3185}{2} - \frac{19}{2} = \frac{0.3185}{2}$) $= 0.1592$. Therefore 0.1592 is the second approximation to the value of z in the equation $\frac{z}{10} - \sqrt{\frac{z - zz}{200}} + \frac{2z - 3zz + z^3}{6000} - \sqrt{\frac{6z - 11zz + 6z^3 - z^4}{240,000}} + \mathcal{E}c = 0.015,255,979,947,706,347,941, \mathcal{E}c$, or in the original equation $1 + \frac{1}{10}^z = 1.015,255,979,947,706,347,941, \mathcal{E}c$.

23. In the third place let us suppose the three first terms of the foregoing series, to wit, the three terms $\frac{z}{10} - \sqrt{\frac{z - zz}{200}} + \frac{2z - 3zz + z^3}{6000}$ to be equal to the whole series, and consequently to the absolute term 0.015,255,979,947,706,347,941, &c.

We shall then have $\frac{19z + zz}{200} + \frac{2z - 3zz + z^3}{6000}$, or $\frac{30 \times 19z + 30zz}{6000} + \frac{2z - 3zz + z^3}{6000}$, or $\frac{570z + 30zz}{6000} + \frac{2z - 3zz + z^3}{6000}$, or $\frac{572z + 27zz + z^3}{6000} = 0.015,255,979,947,706,347,941, \mathcal{E}c$, and consequently $572z + 27zz + z^3$ ($= 6000 \times 0.015,255,979,947,706,347,941, \mathcal{E}c$) $= 91.535,879,686,238,087,646, \mathcal{E}c$. We must therefore now resolve this cubick equation, $572z + 27zz + z^3 = 91.535,879,686,238,087,646, \mathcal{E}c$.

Now, if we substitute 0.1592, or the value of z derived from the two first terms of the series, (and which we know to be not very different from the true value of z in this cubick equation,) instead of z in the compound quantity $572z + 27zz + z^3$ (which forms the left-hand side of this equation) we shall have

$$\begin{aligned} zz & (= 0.1592^2) = 0.025,344,64, \\ \text{and } z^3 & (= 0.1592^3) = 0.004,034,866,688, \\ \text{and } 27zz & (= 27 \times 0.025,344,64) = 0.684,305,28, \\ \text{and } 572z & (= 572 \times 0.1592) = 91.0624, \\ \text{and consequently } 572z + 27zz + z^3 & = \\ & 91.062,4 \\ & + 0.684,305,28 \\ & + 0.004,034,866,688 = \\ & 91.750,740,146,688; \end{aligned}$$

which is greater than 91.535,879,686,238,087,646, &c, or the absolute term of the cubick equation $572z + 27zz + z^3 = 91.535,879,686,238,087,646, \mathcal{E}c$. Therefore 0.1592 must be greater than the true value of z in that equation.

Let us therefore suppose the value of z in that equation to be $0.1592 - v$.

Then we shall have

$$\begin{aligned} zz & (= 0.1592 - v)^2 = 0.1592^2 - 2 \times 0.1592 \times v + \mathcal{E}c \\ & = 0.1592^2 - 0.3184 \times v + \mathcal{E}c \\ & = 0.025,344,64 - 0.3184 \times v + \mathcal{E}c, \end{aligned}$$

and

$$\begin{aligned} \text{and } z^3 & (= \overline{0.1592 - v})^3 = \overline{0.1592}^3 - 3 \times \overline{0.1592}^2 \times v + \&c \\ & = \overline{0.1592}^3 - 3 \times 0.025,344,64 \times v + \&c \\ & = \overline{0.1592}^3 - 0.076,033,92 \times v + \&c \\ & = 0.004,034,866,688 - 0.076,033,92 \times v + \&c, \end{aligned}$$

$$\begin{aligned} \text{and consequently } 572z & (= 572 \times 0.1592 - v \\ & = 572 \times 0.1592 - 572 \times v) \\ & = 91.0624 - 572 \times v, \end{aligned}$$

$$\begin{aligned} \text{and } 27zz & (= 27 \times 0.025,344,64 - 0.3184 \times v + \&c \\ & = 27 \times 0.025,344,64 - 27 \times 0.3184 \times v + \&c) \\ & = 0.684,305,28 - 8.5968 \times v + \&c, \end{aligned}$$

$$\text{and } 572z + 27zz + z^3 =$$

$$\begin{aligned} & 91.062,4 - 572 \times v \\ & + 0.684,305,28 - 8.5968 \times v + \&c \\ & + 0.004,034,866,688 - 0.076,033,92 \times v + \&c \end{aligned}$$

$= 91.750,740,146,688 - 580.672,833,92 \times v + \&c$. Therefore this last quantity $91.750,740,146,688 - 580.672,833,92 \times v + \&c$ will be equal to the absolute term $91.535,879,686,238,087,646, \&c$; and consequently (adding $580.672,833,92 \times v$ to both sides,) $91.750,740,146,688$ will be $= 91.535,879,686,238,087,646, \&c + 580.672,833,92 \times v$, and (subtracting $91.535,879,686,238,087,646, \&c$ from both sides,) $580.672,833,92 \times v$ will be $= 0.214,860,460,449,912,354$, and consequently z will be $(= \frac{0.214,860,460,449,912,354}{580.672,833,92})$

$= 0.000,370$. Therefore z , or $0.1592 - v$, will be $(= 0.1592 - 0.000,370) = 0.15883$. Therefore 0.15883 is a third approximation to the true value of z in the equation $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} +$

$\&c = 0.015,255,979,947,706,347,941, \&c$, or in the original equation $1 + \frac{1}{10}z = 1.015,255,979,947,706,347,941, \&c$. And it is a pretty near approximation to it, the four first figures of it, 0.1588 , being exact. For, if we suppose z to be $= 0.15883$, we shall have x , or $24 + z$, $= 24.15883$, of which the first six figures 24.1588 are true, the more accurate value of x being (as has been already observed,) $24.158,857,928,096,805,5$.

24. In order to obtain a still nearer approximation to the value of z in the equation $\frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} + \&c = 0.015,255,979,947,706,347,941, \&c$, let us suppose the four first terms of the said series to be equal to the whole series, and consequently to the absolute term $0.015,255,979,947,706,347,941, \&c$, and resolve the biquadratick equation that will result from that supposition.

$$\begin{aligned} \text{The four terms } \frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} & \text{ are equal} \\ \text{to } \frac{572z + 27zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} & = \frac{40 \times 572z + 40 \times 27zz + 40z^3}{240,000} - \\ \frac{6z - 11zz + 6z^3 - z^4}{240,000} & = \frac{22880z + 1080zz + 40z^3}{240,000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} = \end{aligned}$$

$\frac{22874z + 10912z + 34z^3 + z^4}{240,000}$. Therefore this last quantity $\frac{22874z + 10912z + 34z^3 + z^4}{240,000}$ is $= 0.015,255,979,947,706,347,941, \&c$; and consequently $22874z + 10912z + 34z^3 + z^4$ is $(= 240,000 \times 0.015,255,979,947,706,347,941, \&c) = 3661.435,187,449,523,505,840,000, \&c$. We must therefore now endeavour to resolve this biquadratic equation $22874z + 10912z + 34z^3 + z^4 = 3661.435,187,449,523,505,840,000, \&c$.

25. Now we know that the root of this equation cannot differ much from the root of the last cubick equation $572z + 27zz + z^3 = 91.535,879,686,238,087,646, \&c$, which was found to be 0.15883 . We will therefore substitute 0.15883 instead of z in the compound quantity $22874z + 10912z + 34z^3 + z^4$ (which forms the left-hand side of the said biquadratic equation,) in order to see whether the value of the said compound quantity resulting from such substitution will be greater, or less, than the absolute term, $3661.435,187,449,523,505,840,000, \&c$, of the said biquadratic equation, and consequently whether 0.15883 is greater, or less, than the root of the said equation.

Now, if z be supposed to be $= 0.15883$, we shall have $zz = 0.025,226,968,9$, and $z^3 = 0.004,006,799,470,387$, and $z^4 = 0.000,636,399,959,881,567,21$, and

$$22,874z (= 22,874 \times 0.15883) = 3633.077,42,$$

$$\text{and } 10912z (= 10912 \times 0.025,226,968,9) = 27.522,623,069,9,$$

$$\text{and } 34z^3 (= 34 \times 0.004,006,799,470,387) = 0.136,231,181,993,158,$$

$$\text{and consequently } 22,874z + 10912z + 34z^3 + z^4 =$$

$$3633.077,42$$

$$+ 27.522,623,069,9$$

$$+ 0.136,231,181,993,158$$

$$+ 0.000,636,399,959,881,567,21$$

$$= 3660.736,910,651,853,039,567,21; \text{ which is less than } 3661.435,187,449,$$

$$523,505,840,000, \&c, \text{ or the absolute term of the equation } 22,874z + 10912z$$

$$+ 34z^3 + z^4 = 3661.435,187,449,523,505,840,000, \&c. \text{ Therefore } 0.15883$$

$$\text{is less than the true value of } z \text{ in that equation.}$$

26. Let us therefore suppose z to be $= 0.15883 + w$, and let this quantity

be substituted instead of z in the terms of the said biquadratic equation, but

with an omission of all the quantities that shall be found to involve either w^2 ,

or w^3 , or w^4 . And we shall then have

$$zz (= \overline{0.15883 + w}^2 = \overline{0.15883}^2 + 2 \times \overline{0.15883} \times w + \&c$$

$$= \overline{0.15883}^2 + 0.31766 \times w + \&c)$$

$$= 0.025,226,968,9 + 0.31766 \times w + \&c,$$

$$\text{and } z^3 (= \overline{0.15883 + w}^3 = \overline{0.15883}^3 + 3 \times \overline{0.15883}^2 \times w + \&c$$

$$= \overline{0.15883}^3 + 3 \times 0.025,226,968,9 \times w + \&c$$

$$= \overline{0.15883}^3 + 0.075,680,906,7 \times w + \&c)$$

$$= 0.004,006,799,470,387 + 0.075,680,906,7 \times w + \&c,$$

$$\text{and}$$

and $z^4 (= \overline{0.15883 + w})^4 = \overline{0.15883}^4 + 4 \times \overline{0.15883}^3 \times w + \&c$
 $= \overline{0.15883}^4 + 4 \times 0.004,006,799,470,387 \times w + \&c$
 $= \overline{0.15883}^4 + 0.016,027,197,881,548 \times w + \&c)$
 $= 0.000,636,399,959,881,567,21 + 0.016,027,197,881,548 \times w + \&c,$
and $22874z (= 22,874 \times \overline{0.15883 + w})$
 $= 22,874 \times \overline{0.15883} + 22,874 \times w)$
 $= 3633.077,42 + 22,874 \times w,$
and $10912z (= 1091 \times 0.025,226,968,9 + 1091 \times 0.317,66 \times w)$
 $= 27.522,623,069,9 + 346.567,06 \times w + \&c$
and $34z^3 (= 34 \times 0.004,006,799,470,387 + 34 \times 0.075,680,906,7 \times w + \&c)$
 $= 0.136,231,181,993,158 + 2.573,150,827,8 \times w + \&c,$
and consequently $22,874z + 10912z + 34z^3 + z^4 =$
 $3633.077,42 + 22,874 \times w$
 $+ 27.522,623,069,9 + 346.567,06 \times w + \&c$
 $+ 0.136,231,181,993,158 + 2.573,150,827,8 \times w + \&c$
 $+ 0.000,636,399,959,881,567,21 + 0.016,027,197,881,548 \times w + \&c$
 $= 3660.736,910,651,853,039,567,21 + 23,223.156,238,025,681,548 \times w +$
 $\&c.$ Therefore this last quantity $3660.736,910,651,853,039,567,21 + 23,223.$
 $156,238,025,681,548 \times w + \&c$ will be $=$ the absolute term $3661.435,187,449,$
 $523,505,840,000, \&c$; and consequently (subtracting $3660.736,910,651,853,$
 $039,567,21$ from both sides,) $23,223.156,238,025,681,548 \times w$ will be $= 0.698,$
 $276,797,670,466,272,790,$ and therefore w will be $= \frac{0.698,276,797,670,466,272,790}{23,223.156,238,025,681,548}$
 $= 0.000,030,0.$ Therefore z , or $\overline{0.15883 + w}$, will be $(= \overline{0.15883} + 0.000,$
 $030,0) = 0.158,860,0,$ or 0.15886 , and x , or $24 + z$, will be $= 24 + 0.15886,$
or $24.158,86$; that is, the index x in the original equation $1 + \frac{1}{10}^x = 10$ is $=$
 $24.15886.$

27. This value of x is exact in the first six figures 24.1588 , and differs only by an unit in the next figure from the true value of x , which is (as we have before observed,) $24.158,857,928,096,805,5.$

28. The labour of resolving the last equation was so considerable, and the progress made by it in our approach to the value of z was so small, (the value of z obtained by it, to wit, 0.15886 , being not more exact than the next preceding value of it, to wit, 0.15883 , by so much as one figure, but only approaching nearer to the truth in the fifth, or last, figure,) that it would be by no means expedient to seek for a nearer approximation to the value of z by taking in more terms of the series $\frac{z}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000} - \frac{6z-11zz+6z^3-z^4}{240,000}$
 $+ \&c$, (as, for example, five terms, or six terms, or seven terms of it,) and supposing them to be equal to the whole series, and consequently to the absolute term $0.015,255,979,947,706,347,941, \&c$, and resolving the equations resulting from such suppositions. But this purpose will be much better answered by

by dropping all further consideration of the equation $\frac{x}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000}$
 $- \frac{6z-11zz+6z^3-z^4}{240,000} + \&c = 0.015,255,979,947,706,347,941, \&c$, and
 deriving from the value of z already found, to wit, 24.15886, another equation
 in which the unknown quantity shall be a quantity much less than z , or 0.15886;
 as we before dropped the consideration of the first equation $\frac{x}{10} + \frac{xx-x}{200} +$
 $\frac{x^3-3xx+6x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \&c = 9$, in order to enter upon that
 of the second equation $\frac{x}{10} - \frac{z-zz}{200} + \frac{2z-3zz+z^3}{6000} - \frac{6z-11zz+6z^3-z^4}{240,000} +$
 $\&c = 0.015,255,979,947,706,347,941, \&c$. This may be done by com-
 puting the value of the 24.15886th power of the binomial quantity $1 + \frac{1}{10}$ by
 means of Sir Isaac Newton's binomial theorem, in order to discover whether
 the said power is greater, or less, than 10, or $1 + \frac{1}{10}^x$, and consequently whe-
 ther the index 24.15886 is greater, or less, than the index x ; after which we
 may suppose x to be equal either to 24.15886 - y , or 24.15886 + y , (that is,
 to 24.15886 - y , if it is less than 24.15886, and to 24.15886 + y , if it is
 greater than 24.15886) and thereby obtain another equation of which y shall
 be the root. This may be done in the manner following.

29. By the binomial theorem $1 + \frac{1}{10}^{24.15886}$ is = the series $1 + 24.15886$
 $\times \frac{1}{10} + 24.15886 \times \frac{23.15886}{2} \times \frac{1}{100} + 24.15886 \times \frac{23.15886}{2} \times \frac{22.15886}{3} \times$
 $\frac{1}{1000} + \&c =$ (if the first, second, third, fourth, and other following terms of the
 series be denoted by the capital letters A, B, C, D, &c, respectively) $1 +$
 $\frac{24.15886}{10} \times A + \frac{23.15886}{2 \times 10} \times B + \frac{22.15886}{3 \times 10} \times C + \frac{21.15886}{4 \times 10} \times D + \frac{20.15886}{5 \times 10}$
 $\times E + \frac{19.15886}{6 \times 10} \times F + \frac{18.15886}{7 \times 10} \times G + \frac{17.15886}{8 \times 10} \times H + \frac{16.15886}{9 \times 10} \times I +$
 $\frac{15.15886}{10 \times 10} \times K + \frac{14.15886}{11 \times 10} \times L + \frac{13.15886}{12 \times 10} \times M + \frac{12.15886}{13 \times 10} \times N + \frac{11.15886}{14 \times 10}$
 $\times O + \frac{10.15886}{15 \times 10} \times P + \frac{9.15886}{16 \times 10} \times Q + \frac{8.15886}{17 \times 10} \times R + \frac{7.15886}{18 \times 10} \times S +$
 $\frac{6.15886}{19 \times 10} \times T + \frac{5.15886}{20 \times 10} \times V + \frac{4.15886}{21 \times 10} \times W + \frac{3.15886}{22 \times 10} \times X + \frac{2.15886}{23 \times 10}$
 $\times Y + \frac{1.15886}{24 \times 10} \times Z + \frac{0.15886}{25 \times 10} \times A' + \&c \text{ ad infinitum}$
 $= 1.000,000,000,000,000,000,$
 $+ 2.415,886,000,000,000,000,$
 $+ 2.797,458,282,498,000,000,$
 $+ 2.066,282,881,257,121,076,$
 $+ 1.093,004,755,122,901,221,$
 $+ 0.440,674,596,757,136,970,$

$$\begin{aligned}
&+ 0.140,713,715,080,440,686, \\
&+ 0.036,502,866,460,365,873, \\
&+ 0.007,829,344,689,901,419, \\
&+ 0.001,405,703,163,731,782, \\
&+ 0.000,213,088,574,605,671, \\
&+ 0.000,027,428,102,685,829, \\
&+ 0.000,003,007,688,027,570, \\
&+ 0.000,000,281,308,135,776, \\
&+ 0.000,000,022,421,986,457, \\
&+ 0.000,000,001,518,545,475, \\
&+ 0.000,000,000,086,925,908, \\
&+ 0.000,000,000,004,175,519, \\
&+ 0.000,000,000,000,166,066, \\
&+ 0.000,000,000,000,005,383, \\
&+ 0.000,000,000,000,000,138, \\
&+ 0.000,000,000,000,000,002, \\
&+ \&c
\end{aligned}$$

$= 10.000,001,974,734,858,821, \&c$; which is somewhat greater than 10 .
Therefore 24.15886 is a little greater than the index x in the equation $1 + \frac{1}{10}^x = 10$.

30. In order to obtain the value of x to a still greater degree of exactness, let us suppose y to be the difference by which 24.15886 exceeds it.

Then will $1 + \frac{1}{10}^{24.15886-y}$ be $= 1 + \frac{1}{10}^x = 10$.

But $1 + \frac{1}{10}^{24.15886-y}$ is $= 1 + \frac{1}{10}^{24.15886} \times 1 + \frac{1}{10}^{-y} = 1 + \frac{1}{10}^{24.15886} \times \frac{1}{1 + \frac{1}{10}^y}$. Therefore $1 + \frac{1}{10}^{24.15886} \times \frac{1}{1 + \frac{1}{10}^y}$ is $= 10$, and consequently $1 + \frac{1}{10}^{24.15886}$ is $= 10 \times 1 + \frac{1}{10}^y$.

But we have just seen that $1 + \frac{1}{10}^{24.15886}$ is $= 10.000,001,974,734,858,821, \&c$. Therefore $10.000,001,974,734,858,821, \&c$, is $= 10 \times 1 + \frac{1}{10}^y$, and consequently $1 + \frac{1}{10}^y$ is $= \frac{10.000,001,974,734,858,821, \&c}{10} = 1.000,000,197,473,485,882,1, \&c$. We must therefore now endeavour to find the value of the index y in the equation $1 + \frac{1}{10}^y = 1.000,000,197,473,485,882,1, \&c$.

31. Now, by the binomial theorem, $1 + \frac{1}{10}^y$ is $=$ the series $1 + y \times \frac{1}{10} + y \times \frac{y-1}{2} \times \frac{1}{100} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{1}{1000} + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{1}{10,000} + \&c = 1 + \frac{y}{10} + \frac{y^2-y}{200} + \frac{y^3-3yy+2y}{6000} + \frac{y^4-6y^3+11yy-6y}{240,000} + \&c =$

$1 + \frac{y}{10} - \frac{\sqrt{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{\sqrt{6y-11yy+6y^3-y^4}}{240,000} + \&c.$ Therefore the series $1 + \frac{y}{10} - \frac{\sqrt{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{\sqrt{6y-11yy+6y^3-y^4}}{240,000} + \&c$ will be $= 1.000,000,197,473,485,882,1, \&c$; and consequently (subtracting 1 from both sides,) the series $\frac{y}{10} - \frac{\sqrt{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{\sqrt{6y-11yy+6y^3-y^4}}{240,000} + \&c$ will be $= 0.000,000,197,473,485,882,1, \&c.$ This equation we must now endeavour to resolve.

32. Let us therefore, in the first place, suppose the first term $\frac{y}{10}$ alone of the last-mentioned series to be equal to the whole series, and consequently to the absolute term $0.000,000,197,473,485,882,1, \&c.$ Then will y be $(= 10 \times 0.000,000,197, \&c) = 0.000,001,97, \&c$, and consequently x , or $24.15886 - y$, will be $(= 24.15886 - 0.000,001,97) = 24.158,858,03$; of which the first seven figures, $24.158,85$, are exact: so that this first approximation to the value of y (which has been obtained by the resolution of an easy simple equation,) enables us to determine the value of $24.15886 - y$, or x , to one figure more than we had found before.

33. In the second place, let us suppose the two first terms of the last-mentioned series, to wit, the terms $\frac{y}{10} - \frac{\sqrt{y-yy}}{200}$, to be equal to the whole series, and consequently to $0.000,000,197,473,485,882,1, \&c$, and then seek the value of y resulting from this supposition.

We shall then have $\frac{y}{10} - \frac{\sqrt{y-yy}}{200} = \frac{20y}{200} - \frac{\sqrt{y-yy}}{200} = \frac{19y + yy}{200}$, and consequently $\frac{19y + yy}{200} = 0.000,000,197,473,485,882,1, \&c$, and $19y + yy (= 200 \times 0.000,000,197,473,485,882,1, \&c) = 0.000,039,494,697,176,420,0, \&c.$ Therefore $\frac{361}{4} + 19y + yy$ will be $(= \frac{361}{4} + 0.000,039,494,697,176,420,0, \&c = \frac{361}{4} + \frac{4 \times 0.000,039,494,697,176,420,0, \&c}{4} = \frac{361}{4} + \frac{0.000,157,978,788,705,680,0}{4}) = \frac{361.000,157,978,788,705,680,0}{4}$; and consequently $\frac{19}{2} + y$ will be $(= \sqrt{\frac{361.000,157,978,788,705,680,0}{4}} = \frac{19.000,004,157}{2}$, and y will be $(= \frac{19.000,004,157}{2} - \frac{19}{2} = \frac{0.000,004,157}{2}) = 0.000,002,078$; and consequently x , or $24.15886 - y$, will be $(= 24.158,860,000 - 0.000,002,078) = 24.158,857,922$; of which number the first ten figures, $24.158,857,92$, are exact, the more accurate value of the index x in the equation $1 + \frac{1}{10}x = 10$ being (as we have already observed,) $24.158,857,928,096,805,5$.

34. Of these ten figures, $24.158,857,92$, which are exact, the 7th figure 5 was obtained by the resolution of the simple equation $\frac{y}{10} = 0.000,000,197,473,485,882,1, \&c$, and the three next figures 7,92 have been obtained by the resolution

tion of the quadratick equation $\frac{y}{10} - \frac{\sqrt{y-yy}}{200} = 0.000,000,197,473,485,882,1,$ &c. And more figures of the value of x might in like manner be obtained by retaining three, or four, or more terms of the series $\frac{y}{10} - \frac{\sqrt{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{6y-11yy+6y^3-y^4}{240,000} + \&c$, and supposing them to be equal to the whole series, and consequently to $0.000,000,197,473,485,882,1,$ &c, and resolving the equations resulting from such suppositions. But the labour of performing these resolutions would be very considerable, and the number of new figures of the value of x which we should thereby obtain would be but small. For I have tried the two next equations, to wit, the cubick equation that results from a supposition that the three first terms of the said series are equal to the whole, and consequently to $0.000,000,197,473,485,882,1,$ &c, and the biquadratick equation that results from a supposition that the four first terms of the said series are equal to the same quantity, and I have found that the root of such cubick equation has been $= 0.000,002,071,400$, and that the root of such biquadratick equation has been $= 0.000,002,071,941,6$, and consequently that the value of x , or $24.15886 - y$, obtained by means of the said cubick equation, has been $(= 24.158,860,000,000 - 0.000,002,071,400) = 24.158,857,928,600$, of which the first eleven figures, $24.158,857,928$, are exact, and that the value of x , or $24.15886 - y$, obtained by means of the said biquadratick equation, has been $(= 24.158,860,000,000,0 - 0.000,002,071,941,6) = 24.158,857,928,058,4$, of which the first twelve figures, $24.158,857,928,0$, are exact. So that the resolution of these two equations gives us only two figures of the true value of x more than we had before obtained by means of the foregoing quadratick equation. In order therefore to obtain the value of x exact to a few more figures than the ten figures $24.158,857,92$, which were obtained by the resolution of the foregoing quadratick equation, we will have recourse to another method of proceeding, which will be less laborious than the resolution of either the cubick or the biquadratick equation that have been just mentioned, and, *à fortiori*, less laborious than the resolution of any of the higher equations that would result from the supposition that more than four terms of the foregoing series were equal to $0.000,000,197,473,485,882,1,$ &c, and which will give us four additional figures of the true value of x above the ten figures, $24.158,857,92$, already obtained by means of the foregoing quadratick equation. This method of proceeding is as follows:

35. Let all the quantities that form the left-hand side of the equation $\frac{y}{10} - \frac{\sqrt{y-yy}}{200} + \frac{2y-3yy+y^3}{6000} - \frac{6y-11yy+6y^3-y^4}{240,000} + \&c = 0.000,000,197,473,485,882,1,$ &c, be ranged in separate lines according to the several powers of y , those involving the simple power of y being placed in the first, or highest line, and those involving yy , or the square of y , being placed in the second line, and those involving y^3 , or the cube of y , being placed in the third line, and those involving y^4 , or the fourth power of y , being placed in the fourth line, and so on of the following powers of y . And we shall then have

$$\begin{aligned}
& \frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000} \\
& + \frac{yy}{200} - \frac{3yy}{6000} + \frac{11yy}{240,000} - \frac{50yy}{12,000,000} + \frac{274yy}{720,000,000} \\
& + \frac{y^3}{6000} - \frac{6y^3}{240,000} + \frac{35y^3}{12,000,000} - \frac{225y^3}{720,000,000} \\
& + \frac{y^4}{240,000} - \frac{10y^4}{12,000,000} + \frac{85y^4}{720,000,000} \\
& + \frac{y^5}{12,000,000} - \frac{15y^5}{720,000,000} \\
& + \frac{y^6}{720,000,000}
\end{aligned}$$

= 0.000,000,197,473,485,882,1, &c.

But, because y is an exceeding small quantity in comparison of unity, (being = to about 0.000,002, or $\frac{2}{1000,000}$, or $\frac{1}{500,000}$;) it is evident that all the powers of y will be extremely small in comparison of y itself, and consequently that all the quantities contained in the second, third, fourth, fifth, sixth, and other following lines of the compound quantity that forms the left-hand side of the foregoing equation, will be extremely small in comparison of the quantities contained in the first, or upper, line of it, which involve only the simple power of y . We may therefore, without erring much from the truth, consider the quantities contained in the upper line alone as being equal to all the quantities contained in all the lines together, and consequently to the absolute term 0.000,000,197,473,485,882,1, &c; and then the foregoing equation will be converted into the following one, to wit,

$$\begin{aligned}
& \frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} \\
& - \frac{120y}{720,000,000} + \&c = 0.000,000,197,473,485,882,1, \&c, \text{ or } \frac{y}{10} - \frac{y}{2 \times 100} \\
& + \frac{2y}{2 \times 3 \times 1000} - \frac{6y}{2 \times 3 \times 4 \times 10,000} + \frac{24y}{2 \times 3 \times 4 \times 5 \times 100,000} - \frac{120y}{2.3.4.5.6 \times 1,000,000} \\
& + \&c = 0.000,000,197,473,485,882,1, \&c, \text{ or } \frac{y}{10} - \frac{y}{2 \times 100} + \frac{y}{3 \times 1000} - \\
& \frac{y}{4 \times 10,000} + \frac{y}{5 \times 100,000} - \frac{y}{6 \times 1,000,000} + \&c = 0.000,000,197,473,485,882, \\
& 1, \&c, \text{ or } y \times \text{the infinite series } \frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \\
& \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1,000,000} + \&c = 0.000,000,197,473,485,882,1, \&c, \text{ or (if}
\end{aligned}$$

we denote the several terms $\frac{1}{10}$, $\frac{1}{2 \times 100}$, $\frac{1}{3 \times 1000}$, $\frac{1}{4 \times 10,000}$, $\frac{1}{5 \times 100,000}$, $\frac{1}{6 \times 1,000,000}$, &c, of the foregoing infinite series by the capital letters A, B, C, D, E, F, &c, respectively,) $y \times$ the infinite series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B$

- $\frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E + \&c = 0.000,000,197,473,$

485,882,1, &c. We must therefore, in the next place, compute the value of

this infinite series to as great a degree of exactness as we shall think necessary;

which will not be very difficult, because the said series evidently converges with

a considerable degree of swiftness, every new term of it being less than a tenth

part

part of the foregoing term. And when this value, which we will call S , is obtained, we shall have $y \times S = 0.000,000,197,473,485,882,1, \&c$, and consequently $y = \frac{0.000,000,197,473,485,882,1, \&c}{S}$.

The Computation of the infinite Series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E + \&c \text{ ad infinitum}.$

36. NOW the infinite series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C + \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E + \&c$ may be computed in the following manner.

$$A \text{ is } = \frac{1}{10} = 0.100,000,000,000,000,000,$$

$$B \text{ is } = \frac{1}{2 \times 10} \times A = 0.005,000,000,000,000,000,$$

$$C = \frac{2}{3 \times 10} \times B = 0.000,333,333,333,333,333,$$

$$D = \frac{3}{4 \times 10} \times C = 0.000,024,999,999,999,999,$$

$$E = \frac{4}{5 \times 10} \times D = 0.000,001,999,999,999,999,$$

$$F = \frac{5}{6 \times 10} \times E = 0.000,000,166,666,666,666,$$

$$G = \frac{6}{7 \times 10} \times F = 0.000,000,014,285,714,285,$$

$$H = \frac{7}{8 \times 10} \times G = 0.000,000,001,249,999,999,$$

$$I = \frac{8}{9 \times 10} \times H = 0.000,000,000,111,111,111,$$

$$K = \frac{9}{10 \times 10} \times I = 0.000,000,000,009,999,999,$$

$$L = \frac{10}{11 \times 10} \times K = 0.000,000,000,000,909,090,$$

$$M = \frac{11}{12 \times 10} \times L = 0.000,000,000,000,083,333,$$

$$N = \frac{12}{13 \times 10} \times M = 0.000,000,000,000,007,692,$$

$$O = \frac{13}{14 \times 10} \times N = 0.000,000,000,000,000,714,$$

$$P = \frac{14}{15 \times 10} \times O = 0.000,000,000,000,000,066,$$

$$Q = \frac{15}{16 \times 10} \times P = 0.000,000,000,000,000,006,$$

$$R = \frac{16}{17 \times 10} \times Q = 0.000,000,000,000,000,000.$$

Therefore $A + C + E + G + I + L + N + P + R$ are =
 $0.100,000,000,000,000,000,$
 $+ \dots, 333,333,333,333,333,$

$$\begin{aligned}
& + \dots, \dots, 1,999,999,999,999, \\
& + \dots, \dots, 14,285,714,285, \\
& + \dots, \dots, \dots, 111,111,111, \\
& + \dots, \dots, \dots, 909,090, \\
& + \dots, \dots, \dots, 7,692, \\
& + \dots, \dots, \dots, 66, \\
& + \dots, \dots, \dots, \dots, \\
& = 0.100,335,347,731,075,576; \text{ and } B + D + F + H + K + M + O + Q \\
& \text{are} =
\end{aligned}$$

$$\begin{aligned}
& 0.005,000,000,000,000,000, \\
& + \dots, 24,999,999,999,999, \\
& + \dots, 166,666,666,666, \\
& + \dots, 1,249,999,999, \\
& + \dots, 9,999,999, \\
& + \dots, 83,333, \\
& + \dots, 714, \\
& + \dots, 6, \\
& = 0.005,025,167,926,750,716; \text{ and consequently } A + C + E + G + I + \\
& L + N + P + R - B - D - F - H - K - M - O - Q \text{ \&c, or, its} \\
& \text{equal, } A - B + C - D + E - F + G - H + I - K + L - M + N - O \\
& + P - Q + R - \text{\&c, will be} =
\end{aligned}$$

$$\begin{aligned}
& 0.100,335,347,731,075,576 \\
& - 0.005,025,167,926,750,716 \\
& = 0.095,310,179,804,324,860, \text{\&c.}
\end{aligned}$$

Therefore the infinite series $\frac{1}{10} - \frac{1}{2 \times 10} \times A + \frac{2}{3 \times 10} \times B - \frac{3}{4 \times 10} \times C$
 $+ \frac{4}{5 \times 10} \times D - \frac{5}{6 \times 10} \times E + \text{\&c, or } \frac{1}{10} - \frac{A}{2 \times 10} + \frac{2B}{3 \times 10} - \frac{3C}{4 \times 10} +$
 $\frac{4D}{5 \times 10} - \frac{5E}{6 \times 10} + \text{\&c, or the series } S, \text{ is } = 0.095,310,179,804,324,860, \text{\&c.}$

Q. E. I.

37. Therefore $y \times$ the series S is $= y \times 0.095,310,179,804,324,860, \text{\&c;}$
 and consequently $y \times 0.095,310,179,804,324,860, \text{\&c}$ is $= 0.000,000,197,473,$
 $485,882,1, \text{\&c, and } y \text{ is } = \frac{0.000,000,197,473,485,882,1, \text{\&c}}{0.095,310,179,804,324,860, \text{\&c}} = 0.000,002,071,903,$

403. Therefore x , or $24.15886 - y$, is
 $= 24.158,860,000,000,000$
 $- 0.000,002,071,903,403$

$= 24.158,857,928,096,597$, or the value of the index x in the original equation
 $1 + \frac{1}{10}^x = 10$ is $24.158,857,928,096,597$; of which number the first 14
 figures $24.158,857,928,096$, are exact, the more accurate value of x being
 $24.158,857,928,096,805,5$.

38. Therefore the proportion of the ratio of 10 to 1 to the ratio of 11 to 10
 is that of the number $24.158,857,928,096,597$ to 1, or (because 597 is nearly
 $= 600$) that of $24.158,857,928,096,6$ to 1.

Q. E. I.

39. Coroll.

39. Coroll. The proportion of 24.158,857,928,096,6 to 1, is equal to that of 1 to $\frac{1}{24.158,857,928,096,6}$, or 0.041,392,685,158,228,29. Therefore, if 1 be taken for the representative of the magnitude of the ratio of 10 to 1, (as it is in Briggs's system of logarithms,) 0.041,392,685,158,228,29 will be the representative of the lesser ratio of 11 to 10; or, in other words, the logarithm of the small ratio of 11 to 10 in Briggs's system of logarithms will be 0.041,392,685,158,228,29; of which number the first thirteen figures 0.041,392,685,158,22, are exact, the more accurate value of this logarithm (as computed by Mr. Abraham Sharp,) being 0.041,392,685,158,225,040,750.

40. And in the same manner we may compute Briggs's logarithms of the ratios of 10 to 9, or of $1 + \frac{1}{9}$ to 1, and of 81 to 80, or of $1 + \frac{1}{80}$ to 1, and of 121 to 120, or of $1 + \frac{1}{120}$ to 1, and of 2401 to 2400, or of $1 + \frac{1}{2400}$ to 1, and, in general, of $m + 1$ to m , (m being any whole number whatsoever,) or of $1 + \frac{1}{m}$ to 1, by first finding the values of the index x in the several equations $1 + \frac{1}{9}^x = 10$, $1 + \frac{1}{80}^x = 10$, $1 + \frac{1}{120}^x = 10$, $1 + \frac{1}{2400}^x = 10$, and, in general, $1 + \frac{1}{m}^x = 10$, and then dividing 1 by the values of x so found; the several values of $\frac{1}{x}$, or quotients of such divisions, being the logarithms of the said ratios of $1 + \frac{1}{9}$ to 1, and of $1 + \frac{1}{80}$ to 1, and of $1 + \frac{1}{120}$ to 1, and of $1 + \frac{1}{2400}$ to 1, and, in general, of $1 + \frac{1}{m}$ to 1, or of the ratios of 10 to 9, and of 81 to 80, and of 121 to 120, and of 2401 to 2400, and, in general, of $m + 1$ to m , in Briggs's system.

41. This method of computing logarithms is not to be compared, in point of ease and expedition, to either of the two logarithmick serieses $k - \frac{k^2}{2} + \frac{k^3}{3} - \frac{k^4}{4} + \frac{k^5}{5} - \frac{k^6}{6} + \&c$ *ad infinitum*, and $k + \frac{k^2}{2} + \frac{k^3}{3} + \frac{k^4}{4} + \frac{k^5}{5} + \frac{k^6}{6} + \&c$ *ad infinitum*, which were invented by Mr. Mercator and Dr. Wallis. But it possesses the advantages recommended by Dr. Halley, of being derived from the abstract nature of ratios and the pure principles of arithmetick, without the assistance of the hyperbola, or any other geometrical figure, and without any recourse to the doctrine of infinitesimals, or of fluxions, or of the limits of ratios, or, in general, of the arithmetick of infinites in any of its modifications. And it is, I believe, considerably less laborious in the practice of it, than the methods by which Mr. Henry Briggs himself computed his logarithms; which were likewise purely arithmetical*. And it likewise serves as a notable instance of the

* See above, in page 73, Mr. Euclid Speidell's *Logarithmotechnia*, chapter viii. where, in speaking of Mr. Henry Briggs's computation of the logarithm of 2, he has these words: "Whereby it is apparent that he did produce the logarithm for 2 to 15 places very true; though I have been told it was eight persons' work for a year's time after his method, which was by large and many extractions of the square-root."

extensive utility of Sir Isaac Newton's wonderful theorem for raising the powers of a binomial quantity.

42. As the solution of the foregoing problem consists of a great number of steps, which, for the ease of the reader, have been set forth distinctly and at considerable length, it will not be amiss, before we conclude this discourse, to take a short review of all the foregoing processes, and of the gradual approximations obtained by means of them, to the value of the index x in the equation $1 + \frac{1}{10}^x = 10$, the discovery of which is the object of the Problem.

A Review of the several Steps of the foregoing Solution.

43. The object of the Problem is to find the value of the index x in the equation $1 + \frac{1}{10}^x = 10$.

The first step we took towards finding this value was to expand the quantity $1 + \frac{1}{10}^x$ into an infinite series by means of Sir Isaac Newton's binomial theorem, by which we obtained the equation $1 + \frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} + \&c \text{ ad infinitum} = 10$, and consequently the equation $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000} + \&c = 9$.

44. Then we supposed, first, one term, then two terms, then three terms, and, lastly, four terms of the series which forms the left-hand side of this last equation to be equal to the whole series, and consequently to the absolute term 9, and resolved the equations resulting from those suppositions.

By resolving the simple equation resulting from the first of these suppositions, to wit, the simple equation $\frac{x}{10} = 9$, we found x to be equal to 90; which was therefore our first approximation to the value of x in the original equation $1 + \frac{1}{10}^x = 10$. This approximation is very wide of the true value of x , being more than three times as great.

From the second supposition, to wit, that the two terms $\frac{x}{10} + \frac{xx - x}{200}$ were equal to the whole series, and consequently to 9, there resulted the quadratick equation $19x + xx = 1800$; by the resolution of which x appeared to be $= 33.97$; which was therefore our second approximation to the value of x in the original equation $1 + \frac{1}{10}^x = 10$. This approximation is much nearer to the truth than the former, but yet is considerably too large.

From the third supposition, to wit, that the three terms $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000}$ were equal to the whole series, and consequently to 9, there resulted the cubick equation $572x + 27xx + x^3 = 54,000$, by the resolution of which we had $x = 27.05$. This therefore was the third approximation to the

value of x in the original equation $1 + \frac{1}{10}^x = 10$; and it is true in the first, or highest, figure 2, the more accurate value of x being 24.158,857,928,096,805,5.

From the fourth supposition, to wit, that the four terms $\frac{x}{10} + \frac{xx - x}{200} + \frac{x^3 - 3xx + 2x}{6000} + \frac{x^4 - 6x^3 + 11xx - 6x}{240,000}$ are equal to the whole series, and consequently to 9, there resulted the biquadratick equation $22,874x + 1091xx + 34x^3 + x^4 = 2,160,000$; by the resolution of which we had $x = 25.07$. This therefore was our fourth approximation to the value of x in the original equation $1 + \frac{1}{10}^x = 10$.

45. As the difference of this fourth value of x , to wit, 25.07, from the third value of it, to wit, 27.05, is less than 2, whereas the difference between the third value, 27.05, and the second value, 33.95, was more than 6, we concluded that it was highly probable that the excess of this fourth value, 25.05, above the true value of x was less than 1, and consequently that x was greater than 24, but less than 25. We then tried whether x was greater, or less, than 24, by raising $1 + \frac{1}{10}$, or 1.1, to the 24th power; which we found to be somewhat less than 10, to wit, 9.849,732,675,807,611,094,711,841. We also raised 1.1 to the 25th power, which we found to be 10.834,705,943,388,372,204,183,025,1; which is greater than 10. And we thereby discovered with certainty that the true value of the index x in the equation $1 + \frac{1}{10}^x = 10$ is greater than 24, but less than 25.

46. We then put z for the unknown difference by which the true value of x in the equation $1 + \frac{1}{10}^x = 10$ exceeds 24, so that x was $= 24 + z$. And we thereby had $1 + \frac{1}{10}^{24+z} (= 1 + \frac{1}{10}^x) = 10$.

But $1 + \frac{1}{10}^{24+z}$ is $= 1 + \frac{1}{10}^{24} \times 1 + \frac{1}{10}^z$; and $1 + \frac{1}{10}^{24}$ had been found to be $= 9.849,732,675,807,611,094,711,841$. Therefore $9.849,732,675,807,611,094,711,841 \times 1 + \frac{1}{10}^z$ is $= 10$, and consequently $1 + \frac{1}{10}^z$ is $(= \frac{10}{9.849,732,675,807,611,094,711,841}) = 1.015,255,979,947,706,347,941$, &c. And thus we obtained a new equation in which the unknown index z of the power of the binomial quantity $1 + \frac{1}{10}$ is less than a unit, instead of the original equation $1 + \frac{1}{10}^x = 10$, in which the index x of the power of the same quantity is greater than 24.

47. We then expanded $1 + \frac{1}{10}^z$ by the binomial theorem, and thereby obtained the equation $1 + \frac{z}{10} - \frac{z - zz}{200} + \frac{2z - 3zz + z^3}{6000} - \frac{6z - 11zz + 6z^3 - z^4}{240,000} + \&c = 1.015,$

$= 1.015,255,979,947,706,347,941, \&c$, and (by subtracting 1 from both sides,) the equation $\frac{z}{10} - \frac{(z-zz)}{200} + \frac{2z-3zz+z^3}{6000} - \frac{(6z-11zz+6z^3-z^4)}{240,000} + \&c = 0.015,255,979,947,706,347,941, \&c$.

48. We then proceeded to find approximations to the value of z in this new equation in the same manner as we had before found approximations to the value of x in the equation $\frac{x}{10} + \frac{xx-x}{200} + \frac{x^3-3xx+2x}{6000} + \frac{x^4-6x^3+11xx-6x}{240,000} + \&c = 9$, by first supposing the first term $\frac{z}{10}$ alone of the series that forms the left-hand side of the equation, to be equal to the whole series, and consequently to the absolute term $0.015,255,979,947,706,347,941, \&c$, and then supposing the two first terms, $\frac{z}{10} - \frac{(z-zz)}{200}$, to be equal to the same quantity, and then supposing the three first terms, $\frac{z}{10} - \frac{(z-zz)}{200} + \frac{2z-3zz+z^3}{6000}$, to be equal to the same quantity, and, lastly, supposing the four first terms, $\frac{z}{10} - \frac{(z-zz)}{200} + \frac{2z-3zz+z^3}{6000} - \frac{(6z-11zz+6z^3-z^4)}{240,000}$, to be equal to it, and by resolving the several equations resulting from those suppositions.

From the first of those suppositions we had the simple equation $\frac{z}{10} = 0.015,255,979,947,706,347,941, \&c$; by the resolution of which we had $z = 0.152, \&c$, of which the two first figures 0.15 are exact.

From the second supposition there resulted the quadratick equation $19z + zz = 3.051,195,800, \&c$; by the resolution of which we had $z = 0.1592$.

From the third supposition there resulted the cubick equation $572z + 27zz + z^3 = 91.535,879,686,238,087,646, \&c$; by the resolution of which we had $z = 0.15883$, of which the four highest figures, 0.1588 , are exact.

And from the fourth supposition there resulted the biquadratick equation $22874z + 10912z + 34z^3 + z^4 = 3661.435,187,449,523,505,840,000, \&c$; by the resolution of which we had $z = 0.15886$, of which the four highest figures, 0.1588 , are exact, and the fifth, or last, figure, 6 , differs but by an unit from the fifth figure of the true value of z , which is $= 0.158,857,928,096,805,5, \&c$. This fourth approximation to the value of z enabled us to conclude that x , or $24 + z$, would be very nearly equal to $24 + 0.15886$, or 24.15886 .

49. We then dropped all further consideration of the equation $\frac{z}{10} - \frac{(z-zz)}{200} + \frac{2z-3zz+z^3}{6000} - \frac{(6z-11zz+6z^3-z^4)}{240,000} + \&c = 0.015,255,979,947,706,347,941, \&c$, and made a trial of the exactness of the last value of x obtained by the foregoing processes, to wit, 24.15886 , by raising the binomial quantity $1 + \frac{1}{10}$ to the power of which 24.15886 is the index, which was done by the help of Sir Isaac Newton's binomial theorem. And we found that the said power of $1 + \frac{1}{10}$ was $= 10,000,001,974,734,858,821, \&c$, (which is a little

greater than 10) and consequently that 24.15886 is somewhat, though but a very little, greater than the true value of the index x in the equation $1 + \frac{1}{10}^x = 10$.

50. We then supposed x to be $= 24.15886 - y$, and consequently $1 + \frac{1}{10}^{24.15886 - y}$ to be $= 10$.

Then, since $1 + \frac{1}{10}^{24.15886 - y}$ is $= 1 + \frac{1}{10}^{24.15886} \times 1 + \frac{1}{10}^{-y} = 1 + \frac{1}{10}^{24.15886} \times \frac{1}{1 + \frac{1}{10}^y}$, we had $1 + \frac{1}{10}^{24.15886} \times \frac{1}{1 + \frac{1}{10}^y} = 10$, and consequently $1 + \frac{1}{10}^{24.15886} = 10 \times 1 + \frac{1}{10}^y$, and $1 + \frac{1}{10}^y = \frac{1 + \frac{1}{10}^{24.15886}}{10}$

$$= \frac{10.000,001,974,734,858,821, \&c}{10} = 1.000,000,197,473,485,882,1, \&c.$$

51. Having thus obtained a third equation $1 + \frac{1}{10}^y = 1.000,000,197,473,485,882,1, \&c$, in which y is much smaller than x in the former equation, we proceeded to expand $1 + \frac{1}{10}^y$ into an infinite series by means of Sir Isaac Newton's binomial theorem, and thereby obtained the equation $1 + \frac{y}{10} - \frac{\frac{y-y^2}{200}}{200} + \frac{2y-3y^2+y^3}{6000} - \frac{6y-11y^2+6y^3-y^4}{240,000} + \&c = 1.000,000,197,473,485,882,1, \&c$, and (by subtracting 1 from both sides,) the equation $\frac{y}{10} - \frac{\frac{y-y^2}{200}}{200} + \frac{2y-3y^2+y^3}{6000} - \frac{6y-11y^2+6y^3-y^4}{240,000} + \&c = 0.000,000,197,473,485,882,1, \&c$.

52. We then proceeded to approximate to the value of y in this equation, by first supposing the first term $\frac{y}{10}$ alone of the series $\frac{y}{10} - \frac{\frac{y-y^2}{200}}{200} + \&c$, (which forms the left-hand side of this equation,) to be equal to the whole series, and consequently to the absolute term 0.000,000,197,473,485,882,1, &c, and, secondly, by supposing the two first terms $\frac{y}{10} - \frac{\frac{y-y^2}{200}}{200}$ of the said series to be equal to the same quantity, and resolving the equations resulting from these suppositions.

From the first of these suppositions we had the simple equation $\frac{y}{10} = 0.000,000,197,473,485,882,1, \&c$; by the resolution of which we had $y = 0.000,001,97$, and consequently $x (= 24.15886 - y = 24.158,860,00 - 0.000,001,97) = 24.158,858,03$; of which number the first seven figures, 24.158,85, are exact.

And from the second of these suppositions there resulted the quadratick equation $19y + yy = 0.000,039,494,697,176,420,0, \&c$; by the resolution of which

which we had $y = 0.000,002,078$, and consequently $x (= 24.15886 - y = 24.158,860,000 - 0.000,002,078) = 24.158,857,922$; of which number the first ten figures, 24.158,857,92, are exact.

53. This is a very considerable degree of exactness. But, in order to obtain the value of y exact to a few more places of figures, we had recourse to a different method of resolving the equation $\frac{y}{10} - \frac{y-yy}{200} + \frac{2y-3yy+y^3}{6000} - \frac{6y-11yy+6y^3-y^4}{240,000} + \&c = 0.000,000,197,473,485,882,1, \&c$, which consisted in the omission of all the members in each term of the series that involved either yy , or y^3 , or y^4 , or any other power of y , except the simple power, or y itself: by which means the said equation was changed into the following simple equation, to wit, $\frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000} + \&c = 0.000,000,197,473,485,882,1, \&c$, or $\frac{y}{10} - \frac{y}{2 \times 100} + \frac{2 \times y}{2 \times 3 \times 1000} - \frac{6 \times y}{2 \times 3 \times 4 \times 10,000} + \frac{24 \times y}{2 \times 3 \times 4 \times 5 \times 100,000} - \frac{120 \times y}{2 \times 3 \times 4 \times 5 \times 6 \times 1,000,000} + \&c = 0.000,000,197,473,485,882,1, \&c$, or $\frac{y}{10} - \frac{y}{2 \times 100} + \frac{y}{3 \times 1000} - \frac{y}{4 \times 10,000} + \frac{y}{5 \times 100,000} - \frac{y}{6 \times 1,000,000} + \&c = 0.000,000,197,473,485,882,1, \&c$, or $y \times$ the infinite series $\frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1,000,000} + \&c = 0.000,000,197,473,485,882,1, \&c$. We then computed the value of the infinite series $\frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1,000,000} + \&c$, or $\frac{1}{10} - \frac{A}{2 \times 10} + \frac{2B}{3 \times 10} - \frac{3C}{4 \times 10} + \frac{4D}{5 \times 10} - \frac{5E}{6 \times 10} + \frac{6F}{7 \times 10} - \frac{7G}{8 \times 10} + \&c$, and found it to be $= 0.095,310,179,804,324,860, \&c$; which gave us the simple equation $y \times 0.095,310,179,804,324,860, \&c = 0.000,000,197,473,485,882,1, \&c$, by the resolution of which we had $y = \frac{0.000,000,197,473,485,882,1, \&c}{0.095,310,179,804,324,860, \&c} = 0.000,002,071,903,403$. And from this value of y we concluded x , or $24.158,86 - y$, to be $(= 24.158,860,000,000,000 - 0.000,002,071,903,403) = 24.158,857,928,096,597$, or (because 597 is nearly $= 600$) $24.158,857,928,096,6$; which number is true in the first fourteen figures, 24.158,857,928,096, the more accurate value of x being (as has been before observed,) $24.158,857,928,096,805,5$. And thus we obtained the value of the index x of the binomial quantity $1 + \frac{1}{10}$ in the original equation $1 + \frac{1}{10}^x = 10$ exact to fourteen places of figures.

Q. E. I.

54. If any lover of this subject should be inclined to compute the value of x to still more than 14 figures, he may find it to nearly as many figures more, or 28 figures in all, by raising $1 + \frac{1}{10}$ to the $24.158,857,928,096$ th power by means of the binomial theorem, and then proceeding in the manner following.

following. Let the number that shall be found to be equal to this power of $1 + \frac{1}{10}$ be denoted by the letter P. And, as this number will be somewhat less than 10, and consequently the index 24.158,857,928,096 is somewhat less than the index x , let 24.158,857,928,096 + v be substituted instead of x in the equation $1 + \frac{1}{10}^x = 10$; and the said equation will be thereby converted into the equation $1 + \frac{1}{10}^{24.158,857,928,096 + v} = 10$. But $1 + \frac{1}{10}^{24.158,857,928,096 + v}$ is $= 1 + \frac{1}{10}^{24.158,857,928,096} \times 1 + \frac{1}{10}^v = P \times 1 + \frac{1}{10}^v$. Therefore $P \times 1 + \frac{1}{10}^v$ will be $= 10$, and $1 + \frac{1}{10}^v$ will be $= \frac{10}{P}$.

Now let $1 + \frac{1}{10}^v$ be expanded into an infinite series by means of the binomial theorem; and we shall have $1 + \frac{v}{10} - \frac{v^2}{200} + \frac{2v^3 - 3vv^2 + v^3}{6000} - \frac{6v^4 - 11vv^3 + 6v^3 - v^4}{240,000} + \&c \text{ ad infinitum} (= 1 + \frac{1}{10}^v) = \frac{10}{P}$, and $\frac{v}{10} - \frac{v^2}{200} + \frac{2v^3 - 3vv^2 + v^3}{6000} - \frac{6v^4 - 11vv^3 + 6v^3 - v^4}{240,000} + \&c = \frac{10}{P} - 1$, and (omitting all the quantities that involve vv , or v^3 , or v^4 , or any of the powers of v except the simple power, or v itself, on account of their extreme smallness in comparison of the terms that involve only the simple power of v ,) $\frac{v}{10} - \frac{v^2}{200} + \frac{2v^3}{6000} - \frac{6v^4}{240,000} + \&c$, or $v \times$ the infinite series $\frac{1}{10} - \frac{1}{2 \times 100} + \frac{1}{3 \times 1000} - \frac{1}{4 \times 10,000} + \frac{1}{5 \times 100,000} - \frac{1}{6 \times 1,000,000} + \&c$, or $v \times$ the infinite series $\frac{1}{10} - \frac{A}{2 \times 10} + \frac{2B}{3 \times 10} - \frac{3C}{4 \times 10} + \frac{4D}{5 \times 10} - \frac{5E}{6 \times 10} + \frac{6F}{7 \times 10} - \frac{7G}{8 \times 10} + \&c = \frac{10}{P} - 1$, or (putting S for the value of the said infinite series computed to a sufficient number of figures,) $v \times S = \frac{10}{P} - 1$; and consequently $v = \frac{\frac{10}{P} - 1}{S}$, and $x (=$

24.158,857,928,096 + $v) = 24.158,857,928,096 + \frac{\frac{10}{P} - 1}{S}$; which will be

true to 27 or 28 places of figures.

55. Or, if it should be required to find the value of x to only a few figures more than those which were found in Art. 37, to wit, 24.158,857,928,096,597, (of which the first 14 figures, 24.158,857,928,096, were exact,) it may be done without raising the binomial quantity $1 + \frac{1}{10}$ to the 24.158,857,928,096th power, (which is a very laborious operation,) by retaining some of the terms that involve the square of y , as well as those that involve the simple power of y , in the last equation $\frac{y}{10} - \frac{y^2}{200} + \frac{2y^3 - 3yy^2 + y^3}{6000} - \frac{6y^4 - 11yy^3 + 6y^3 - y^4}{240,000} + \&c = 0.000,000,197,473,485,882,1, \&c$, and supposing the terms so retained to be

be equal to the whole series that forms the left-hand side of the said equation, and consequently to be equal to the absolute term 0.000,000,197,473,485,882,1, &c, and then resolving the quadratick equation which will result from such supposition. This may be done in the manner following.

56. If we retain only the terms that involve y and yy in the foregoing equation, it will be converted into the following equation, to wit,

$$\begin{aligned} \frac{y}{10} - \frac{y}{200} + \frac{2y}{6000} - \frac{6y}{240,000} + \frac{24y}{12,000,000} - \frac{120y}{720,000,000} + &c \\ + \frac{yy}{200} - \frac{3yy}{6000} + \frac{11yy}{240,000} - \frac{50yy}{12,000,000} + \frac{274yy}{720,000,000} - &c \\ = 0.000,000,197,473,485,882,1, &c, \end{aligned}$$

$$\begin{aligned} \text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6000,000} + &c \\ + \frac{yy}{200} - \frac{yy}{2000} + \frac{11yy}{240,000} - \frac{yy}{240,000} + \frac{274yy}{720,000,000} - &c \\ = 0.000,000,197,473,485,882,1, &c, \end{aligned}$$

$$\begin{aligned} \text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6000,000} + &c \\ + 0.005,000,000,000,000,000 \times yy \\ - 0.000,500,000,000,000,000 \times yy \\ + 0.000,045,833,333,333,333 \times yy \\ - 0.000,004,166,666,666,666 \times yy \\ + 0.000,000,380,555,555,555 \times yy \\ - 0.000,000,035,000,000,000 \times yy \\ + &c \\ = 0.000,000,197,473,485,882,1, &c, \end{aligned}$$

$$\begin{aligned} \text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6000,000} + &c \\ + 0.005,046,213,888,888,888 \times yy \\ - 0.000,504,201,666,666,666 \times yy \\ = 0.000,000,197,473,485,882,1, &c \end{aligned}$$

$$\begin{aligned} \text{or } \frac{y}{10} - \frac{y}{200} + \frac{y}{3000} - \frac{y}{40,000} + \frac{y}{500,000} - \frac{y}{6,000,000} + &c \\ + 0.004,542,012,222,222,222 \times yy \\ = 0.000,000,197,473,485,882,1, &c, \text{ or} \\ 0.095,310,179,804,324,860, &c \times y \\ + 0.004,542,012,222,222,222 \times yy \end{aligned}$$

$= 0.000,000,197,473,485,882,1, &c$, or (neglecting all but the four highest figures of the co-efficient of yy ,) $0.095,310,179,804,324,860, &c \times y + 0.004,542 \times yy = 0.000,000,197,473,485,882,1, &c$.

57. This quadratick equation may be most conveniently resolved by approximation, by substituting, instead of y , in the quantity $0.004,542 \times yy$ the value of y derived from the simple equation $0.095,310,179,804,324,860, &c \times y = 0.000,000,197,473,485,882,1, &c$, which is $= \left(\frac{0.000,197,473,485,882,1, &c}{0.095,310,179,804,324,860, &c} \right)$, or)

$0.000,002,071,903,403$. We shall then have $yy = 0.000,002,071,903,403^2 = 0.000,000,000,004,292,783,711,362,980,409$, and consequently $0.004,542 \times yy = 0.004,542 \times 0.000,000,000,004,292,783, &c = 0.000,000,000,000,$

019,497,820, &c. Therefore $0.095,310,179,804,324,860 \times y + 0.000,000,000,000,019,497,820, \&c$, will be $= 0.000,000,197,473,485,882,1, \&c$, and (dividing all the terms by $0.095,310,179,804,324,860$,) $y + 0.000,000,000,000,204$, will be $= 0.000,002,071,903,403$, and (subtracting $0.000,000,000,000,204$, from both sides) y will $= 0.000,002,071,903,199$; and consequently x , or $24.158,86 - y$, will be $=$

$24.158,860,000,000,000$
 $- 0.000,002,071,903,199$

$= 24.158,857,928,096,801$; of which all the figures, except the last, are exact, the more accurate value of x being (as we have before observed in Art. 6,) $24.158,857,928,096,805,5$. We have therefore now found the value of the index x in the equation $1 + \frac{1}{10}^x = 10$ exact to sixteen places of figures.

Q. E. I.

58. The value of x just now obtained, to wit, $24.158,857,928,096,801$, is to 1, as 1 is to $\frac{1,000,000,000,000,000,000, \&c}{24,158,857,928,096,801} = 0.041,392,685,158,225,484, \&c$.

Therefore the proportion of the ratio of 10 to 1 to the ratio of $1 + \frac{1}{10}$ to 1, or to the ratio of 11 to 10, is that of 1 to $0.041,392,685,158,225,484, \&c$, or, in other words, Briggs's logarithm of the ratio of 11 to 10 is $= 0.041,392,685,158,225,484, \&c$; of which number the first fifteen figures, (reckoning from the place of units,) to wit, $0.041,392,685,158,225$, are exact, the more accurate value of this logarithm (as computed by Mr. Abraham Sharp,) being $0.041,392,685,158,225,040,750$.

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D E M O N S T R A T I O N

O F

S I R I S A A C N E W T O N ' S

B I N O M I A L T H E O R E M

In the Case of Integral Powers, or Powers of which the Indexes are whole Numbers.

B Y *FRANCIS MASERES*, Esq.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

A R T I C L E I.

THE binomial theorem of the great Sir Isaac Newton has been so much resorted to in the preceeding Discourse of Dr. Halley, and in the last foregoing Tract, intituled, "An Appendix to the said Tract," for the purpose of computing logarithms, and is so closely connected with that subject, that it will probably be agreeable to the readers of this collection to see a demonstration given of it in the same volume with the other tracts, of which it is the chief foundation. I shall therefore now proceed to give a demonstration of it in the first and simplest case, or that in which the indexes of the powers, to which the binomial quantity is raised, are positive whole numbers; which may be done with sufficient perspicuity and exactness within the compass of a few pages.

2. This theorem is as follows.

T H E O R E M.

If m be any whole number whatsoever, $\overline{a + b}^m$, or the m th power of the binomial quantity $a + b$, will be equal to the following series of terms, to wit, $a^m + \frac{m}{1} \times a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} \times a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times a^{m-5} b^5 + \&c$ continued to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$

$\times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c \times \frac{m-\sqrt{m-1}}{m} \times a^{m-m} \times b^m$, or to $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c \times \frac{1}{m} \times 1 \times b^m$, or to $\frac{1}{m} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c \times \frac{b^m}{m}$; in which series the law by which the several terms, after the first term a^m , are generated from the said first term, is as follows. The literal parts of the second, and third, and fourth, and other following terms, to wit, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, &c, are generated from the first term a^m by the continual multiplication of the fraction $\frac{b}{a}$, the index of the power of a (the first term of the binomial quantity $a + b$) in every new term decreasing continually by an unit, and the index of the power of b (the second term of the binomial quantity $a + b$) increasing by an unit at the same time. And the several co-efficients of the second, third, fourth, and other following terms of the series, to wit, $\frac{m}{1}$, $\frac{m}{1} \times \frac{m-1}{2}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, &c, are generated from 1, (the co-efficient of the first term a^m , or $1 \times a^m$), by the continual multiplication of the fractions $\frac{m}{1}$, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, of which the denominators are the numbers 1, 2, 3, 4, 5, &c, in their natural order, and the numerator of the first fraction $\frac{m}{1}$ is the index m , and the numerators of the following fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, are derived from that of the first, to wit, m , by the continual subtraction of an unit.

Examples of the said Theorem.

3. Thus, for example, if m is = 2, or it is required to find, by means of this theorem, the square of the binomial quantity $a + b$, we shall have $\overline{a + b}^2 = a^2 + \frac{2}{1} \times a^{2-1}b + \frac{2}{1} \times \frac{2-1}{2} \times a^{2-2}b^2 = a^2 + \frac{2}{1} \times a^1b + \frac{2}{1} \times \frac{1}{2} \times a^0 \times b^2 = a^2 + 2ab + \frac{2}{2} \times 1 \times b^2 = a^2 + 2ab + b^2$.

And, in like manner, if m is = 3, we shall have $\overline{a + b}^3 = a^3 + \frac{3}{1} \times a^{3-1}b + \frac{3}{1} \times \frac{3-1}{2} \times a^{3-2}b^2 + \frac{3}{1} \times \frac{3-1}{2} \times \frac{3-2}{3} \times a^{3-3}b^3 = a^3 + \frac{3}{1} \times a^2b + \frac{3}{1} \times \frac{2}{2} \times a^1b^2 + \frac{3}{1} \times \frac{2}{2} \times \frac{1}{3} \times a^0b^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

And, if m is = 4, we shall have $\overline{a + b}^4 = a^4 + \frac{4}{1} \times a^{4-1}b + \frac{4}{1} \times \frac{4-1}{2} \times a^{4-2}b^2 + \frac{4}{1} \times \frac{4-1}{2} \times \frac{4-2}{3} \times a^{4-3}b^3 + \frac{4}{1} \times \frac{4-1}{2} \times \frac{4-2}{3} \times \frac{4-3}{4} \times a^{4-4}b^4 = a^4 + \frac{4}{1} \times a^3b + \frac{4}{1} \times \frac{3}{2} \times a^2b^2 + \frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times a^1b^3 + \frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} \times a^0b^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

$$\frac{2}{3} \times a^1 b^3 + \frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} \times 1 \times b^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

And, if m is $= 5$, we shall have $\overline{a+b}^5 = a^5 + \frac{5}{1} \times a^{5-1}b + \frac{5}{1} \times \frac{5-1}{2} \times a^{5-2}b^2 + \frac{5}{1} \times \frac{5-1}{2} \times \frac{5-2}{3} \times a^{5-3}b^3 + \frac{5}{1} \times \frac{5-1}{2} \times \frac{5-2}{3} \times \frac{5-3}{4} \times a^{5-4}b^4 + \frac{5}{1} \times \frac{5-1}{2} \times \frac{5-2}{3} \times \frac{5-3}{4} \times \frac{5-4}{5} \times a^{5-5}b^5 = a^5 + \frac{5}{1} \times a^4b + \frac{5}{1} \times \frac{4}{2} \times a^3b^2 + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times a^2b^3 + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times a^1b^4 + \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times \frac{1}{5} \times a^0b^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$

4. These examples are sufficient both to illustrate the meaning of the foregoing theorem, and to prove the truth of it in these few easy cases. For "that the theorem is true in these cases," will appear by raising the second, third, fourth, and fifth, powers of the binomial quantity $a+b$ in the common way by multiplication; which will produce the very same quantities for the said powers as have been just now obtained by means of the foregoing theorem. This may be done in the manner following.

$$\begin{array}{r} a+b \\ a+b \\ \hline aa+ab \\ +ab+bb \\ \hline aa+2ab+bb = \overline{a+b}^2 \\ a+b \\ \hline a^3+2aab+abb \\ +aab+2abb+b^3 \\ \hline a^3+3aab+3abb+b^3 = \overline{a+b}^3 \\ a+b \\ \hline a^4+3a^3b+3a^2b^2+ab^3 \\ +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline a^4+4a^3b+6a^2b^2+4ab^3+b^4 = \overline{a+b}^4 \\ a+b \\ \hline a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\ +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\ \hline a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 = \overline{a+b}^5 \end{array}$$

These values of the several powers $\overline{a+b}^2$, $\overline{a+b}^3$, $\overline{a+b}^4$, and $\overline{a+b}^5$, are the very same with those that have been obtained just above in Art. 3 by means of the foregoing theorem. And consequently the said theorem is true in these four instances.

Of the literal Parts of the Terms of the foregoing Products, and the Law of their Generation one from another.

5. The law by which the literal parts of the second and other following terms of the series that is equal to $\overline{a+b}^m$ are derived from the first term a^m , and

from each other, to wit, "that they are generated by the continual multiplication of the fraction $\frac{b}{a}$," will be sufficiently evident from the foregoing multiplications of $a + b$ and its powers by $a + b$. For it is evident that the indexes of the powers of a in those several products decrease continually by an unit, and that the indexes of the powers of b in the same products increase by an unit at the same time; so that the literal part of the second and every following term in each of the said products is generated from the term next before it by multiplying the said preceeding term into $\frac{b}{a}$. And it is easy to see that the same thing must take place in any higher powers of $a + b$ whatsoever, if the said multiplications by $a + b$ were to be continued till such higher powers were produced. This part, therefore, of the aforesaid binomial theorem stands in need of no further demonstration.

Of the numeral Co-efficients of the two first Terms of the said Products.

6. And with respect to the numeral co-efficients of the several terms of these products obtained by the foregoing multiplications by $a + b$, (which co-efficients are in the square 1, 2, and 1, and in the cube 1, 3, 3, 1, and in the fourth power 1, 4, 6, 4, 1, and in the fifth power 1, 5, 10, 10, 5, 1,) it is easy to see that the co-efficient of the first term of every new power of $a + b$ must always be 1; because it arises by multiplying the next preceeding power of a by a , or $1 \times a$, which cannot alter the co-efficient of the said next preceeding power, which at first was 1. And it is also manifest that the co-efficient of the second term of every new power of $a + b$ must always be the index of the said new power, or, in our present notation, must be m ; because it is produced by adding the product of the multiplication of the first term of the next preceeding power of $a + b$, (of which first term 1 is always the co-efficient,) by b to the product of the multiplication of the second term of the said next preceeding power of $a + b$ by a ; the effect of which addition is, to increase the co-efficient of the second term of every new power of $a + b$ by an unit. And this, it is easy to see, must be the case in any higher powers whatsoever of $a + b$, if we were to continue the said multiplications by $a + b$ till such higher powers were produced. We may, therefore, conclude that the co-efficients of the two first terms of the series that is equal to $(a + b)^m$ must always be 1 and m , and consequently that the two first terms of the said series must always be $1 \times a^m$ and $m \times a^{m-1} b$, or a^m and $\frac{m}{1} \times a^{m-1} b$.

7. It remains that we shew that the co-efficients of the third, fourth, fifth, sixth, and other following terms of the series that is equal to $(a + b)^m$ are generated from the co-efficient m , or $\frac{m}{1}$, of the second term by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$ &c; which is indeed by no means obvious.

Of the numeral Co-efficients of the third and fourth, and other following Terms of the said Products, and the Law of their Generation from the second Term, and from each other.

8. Now, in order to demonstrate the law of the generation of these co-efficients, it will be convenient to get rid of the powers of a and b in the terms of the

the series that is equal to $\overline{a+b}^m$, and to fix our attention only on the generation of the numeral co-efficients of the third, fourth, fifth, sixth, and other following terms of the said series. This may be done by supposing a and b to be, each of them, equal to 1, and consequently $a+b$ to be equal to $1+1$, and $\overline{a+b}^m$ to be equal to $\overline{1+1}^m$. For, as all the powers of both a and b will, on this supposition, be equal to 1, our theorem will then be reduced to this, to wit, that $\overline{1+1}^m$ will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c$, continued to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c \times \frac{m-(m-1)}{m}$, or to the term $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c \times \frac{1}{m}$, or to the term 1. For the last term of this series must always be 1; because the numerators of the several factors in it form a decreasing progression of numbers from m to 1, and the denominators of the same factors form an increasing progression of numbers from 1 to m , and consequently, the product of all the denominators is equal to the product of all the numerators, and therefore the product of all the said factors, or the said last term, must always be equal to 1; which we have seen to be the case in the last terms of the values of $\overline{a+b}^2$, $\overline{a+b}^3$, $\overline{a+b}^4$, $\overline{a+b}^5$, as derived from the series in the theorem in Art. 3, which last terms were $\frac{2}{1} \times \frac{1}{2}$, and $\frac{3}{1} \times \frac{2}{2} \times \frac{1}{3}$, and $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4}$, and $\frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{2}{4} \times \frac{1}{5}$, which are, each of them, equal to 1. We are therefore now to demonstrate that $\overline{1+1}^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-2}{1} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c + 1$.

Of Mr. James Bernouilli's Demonstration of the Law of the Generation of the said numeral Co-efficients.

9. Now the clearest and best demonstration that I have ever seen of this useful proposition is, that which is given us by the learned and sagacious *Mr. James Bernouilli* in the third chapter of the second book of his excellent Treatise on the Doctrine of Chances, intitled, "*De Arte Conjectandi*," which is written in Latin, and was published in a small quarto volume at *Basil*, or *Basle*, in Switzerland, in the year 1713, eight years after the author's death. This demonstration is founded on the doctrine of combinations and the properties of the figurate numbers, which are there shewn to involve in them the generation of these co-efficients. And the most important properties of the said numbers are in the same chapter set forth and demonstrated by that great author in a very perspicuous and masterly manner, though with rather too much conciseness to be easily understood by beginners in these studies. Those readers, therefore, who are desirous of seeing this theorem demonstrated from its natural and fundamental principles, and in the clearest and most satisfactory manner

manner that, as I believe, the nature of the thing will admit of, must be referred to the said third chapter of the second book of that learned treatise; which, together with the two preceeding chapters of the same book, I would advise them to peruse with the closest attention, and to make themselves thorough masters of their contents; to do which they will find, will require a considerable exertion of their diligence.

Another Demonstration will be here given of the said Law.

10. But, though that is the best and most satisfactory demonstration that has been given, and, probably that can be given, of this theorem, yet it may also be demonstrated in a shorter and easier manner, and with the same degree of certainty as by that method of Mr. James Bernouilli, though not with the same degree of originality and elegance. And such a demonstration I now propose to give of it in the course of the following pages.

11. The demonstration which is here intended to be given of this important theorem is founded on the observation "that the said theorem is found by trial to be true in some of the lowest powers of the binomial quantity $1 + 1$;" as has been seen above, in Art. 4, in the cases of the second, third, fourth, and fifth, powers of $a + b$. For, if this theorem is true when the index m is of any particular value, as, for example, when it is equal to 5, it may be shewn by abstract and general reasonings, derived from the nature of multiplication, that it must likewise be true when the index m is increased by an unit, or that, if n be taken $= m + 1$, the quantity $1 + 1^n$, or the n th power of the binomial quantity $1 + 1$, will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c$ continued to the term $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \&c \times \frac{1}{n}$, or 1. And this is what I shall now endeavour to demonstrate.

12. To facilitate the demonstration of this proposition it will be convenient to premise the following Lemma.

A L E M M A.

If the terms of the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c + 1$ (in which m represents any whole number whatsoever,) be set down twice together in two parallel lines, or rows, one under the other, but with the terms in the lower row advanced one step further to the right-hand than the terms in the upper row, so that the first term in the lower row shall stand under the second term of the upper row, and the second term in the lower row shall stand under the third term in the upper row, and the third, fourth, fifth, sixth, and other following terms in the lower row shall stand under the fourth, fifth, sixth, seventh, and other following terms in the upper row, respectively;

and both rows are continued to the same number of terms, namely, to the whole number of terms in the said series, or to $m + 1$ terms; and then the terms in the lower row, (which, it is evident, will consist of one factor less than the corresponding terms, or terms standing immediately above them in the upper row,) be reduced to the same denomination as the terms that stand immediately above them in the upper row, and, after being so reduced, are added to the said terms that stand immediately above them in the said upper row;—upon these suppositions the new series of terms arising from this addition of the said two rows of terms to each other, will be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m+1}{5} + \&c + 1$; in which series the last term is 1, as well as in the two serieses from the addition of which this series arises; and the numerators of the last factors in all the terms, except the last, are always equal to $m + 1$, instead of being equal to $m - 1, m - 2, m - 3, m - 4, \&c$, as in the two foregoing serieses; and the number of terms in the said new series is $m + 2$, instead of $m + 1$, which is the number of terms in each of the said foregoing serieses.

DEMONSTRATION.

13. This will appear by setting down the said series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$ twice over, in the manner that has been just described; which may be done as follows:

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$$

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \&c.$$

In these two rows of terms it is evident, in the first place, that the terms in the upper row, after the two first terms 1 and $\frac{m}{1}$, consist of two, three, and four, and more, factors, every new term having one more factor than the term next before it; and, 2dly, that the terms in the lower row that stand immediately under the third, fourth, fifth, and other following terms in the upper row, consist of one factor less than the corresponding terms, or terms immediately over them in the upper row; and, 3dly, that the terms in the lower row consist of the very same factors as the corresponding terms in the upper row, excepting that they want the last factors of the said terms in the upper row. And hence it follows, that, in order to reduce the terms in the lower row to the same denomination as the terms in the upper row, we must multiply them by factors that shall have the same denominators as the last, or additional factors in the upper row, and which must have their numerators equal to their denominators, so as to make each of them equal to 1, to the end that the magnitudes of the said lower terms may not be altered by the multiplication of them by the said

faid new factors. Thus, for example, the second term of the lower row, to wit, $\frac{m}{1}$, must be multiplied into the factor $\frac{2}{2}$, in order to bring it to the same denomination as the third term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, without altering its magnitude; and the third term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2}$, must be multiplied into the factor $\frac{3}{3}$, in order to bring it to the same denomination as the fourth term of the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, without altering its magnitude; and the fourth term in the lower row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, must be multiplied into the factor $\frac{4}{4}$, in order to bring it to the same denomination as the fifth term in the upper row, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, without altering its magnitude; and, for the like reason, the fifth, and sixth, and seventh, and other following terms in the lower row must be multiplied into the several factors $\frac{5}{5}$, and $\frac{6}{6}$, and $\frac{7}{7}$, &c, respectively; after which multiplications the two rows of terms that are to be added to each other, will be as follows, to wit,

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c \\ + 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c.$$

14. And, if these two rows of terms, (being now brought to the same denominations,) are added together in the manner above described; that is, every term in the lower row to the term that is immediately above it, the sum thence resulting will be the series

$$1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c,$$

in which the numerator of the last factor in every term is always $m+1$, instead of $m-1$, $m-2$, $m-3$, $m-4$, &c.

And “That this must be the case in all the following terms of the said new series as well as in the few terms of it that have been here set down,” will be evident from this consideration, to wit, That the denominator of the last factor of every term in the upper of the two rows of terms that are added together is always greater by an unit than the number which is subtracted from m in the numerator of the same factor. For from thence it follows that the denominator of the new multiplying fraction in the corresponding term of the lower row (which is always equal to the denominator of the said last factor in the upper row,) must always be greater by an unit than the number which is subtracted from m in the numerator of the last factor of the said upper term. And, therefore, the numerator of the said new multiplying fraction in the lower row (which is always equal to its denominator,) must also always be greater by an unit than the number which is subtracted from m in the numerator of the last factor of the said upper term; the consequence of which, in adding the
lower

lower term to the upper term, is to convert the numerator of the last factor in the upper term from $m - 1$, or $m - 2$, or $m - 3$, or the excess of m above some other number, into $m + 1$. Q. E. D.

15. And the number of terms in the new series, arising from the addition of the two former in the manner that has been described, will be greater by one than the number of the terms in either of the two added series: because the lower row of terms, consisting of the same number of terms as the upper row, and being placed one term further to the right-hand, must extend one term beyond it; and consequently, as the number of terms in each of the two rows of terms is $m + 1$, the number of terms in the new series, arising from the addition of the two rows together, must be $m + 2$. Q. E. D.

16. And, lastly, the last term of the said new series must be the same as the last term of the old series, or of the lower row of terms; because, as the lower row of terms extends one term beyond the upper row, the last term in the lower row will not have any term over it in the upper row to which it is to be added, and consequently will continue the same in the new series $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$ as in the old series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$. But we have seen above, in Art. 8, that the last term of the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$ is 1. Therefore the last term in the new series $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$ will also be 1. Q. E. D.

17. Coroll. 1. Now let the order of the numerators $m, m - 1, m - 2, m - 3, m - 4, \&c$, and $m + 1$, of the factors of the several terms of the last series $1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$, after the two first terms, be changed, by making $m + 1$ the numerator of the first factor of every term instead of being the numerator of the last factor. The said series will then be as follows, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c$. Now this change in the order of the numerators of the several factors of the terms will create no change in the values, or magnitudes, of the several terms themselves; because the products arising from the multiplication of the same numbers are always the same, in whatever order the numbers are multiplied. Therefore the foregoing series, after this change in the order of the numerators

of the several factors of its terms, will still be of the same magnitude as before, and consequently will be equal to the sum that arises from the addition of the aforesaid two rows of terms in the manner above described; that is, the series $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c + 1$ will be equal to the sum that arises from the addition of the aforesaid two rows of terms in the manner above described.

18. Coroll. 2. Now let n be $= m + 1$. Then will $n - 1$ be $= m$, and $n - 2$ will be $= m - 1$, and $n - 3$ will be $= m - 2$, and $n - 4$ will be $= m - 3$, and, in like manner, $n - 5$, $n - 6$, $n - 7$, &c, will be equal to $m - 4$, $m - 5$, $m - 6$, &c. respectively. And consequently the series obtained in the foregoing Corollary, to wit, $1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c + 1$, consisting of $m + 2$ terms, will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n + 1$ terms. Therefore the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n + 1$ terms, will be equal to the sum that arises by adding the two aforesaid rows of terms together in the manner above described.

The Demonstration of the principal Proposition.

19. These things being premised, the main proposition stated at the end of Art. 8, to wit, that, if m denote any whole number whatsoever, the quantity $\overline{1+1}^m$, or the m th power of the binomial quantity $1 + 1$, will be equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c$ continued to $m + 1$ terms, or to the term 1, may be demonstrated in the manner following.

20. The product that arises by multiplying the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$ into $1 + 1$ is the sum that arises by setting down the said series twice following in two parallel rows, one under the other, with the terms in the lower row advanced one term further to the right-hand than the terms in the upper row, in the manner above described, and then adding the terms in the lower row to the corresponding terms in the upper row. And the $\overline{m+1}^b$ power of $1 + 1$ is the product of the multiplication of the m th power of $1 + 1$ into $1 + 1$. Therefore, if in any particular value of m the m th power of $1 + 1$ is equal to the

the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$, consisting of $m+1$ terms, the $\overline{m+1}^{th}$ power of $1+1$ will be equal to the sum that arises by setting down the said series twice following in two parallel rows in the manner above described, and adding the said two rows of terms together. But, by the second Corollary of the foregoing Lemma, if n be $= m+1$, the sum arising from the addition of the said two rows of terms is the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$ consisting of $n+1$ terms. Therefore, if in any particular value of m the m th power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$, consisting of $m+1$ terms, it will follow that the $\overline{m+1}^{th}$, or n th, or next higher power, of $1+1$ will be equal to the series $1 + \frac{n}{1} \times \frac{n-1}{1} + \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n+1$ terms. But it has been shewn in Art. 3 and 4 that, when m is equal either to 2, or to 3, or to 4, or to 5, the m th power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$ consisting of $m+1$ terms. Therefore, if n be equal to $5+1$, or 6, the $\overline{5+1}^{th}$ power, or 6th power, or n th power, of $1+1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n+1$, or $6+1$, or 7, terms. And in the same manner it may be proved that, since, when m is $= 6$, the m th power of $1+1$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c + 1$, consisting of $m+1$, or $6+1$, or 7, terms, the $\overline{m+1}^{th}$, or $6+1^{th}$, or 7th, or (putting $n = m+1 = 6+1 = 7$) the n th, power of $1+1$ will be equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c + 1$, consisting of $n+1$, or $7+1$, or 8, terms. And so we may proceed from number to number *ad infinitum*. And consequently, whatever be the whole number denoted by m , it will always be true that $\overline{1+1}^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} + \&c + 1$, consisting of $m+1$ terms.

Q. E. D.

The foregoing Demonstration expressed in a more concise Manner.

21. The foregoing reasonings may be expressed in a more concise manner as follows. If n be $= m + 1$, and it be true in any particular value of m that $\overline{1 + 1}^m$ is $=$ the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$, it will also be true that $\overline{1 + 1}^n$ will be $=$ $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c$.

For $\overline{1 + 1}^n$ is $= \overline{1 + 1}^{m+1} = \overline{1 + 1}^m \times \overline{1 + 1} =$ the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$ multiplied into $1 + 1 =$

$$1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$$

$$+ 1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c.$$

$$= 1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$$

$$+ 1 + \frac{m}{1} \times \frac{2}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{3}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{4}{4} + \&c$$

$$= 1 + \frac{m+1}{1} + \frac{m}{1} \times \frac{m+1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m+1}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m+1}{4} + \&c$$

$$= 1 + \frac{m+1}{1} + \frac{m+1}{1} \times \frac{m}{2} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} + \frac{m+1}{1} \times \frac{m}{2} \times \frac{m-1}{3} \times \frac{m-2}{4} + \&c$$

$$= 1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c.$$

But it has been shewn in Art. 3 and 4 that, when m is equal either to 2, or to 3, or to 4, or to 5, $\overline{1 + 1}^m$ is equal to the series $1 + \frac{m}{1} + \frac{m}{1} \times \frac{m-1}{2} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} + \&c$. Therefore, if n be $= 5 + 1$,

or 6, $\overline{1 + 1}^n$, or $\overline{1 + 1}^6$, will be $=$ the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c$. And it may be shewn in like manner that, if n be put for 7, 8, 9, 10, &c *ad infinitum* successively,

$\overline{1 + 1}^n$ will in all these suppositions be always equal to the series $1 + \frac{n}{1} + \frac{n}{1} \times \frac{n-1}{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c$; and therefore the proposition is universally true, whatever be the whole number denoted by the letter n . Q. E. D.

22. This demonstration of the binomial theorem in the case of integral powers, is nearly the same with that given by Mr. John Stewart, of Aberdeen, in the 6th Section of his Commentary on Sir Isaac Newton's curious little Tract, intitled, *Analysis by Equations of an infinite number of Terms*. See his edition of

of Newton's *Treatise on the Quadrature of Curves*, and of the said Tract intitled *Analysis*, &c, with his learned Comments on both, in one volume, quarto, published at London, in the year 1745, page 471, Art. 155.

Of the Invention of the foregoing Theorem.

23. This famous theorem is usually ascribed to Sir Isaac Newton: and in the case of roots, or fractional powers, of a binomial quantity, it is generally agreed that he was the first inventor of it. But in the case of integral powers, (which has been the subject of this Discourse,) it seems to have been discovered many years before by Mr. Henry Briggs, the ingenious computer of the logarithms that are called by his name. For it has been lately observed by Dr. Hutton, (the learned Professor of Mathematicks at Woolwich Academy,) in his very curious historical Introduction to his new edition of Sherwin's Mathematical Tables, that there is a passage in the said Mr. Briggs's valuable *Treatise*, intitled, *Arithmetica Logarithmica*, (which was published in the year 1624,) in which Mr. Briggs shews how to derive the third, and fourth, and fifth, and other following terms of any integral power of a binomial quantity from the second term of it without raising all the intermediate, or lower powers of the binomial by continual multiplication; which is the operation of the binomial theorem. But Mr. Briggs has only described the method of doing this in words, and has not expressed it in Algebraick symbols, as Sir Isaac Newton has done, by assigning the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$ &c, as the factors, by the continual multiplication of which the co-efficients of the said terms may be produced. The merit, therefore, of being the first inventor, or publisher, of this useful discovery in the case of integral powers, must be allowed to Mr. Briggs, and only that of giving the invention a more convenient form, by expressing it by a short and suitable algebraick notation, must be ascribed to Sir Isaac Newton, together with that of extending it by his sagacious conjectures from the case of powers of which the indexes are positive whole numbers to the various other cases of it in which the indexes of the powers are either fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{17}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{17}$, and $\frac{17}{4}$, or, in general, $\frac{m}{n}$, or negative whole numbers, as -1 , -2 , -3 , -4 , -5 , &c, or, in general, $-m$, or negative fractions, as $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{17}$, $-\frac{2}{3}$, $-\frac{3}{2}$, $-\frac{4}{17}$, or, in general, $-\frac{m}{n}$: which extension of it is of wonderful utility in various branches of the higher parts of Algebra and mathematicks, as we have seen in the remarks that have been published in the former volume of these Tracts on the Logarithmick Serieses invented by Mr. Mercator and Dr. Wallis, and in the foregoing Tract concerning the resolution of the equation $1 + \frac{1}{10}^x = 10$, or the investigation of the proportion of the ratio of 10 to 1 to the lesser ratio of $1 + \frac{1}{10}$ to 1, or of 11 to 10.

24. Yet it may reasonably be conjectured that Sir Isaac Newton was likewise an inventor, as well as Mr. Briggs, of this useful theorem even in the case of integral powers, though not the first inventor of it. For it is well known that he

was

was not an extensive reader of mathematical works; and he appears to have applied himself principally in his younger years to the study of Des Cartes's Geometry, with Schooten's Commentary on it, and the other Tracts published by Schooten with it, and of Dr. Wallis's *Arithmetica Infinitorum*, and his other works on mathematical subjects then published. And therefore he may well be supposed not to have seen Mr. Briggs's *Arithmetica Logarithmica*, in which this method of deriving the co-efficients of the terms of the powers of a binomial quantity one from another is contained, at the time of his discovering this famous theorem himself, which was about the year 1665, or when he was only 23 years old. And, if he had seen that book, and observed this discovery to be contained in it, it is hardly to be conceived that, when he was speaking of this theorem, he would have omitted to make mention of this discovery, and to acknowledge that it contained the substance of the said binomial theorem in the case of integral powers, though not expressed in algebraick symbols. For these reasons I am inclined to think that Sir Isaac Newton had not seen Mr. Briggs's *Arithmetica Logarithmica*, when he invented the binomial theorem, and therefore, that he was truly *an inventor* of it even in the case of integral powers, though *not the first inventor*.

25. But it seems more surprising that Dr. Wallis, who was a much more copious reader of mathematical works than Sir Isaac Newton, and who actually had seen Mr. Briggs's *Arithmetica Logarithmica*, and makes mention of it in his Algebra, chapter xii, page 60, should not have attended to the contents of that ingenious treatise enough to have observed that it contained this most useful theorem. Yet this appears to have been the fact, from what the Doctor tells us in the 85th chapter of his Algebra, page 319, where, in speaking of Mr. Newton's method of generating, or deriving, the co-efficients of the third and fourth, and other following terms of the m th power of a binomial quantity from m , the co-efficient of the second term of it, by the continual multiplication of the fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, &c, he confesses that he had sought after this method of generating, or deriving, these co-efficients himself, but without success. His words are these, after speaking of some other excellent inventions in the mathematicks contained in a letter of Mr. Isaac Newton (at that time Professor of Mathematicks in the University of Cambridge, and who was afterwards better known by the title of Sir Isaac Newton,) to Mr. Oldenburgh, (the Secretary of the Royal Society at London,) dated October 24, 1676. "He [Mr. Newton] then observes (what I had formerly sought after, but unsuccessfully,) that the following numbers are, from the two first, to be found by continual multiplication of this series $1 \times \frac{m-0}{1} \times$ " $\frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times$ &c." These are the words of Dr. Wallis in his Algebra, page 319; from which, I think, we may conclude that, though he had seen Mr. Briggs's *Arithmetica Logarithmica*, he had not read it with sufficient attention to discover that this method of generating the co-efficients of the terms of the m th power of a binomial quantity, when m was any whole number whatsoever, was contained in it: though it seems indeed unaccountably strange that he should not have taken notice of it.

26. Dr. Wallis's Algebra was published in the year 1685. And that was the first time, after Sir Isaac Newton's discovery of it, that the binomial theorem (in Sir Isaac Newton's manner of expressing it) was published in print, and made known to the learned world in general, though Mr. Oldenburgh and Mr. Leibnitz, and probably Dr. Barrow, (who was Sir Isaac Newton's great friend and patron in his youth,) and some other learned mathematicians of that time, had seen it in that letter to Mr. Oldenburgh, of October 24, 1676, soon after the said letter was written. But, how Sir Isaac discovered this theorem, is not known; nor is any demonstration of it, even in this easiest case of it, (in which the index m of the power to which the binomial quantity is to be raised, is a whole number,) any where to be found in all his works.

Of the Powers of a Residual Quantity $a - b$, when their Indexes are whole Numbers.

27. We have hitherto been considering the integral powers of a *binomial* quantity $a + b$, or of the *sum* of two single quantities a and b ; and we have seen that, if the said binomial quantity $a + b$ be raised to any power of which a whole number denoted by m is the index, the quantity $\overline{a + b}^m$, or the said m th power of $a + b$, will be equal to the series $a^m + \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c + b^m$, or (if we put $A = 1$, $B = \frac{m}{1} A$, $C = \frac{m-1}{2} B$, $D = \frac{m-2}{3} C$, $E = \frac{m-3}{4} D$, $F = \frac{m-4}{5} E$, and $G, H, I, K, L, \&c, = \frac{m-5}{6} F, \frac{m-6}{7} G, \frac{m-7}{8} H, \frac{m-8}{9} I, \frac{m-9}{10} K, \&c$, respectively,) to the series $a^m + \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 + \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 + \frac{m-4}{5} E a^{m-5} b^5 + \&c + b^m$; in which all the terms after the first term a^m are marked with the sign $+$, or are added to the said first term. We will now proceed to consider the value of $\overline{a - b}^m$, or the m th power of the *residual* quantity $a - b$, or the *difference* of the two quantities a and b , upon a supposition that a is the greater of the two.

28. Now, if a be supposed to be greater than b , and m be any whole number whatsoever, the quantity $\overline{a - b}^m$, or the m th power of the residual quantity, or difference, $a - b$, will be equal to the series $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c$, or (if we put A , as before, $= 1$, and $B = \frac{m}{1} A$, and $C = \frac{m-1}{2} B$, and $D = \frac{m-2}{3} C$, and $E = \frac{m-3}{4} D$, and $F = \frac{m-4}{5} E$, and $G, H, I, K, L, \&c, = \frac{m-5}{6} F, \frac{m-6}{7} G, \frac{m-7}{8} H, \frac{m-8}{9} I, \frac{m-9}{10} K, \&c$, respectively,) to the series $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 + \frac{m-5}{6} F a^{m-6} b^6 - \frac{m-6}{7} G a^{m-7} b^7 + \frac{m-7}{8} H a^{m-8} b^8 - \frac{m-8}{9} I a^{m-9} b^9 + \frac{m-9}{10} K a^{m-10} b^{10} - \&c + b^m$; in which all the terms after the first term a^m are marked with the sign $-$, or are subtracted from the said first term.

G, $\frac{m-7}{8}$ H, $\frac{m-8}{9}$ I, $\frac{m-9}{10}$ K, &c, respectively,) to the series $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 + \&c$, which consists of exactly the same terms as the series that is equal to $\overline{a+b}^m$, or the same power of the binomial quantity $a+b$, but with the sign — prefixed to the second, and fourth, and sixth, and every following even term in the series, which denotes that the said terms are not to be added to the first term a^m , and to the third, and fifth, and other following odd terms, (as they were in the former series, which was equal to $\overline{a+b}^m$;) but to be subtracted from them.

29. That this must be so, will be evident from considering the manner in which the several powers of the residual quantity $a-b$ are generated from each other by the continual multiplication of $a-b$, of which we will now exhibit a specimen with respect to a few of its lowest powers. The second, third, fourth, and fifth powers of $a-b$ are derived from $a-b$ itself by the following multiplications.

$$\begin{array}{r}
 a-b \\
 a-b \\
 \hline
 aa-ab \\
 -ab+bb \\
 \hline
 aa-2ab+bb = \overline{a-b}^2 \\
 \quad a-b \\
 \hline
 a^3-2a^2b+ab^2 \\
 -a^2b+2ab^2-b^3 \\
 \hline
 a^3-3a^2b+3ab^2-b^3 = \overline{a-b}^3 \\
 \quad a-b \\
 \hline
 a^4-3a^3b+3a^2b^2-ab^3 \\
 -a^3b+3a^2b^2-3ab^3+b^4 \\
 \hline
 a^4-4a^3b+6a^2b^2-4ab^3+b^4 = \overline{a-b}^4 \\
 \quad a-b \\
 \hline
 a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\
 -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\
 \hline
 a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 = \overline{a-b}^5.
 \end{array}$$

30. From these operations it is evident that, wherever the odd powers of b occur in the said powers of $a-b$, the terms are marked with the sign —, and that, wherever the even powers of b occur in the said powers of $a-b$, the terms are marked with the sign +. And the same thing, it is evident, must happen in all higher powers of $a-b$ whatsoever, as well in those that have been here set down, because b is marked with the sign — in the two original factors $a-b$ and $a-b$; whence it follows, from the nature of algebraick multiplication, that, whenever b is multiplied into itself an even number of times, the product will be marked with the sign +; and, whenever it is multiplied into itself an odd number of times, the product will be marked with the sign —. And it is further evident, from the foregoing multiplications, that the odd powers of b occur in the second, and fourth, and sixth, terms of the fore-

foregoing products, and that the even powers of b occur in the third and fifth terms of them. And it is easy to see that the odd powers of b will occur in like manner in the eighth, and tenth, and twelfth, and other following even terms of all higher powers of $a - b$ whatsoever, and that the even powers of b will occur in like manner in the seventh, and ninth, and eleventh, and other following odd terms of the said higher powers of $a - b$. And it is also evident, from the foregoing multiplications, that the terms themselves of which the several powers of $a - b$ will be composed, are exactly the same with the terms of which the same powers of $a + b$ are composed. And hence it follows that the series which is equal to $a - b^m$ will be the same with the series which is equal to $a + b^m$, when the sign $-$ has been prefixed to the second, and fourth, and sixth, and other following even terms of it, instead of the sign $+$, or that $a - b^m$, or the m th power of the residual quantity $a - b$, will be equal to the series $a^m - \frac{m}{1} a^{m-1} b + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3} b^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} a^{m-5} b^5 + \&c$, or $a^m - \frac{m}{1} A a^{m-1} b + \frac{m-1}{2} B a^{m-2} b^2 - \frac{m-2}{3} C a^{m-3} b^3 + \frac{m-3}{4} D a^{m-4} b^4 - \frac{m-4}{5} E a^{m-5} b^5 + \&c$. Q. E. D.

A

D E M O N S T R A T I O N

O F

S I R I S A A C N E W T O N ' S

B I N O M I A L T H E O R E M,

In the Cases of Roots and the Powers of Roots, as well as in the Case of
Integral Powers; published by Mr. John Landen in the Year 1758.

IN the year 1758 the very learned Mr. John Landen, of Walton, near Peterborough, in Northamptonshire, published a mathematical Tract in quarto, containing 43 pages, intitled, *A Discourse concerning the Residual Analysis: a new Branch of the Algebraick Art, of very extensive Use both in Pure Mathematicks and Natural Philosophy*. In this Discourse he has given us a demonstration of the famous binomial theorem of Sir Isaac Newton, that extends to the cases of roots and the powers of roots of a binomial quantity, as well as to the case of its integral powers; and this, without having recourse to the doctrine of fluxions, which had usually been employed for this purpose by the writers that had gone before him. The proposition he demonstrates is as follows, to wit, "That, if m and n be any two whole numbers whatsoever, the quantity $\sqrt[n]{1+x}$, or that power of the binomial quantity $1+x$ which is denoted by the fractional index $\frac{m}{n}$, will be equal to the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 + \&c$, or (putting $A = 1$, and $B = \frac{m}{n}$, and $C = \frac{m-n}{2n}B$, and $D = \frac{m-2n}{3n}C$, and $E = \frac{m-3n}{4n}D$, and $F = \frac{m-4n}{5n}E$, and $G, H, I, K, L, \&c$, for $\frac{m-5n}{6n}F, \frac{m-6n}{7n}G, \frac{m-7n}{8n}H, \frac{m-8n}{9n}I, \frac{m-9n}{10n}K, \&c$, respectively,)

spectively,) to the series $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4$
 $+ \frac{m-4n}{5n} E x^5 + \frac{m-5n}{6n} F x^6 + \frac{m-6n}{7n} G x^7 + \&c.$ And it is the first propo-
 sition contained in the said Discourse. I shall therefore here give the reader both
 the proposition itself and the demonstration of it, in the learned author's own
 words, together with the reflections with which he introduces it; and, as the
 demonstration is in itself rather subtle and difficult, and is also expressed in a very
 concise manner, I shall afterwards subjoin such an explanation of it as will, I
 hope, enable the reader to comprehend it easily. The author's words are as
 follows:

“ Before Sir Isaac Newton invented the *Method of Fluxions*, mathematicians
 “ had made considerable improvements in the algebraic art, and had devised
 “ several curious rules for resolving certain problems relating to the greatest
 “ and least ordinates, points of inflexion, tangents, curvature, and quadrature
 “ of curve lines, and the cubature of solids, &c. Those rules, however, were
 “ esteemed of little value, upon the appearance of the fluxionary method;
 “ which, being of far more extensive use, was, by the mathematical world,
 “ received with great admiration, and studied with great eagerness. Highly
 “ indeed has that method been extolled by many writers; yea, a certain
 “ gentleman has gone so far as to say, The method of fluxions is capable of
 “ resolving such difficulties as raise the wonder and surprise of all mankind, and
 “ which would in vain be attempted by any other method whatsoever. So that
 “ it is justly esteemed the greatest work of genius, and the noblest thought that
 “ ever entered the human mind. *Pref. to EMERS. Flux.*

“ Yet, notwithstanding the method of fluxions is so greatly applauded, I
 “ am induced to think it is not the most natural method of resolving many
 “ problems to which it is usually applied.—The operations therein being chiefly
 “ performed with algebraic quantities, it is, in fact, a branch of the algebraic
 “ art, or an improvement thereof, made by the help of some peculiar principles
 “ borrowed from the doctrine of motion: which principles, I must confess, to
 “ me seem not so properly applicable to algebra as those on which that art was,
 “ before, very naturally founded. We may indeed very naturally conceive a
 “ line to be generated by motion; but there are quantities of various kinds,
 “ which we cannot conceive to be so generated. It is only in a figurative sense
 “ that an algebraic quantity can be said to increase or decrease with some
 “ velocity or degree of swiftness; and, by the fluxion of a quantity of that
 “ kind, we must, I presume, to have a clear idea of its meaning, understand
 “ the velocity of a point supposed to describe a line denoting such quantity.
 “ Fluxions therefore are not immediately applicable to algebraic quantities;
 “ but in fluxionary computations made by means of such quantities, we, to
 “ proceed with perspicuity, must have recourse to the supposition of lines being
 “ put to denote those quantities, and the generation of those lines by motion.
 “ It therefore, to me, seems more proper, in the investigation of propositions by
 “ algebra, to proceed upon the *anciently received* principles of that art, than to
 “ introduce therein, without any necessity, the new fluxionary principles, de-
 “ rived from a consideration of motion; and the rather, as the introduction of
 “ those

“ those new principles is not attended with any peculiar advantage.—That the
 “ borrowing principles from the doctrine of motion, with a view to improve
 “ the analytic art, was done, not only without any necessity, but even without
 “ any peculiar advantage, will appear by shewing that, whatever can be done
 “ by the method of computation, which is founded on those borrowed princi-
 “ ples, may be done as well by another method founded entirely on the
 “ *anciently-received* principles of algebra: And that I shall endeavour to shew,
 “ as soon as I have leisure, in the treatise I lately proposed to publish by sub-
 “ scription.—In the mean time, this essay is intended to give the inquisitive
 “ reader some notion of the new method of computation, which is the subject
 “ of that treatise.—Which method I call the *Residual Analysis*; because, in all
 “ the enquiries wherein it is made use of, the conclusions are obtained by means
 “ of residual quantities.

“ In the application of the *Residual Analysis*, a geometrical or physical pro-
 “ blem is naturally reduced to another purely algebraical; and the solution is
 “ then readily obtained, without any supposition of motion, and without con-
 “ sidering quantities as composed of infinitely small particles.

“ It is by means of the following theorem, *viz.*

$$\frac{x^{\frac{m}{n}} - v^{\frac{m}{n}}}{x - v} = x^{\frac{m}{n} - 1} \times \frac{1 + \frac{v}{x} + \left(\frac{v}{x}\right)^2 + \left(\frac{v}{x}\right)^3 + \dots + \left(\frac{v}{x}\right)^{m/n}}{1 + \left(\frac{v}{x}\right)^{\frac{m}{n}} + \left(\frac{v}{x}\right)^{\frac{2m}{n}} + \left(\frac{v}{x}\right)^{\frac{3m}{n}} + \dots + \left(\frac{v}{x}\right)^m} \quad (m)$$

(where m and n are any integers,)

“ that we are enabled to perform all the principal operations in our said Ana-
 “ lysis; and I am not a little surprised, that a theorem so obvious, and of such
 “ vast use, should so long escape the notice of algebraists!

“ I have no objection against the truth of the method of fluxions, being
 “ fully satisfied, that even a problem purely algebraical may be very clearly
 “ resolved by that method, by bringing into consideration lines, and their
 “ generation by motion. But I must own, I am inclined to think such a
 “ problem would be more naturally resolved by pure algebra, without any such
 “ consideration of lines and motion.—Suppose it required to investigate the

“ binomial theorem; *i. e.* to expand $1 + x^{\frac{m}{n}}$ into a series of terms of x , and
 “ known co-efficients.

“ To do this by the method of fluxions, we first assume $1 + x^{\frac{m}{n}} = 1 + ax +$
 “ $bx^2 + cx^3 + dx^4$ &c. We, to proceed with perspicuity, are next to conceive
 “ x , and each term of that assumed equation, to be denoted by some line, and
 “ that line to be described by the motion of a point: then, supposing x to be
 “ the velocity of the point describing the line x , and taking, by the rules
 “ taught by those who have treated of the said method, the several contemporary
 “ velocities of the other describing points, or the fluxions of the several terms

“ in the said equation*, we get $\frac{m}{n} \times \frac{1}{1 + x^{\frac{m}{n}}} \times \dot{x} = a\dot{x} + 2bxx\dot{x} + 3cx^2\dot{x} +$
 “ $4dx^3\dot{x}$ &c. because, when the space described by a motion is always equal to
 “ the

* See Mac Laurin's Fluxions, vol. 2, page 585, Art. 714.

$$= \frac{1 + \frac{1+y}{1+x} + \frac{1+y}{1+x}}{1 + \frac{1+y}{1+x} + \frac{1+y}{1+x}} \quad (m)$$

This is Mr. Landen's demonstration of the foregoing celebrated theorem, expressed in his own words in the above-mentioned Tract, intitled, *A Discourse concerning the Residual Analysis*, &c, which was published in the year 1758. Six years afterwards, to wit, in the year 1764, Mr. Landen published a second Tract on the same subject, containing 218 pages in quarto, intitled, *The Residual Analysis*; Book I. To which the former Tract, intitled, *A Discourse concerning the Residual Analysis*, &c, had been only an introduction. In this second Tract, pages 5 and 6, Mr. Landen informs us that one of the theorems which chiefly enabled him to perform a certain division of one residual quantity by another, which is of great use in his Residual Analysis, was the following, to

$$\text{wit, } \frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} = \frac{v^{m-1} + v^{m-2}w + v^{m-3}w^2 + v^{m-4}w^3}{v^{\frac{m-m}{r}} + v^{\frac{m-2m}{r}}w^{\frac{m}{r}} + v^{\frac{m-3m}{r}}w^{\frac{2m}{r}} + v^{\frac{m-4m}{r}}w^{\frac{3m}{r}}} \quad (m)$$

$$= v^{\frac{m}{r}-1} \times \frac{1 + \frac{w}{v} + \frac{w^2}{v^2} + \frac{w^3}{v^3}}{1 + \frac{w}{v} + \frac{w^2}{v^2} + \frac{w^3}{v^3}} \quad (r)$$

m and r being positive integers. And in the bottom of the said page 6 he subjoins the following note, containing an investigation of the said theorem.

NOTE.—This theorem may be investigated as follows:

It is well known, that

$$\frac{v^m - w^m}{v - w} \text{ is } = v^{m-1} + v^{m-2}w + v^{m-3}w^2 \quad (m);$$

and that

$$\frac{a^r - b^r}{a - b} \text{ is } = a^{r-1} + a^{r-2}b + a^{r-3}b^2 \quad (r),$$

m and r being positive integers.

In the second equation write $v^{\frac{m}{r}}$ and $w^{\frac{m}{r}}$ instead of a and b respectively; and you will have

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v^{\frac{m}{r}} - w^{\frac{m}{r}}} = \frac{v^{\frac{m-m}{r}} + v^{\frac{m-2m}{r}}w^{\frac{m}{r}} + v^{\frac{m-3m}{r}}w^{\frac{2m}{r}}}{v^{\frac{m-m}{r}} + v^{\frac{m-2m}{r}}w^{\frac{m}{r}} + v^{\frac{m-3m}{r}}w^{\frac{2m}{r}}} \quad (r).$$

Then, from the first and third equations, it will appear by division, that

$$\frac{v^{\frac{m}{r}} - w^{\frac{m}{r}}}{v - w} \text{ is } = \frac{v^{m-1} + v^{m-2}w}{v^{\frac{m-m}{r}} + v^{\frac{m-2m}{r}}w^{\frac{m}{r}}}, \quad (m)$$

which being so obvious, it is matter of surprize to me, that algebraists have not before observed it, and shewn its singular use in analytics.

These are all the passages that I have found in these two Tracts of Mr. Landen concerning the foregoing demonstration, of which I will now proceed to give as full and clear an explanation as I can.

AN EXPLANATION

OF THE FOREGOING DEMONSTRATION

OF THE BINOMIAL THEOREM,

In the Case of the fractional Index $\frac{m}{n}$, invented by Mr. John Landen.

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CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

ARTICLE I.

THE proposition demonstrated by Mr. John Landen is as follows: If m and n be any whole numbers whatsoever, and x be any quantity not greater than 1, the quantity $(1+x)^{\frac{m}{n}}$, or that power of the binomial quantity $1+x$ which has for its index the fraction $\frac{m}{n}$, will be equal to the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n} x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5 + \&c$, or (putting $A = 1$, and $B = \frac{m}{n} A$, and $C = \frac{m-n}{2n} B$, and $D = \frac{m-2n}{3n} C$, and $E = \frac{m-3n}{4n} D$, and $F = \frac{m-4n}{5n} E$, and $G, H, I, K, L, \&c = \frac{m-5n}{6n} F$, and $\frac{m-6n}{7n} G$, and $\frac{m-7n}{8n} H$, and $\frac{m-8n}{9n} I$, and $\frac{m-9n}{10n} K$, &c, respectively,) to the series $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 + \frac{m-5n}{6n} F x^6 + \frac{m-6n}{7n} G x^7$,

+ $\frac{m-7n}{8n} H x^8 + \frac{m-8n}{9n} I x^9 + \frac{m-9n}{10n} K x^{10} + \&c.$ His demonstration of this proposition is deduced from the two following Lemmas.

L E M M A I.

2. If a and b be any two quantities whatsoever, of which a is the greater, and m be any whole number whatsoever, the difference of the m th powers of a and b , to wit, $a^m - b^m$, will be capable of being exactly divided by $a - b$, or the difference of the said quantities themselves, so as to leave no remainder; and the quotient that arises by such division, or the value of the fraction $\frac{a^m - b^m}{a - b}$, will be a series of terms in continued geometrical proportion, consisting of m terms, of which a^{m-1} will be the first term, and b^{m-1} will be the last term, and the common ratio of the terms will be that of a to b .

E X A M P L E S.

Thus, for example, if we divide $a^2 - b^2$ by $a - b$, the quotient will be $a + b$. The division is performed as follows:

$$\begin{array}{r} a-b \overline{) a^2 - b^2} \quad (a+b \\ \underline{a^2 - ab} \\ + ab - b^2 \\ \underline{ + ab - b^2} \\ \end{array}$$

And, if we divide $a^3 - b^3$ by $a - b$, the quotient will be $a^2 + ab + b^2$. The division is performed as follows:

$$\begin{array}{r} a-b \overline{) a^3 - b^3} \quad (a^2 + ab + b^2 \\ \underline{a^3 - a^2 b} \\ + a^2 b - ab^2 \\ \underline{ + a^2 b - ab^2} \\ + ab^2 - b^3 \\ \underline{ + ab^2 - b^3} \\ \end{array}$$

And, if we divide $a^4 - b^4$ by $a - b$, the quotient will be $a^3 + a^2 b + ab^2 + b^3$. The division is performed as follows:

$$\begin{array}{r} a-b \overline{) a^4 - b^4} \quad (a^3 + a^2 b + ab^2 + b^3 \\ \underline{a^4 - a^3 b} \\ + a^3 b - a^2 b^2 \\ \underline{ + a^3 b - a^2 b^2} \\ + a^2 b^2 - ab^3 \\ \underline{ + a^2 b^2 - ab^3} \\ + ab^3 - b^4 \\ \underline{ + ab^3 - b^4} \\ \end{array}$$

Now it is evident that the terms of this quotient, $a^{m-1} + a^{m-2}b + a^{m-3}b^2 + a^{m-4}b^3 + a^{m-5}b^4 + a^{m-6}b^5 + \&c$, decrease in the continual proportion of a to b . For the index of the power of b in every new term of it is greater by an unit than the index of the power of b in the term next before it; and the index of the power of a in every new term of it is less by an unit than the index of the power of a in the term next before it; so that every new term is equal to the term next before it multiplied into the fraction $\frac{b}{a}$. And, further, the first term of this series, or quotient, is a^{m-1} ; and its last term will be b^{m-1} for the following reason. The index of the power of b in every term is less by an unit than the number subtracted from m in the index of the power of a in the same term. Thus, for example, in the fourth term, $a^{m-4}b^3$, the index of b is 3, which is less by an unit than the number 4, which is subtracted from m in the index $m-4$ of the power of a . It follows therefore that, by continuing the division, and obtaining more and more terms of the quotient, we shall come at last (whatever be the magnitude of the number m ,) to the term $a^{m-m} \times b^{m-1}$, or $a^0 \times b^{m-1}$, or $1 \times b^{m-1}$, or b^{m-1} ; and this term, being multiplied into the divisor $a - b$, will produce the terms $ab^{m-1} - b^m$, which being subtracted from the last dividend, (which will also be $ab^{m-1} - b^m$,) will leave no remainder. Therefore b^{m-1} will be the last term of the quotient, as is asserted in the proposition. And, lastly, since b^{m-1} is the last term of the said quotient, and the terms of the said quotient, beginning with the second term $a^{m-2}b$, involve all the powers of b , from b , or b^1 , to b^{m-1} , in their regular order, it follows, that there will be as many terms in the said quotient that will involve some power of b , as there are units in the index $m-1$, that is, $m-1$ such terms; and consequently, the whole quotient, including the first term a^{m-1} , (which does not involve b in it,) will consist of $m-1$ terms and one term, that is, of m terms. It appears therefore, in the first place, that the terms of this quotient will constitute a decreasing geometrical progression, in which the common ratio of the terms will be that of a to b ; and secondly, that the first term of the said progression will be a^{m-1} , and that the last term of it will be b^{m-1} ; and, thirdly, that the number of the terms of the said progression will be m ; which are the several points which were to be demonstrated.

L E M M A II.

5. If m and n be any whole numbers whatsoever, and a and b be any two quantities of which a is the greater, and p and q be any two quantities of which p is

p is the greater, the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$, or the quotient of the division of

$p^{\frac{m}{n}} - q^{\frac{m}{n}}$ by $p - q$, will be equal to the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + \&c, \text{ continued to } m \text{ terms,}}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}}q^{\frac{m}{n}} + p^{\frac{m-3m}{n}}q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}}q^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

DEMONSTRATION.

By the foregoing Lemma we shall have $\frac{a^n - b^n}{a - b} =$ the series $a^{n-1} + a^{n-2}$

$b + a^{n-3}b^2 + a^{n-4}b^3 + a^{n-5}b^4 + \&c, \text{ continued to } n \text{ terms.}$ Therefore,

if we substitute $p^{\frac{m}{n}}$ instead of a , and $q^{\frac{m}{n}}$ instead of b in this equation, we shall

have $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^{\frac{m}{n}} - q^{\frac{m}{n}}} =$ the series $p^{\frac{m}{n}n-1} + p^{\frac{m}{n}n-2} \times q^{\frac{m}{n}} + p^{\frac{m}{n}n-3} \times q^{\frac{m}{n}2} +$

$p^{\frac{m}{n}n-4} \times q^{\frac{m}{n}3} + p^{\frac{m}{n}n-5} \times q^{\frac{m}{n}4} + \&c, \text{ continued to } n \text{ terms; that is, } \frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^{\frac{m}{n}} - q^{\frac{m}{n}}}$

will be $=$ the series $p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} \times q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} \times q^{\frac{3m}{n}}$

$+ p^{\frac{m-5m}{n}} \times q^{\frac{4m}{n}} + \&c, \text{ continued to } n \text{ terms.}$ Therefore, the reciprocal of

this fraction, or the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^{\frac{m}{n}} - q^{\frac{m}{n}}}$; will be equal to the quotient that arises

by dividing 1 by the series $p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} \times q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} \times$

$q^{\frac{3m}{n}} + p^{\frac{m-5m}{n}} \times q^{\frac{4m}{n}} + \&c, \text{ continued to } n \text{ terms; that is, the fraction } \frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^{\frac{m}{n}} - q^{\frac{m}{n}}}$

will be equal to the fraction

$$\frac{1}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} \times q^{\frac{2m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

Further, by the foregoing Lemma, the fraction $\frac{p^m - q^m}{p - q}$ is equal to the

series $p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms.}$

m terms. Therefore, if we multiply the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^m - q^m}$ by the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$, and the fraction

I

$$\frac{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} \times q^{\frac{2m}{n}} + \&c, \text{ continued to } n \text{ terms, by the series } p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms, the products thence arising will be equal. But the product of the multiplication of the fraction } \frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p^m - q^m} \text{ by the fraction } \frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q} \text{ is the fraction}$$

$\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$; and the product of the multiplication of the fraction

I

$$\frac{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} \times q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} \times q^{\frac{2m}{n}} + \&c, \text{ continued to } n \text{ terms, by the series } p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms, is the fraction}$$

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms,}}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

Therefore the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$ will be equal to the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms,}}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} q^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

Q. E. D.

6. Coroll. The said fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$ is also equal to $p^{\frac{m}{n}-1} \times$ the fraction

$$\frac{1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \left(\frac{q}{p}\right)^4 + \left(\frac{q}{p}\right)^5 + \&c, \text{ continued to } m \text{ terms,}}{1 + \left(\frac{q}{p}\right)^{\frac{m}{n}} + \left(\frac{q}{p}\right)^{\frac{2m}{n}} + \left(\frac{q}{p}\right)^{\frac{3m}{n}} + \left(\frac{q}{p}\right)^{\frac{4m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

For the series $p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c$, continued to m terms, is $= p^{m-1} \times$ the series $1 + p^{-1}q + p^{-2}q^2 + p^{-3}q^3 + p^{-4}q^4 + \&c$, continued to m terms,

$$= p^{m-1} \times \text{the series } 1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} + \&c, \text{ continued to } m \text{ terms,}$$

$$= p^{m-1} \times \text{the series } 1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \left(\frac{q}{p}\right)^4 + \&c, \text{ continued to } m \text{ terms.}$$

And the series $p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} q^{\frac{3m}{n}} + \&c$, continued to n terms, is =

$$p^{\frac{m-m}{n}} \times \text{the series } 1 + p^{\frac{-m}{n}} q^{\frac{m}{n}} + p^{\frac{-2m}{n}} q^{\frac{2m}{n}} + p^{\frac{-3m}{n}} q^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms, =}$$

$$p^{\frac{m-m}{n}} \times \text{the series } 1 + \frac{q^{\frac{m}{n}}}{p^{\frac{m}{n}}} + \frac{q^{\frac{2m}{n}}}{p^{\frac{2m}{n}}} + \frac{q^{\frac{3m}{n}}}{p^{\frac{3m}{n}}} + \&c, \text{ continued to } n \text{ terms, =}$$

$$p^{\frac{m-m}{n}} \times \text{the series } 1 + \left(\frac{q}{p}\right)^{\frac{m}{n}} + \left(\frac{q}{p}\right)^{\frac{2m}{n}} + \left(\frac{q}{p}\right)^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms.}$$

Therefore the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms,}}{p^{\frac{m-m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} q^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms,}}$$

$$\text{will be = the fraction } \frac{p^{m-1}}{p^{\frac{m-m}{n}}} \times \text{the fraction}$$

$$\frac{1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \left(\frac{q}{p}\right)^3 + \left(\frac{q}{p}\right)^4 + \&c, \text{ continued to } m \text{ terms,}}{1 + \left(\frac{q}{p}\right)^{\frac{m}{n}} + \left(\frac{q}{p}\right)^{\frac{2m}{n}} + \left(\frac{q}{p}\right)^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms.}}$$

$$\text{But the fraction } \frac{p^{m-1}}{p^{\frac{m-m}{n}}} \text{ is } = p^{m-1} \times \frac{1}{p^{\frac{m-m}{n}}} = p^{m-1} \times p^{\frac{-m+m}{n}} = p^{\frac{m-1-m+m}{n}}$$

$$= p^{\frac{m}{n}-1}.$$

$$\text{Therefore, if we substitute } p^{\frac{m}{n}-1} \text{ instead of } \frac{p^{m-1}}{p^{\frac{m-m}{n}}} \text{ in the last equation, we}$$

$$\text{shall have the fraction}$$

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms,}}{p^{\frac{m}{n}} + p^{\frac{m-2m}{n}} q^{\frac{m}{n}} + p^{\frac{m-3m}{n}} q^{\frac{2m}{n}} + p^{\frac{m-4m}{n}} q^{\frac{3m}{n}} + \&c, \text{ continued to } n \text{ terms,}}$$

$$= p^{\frac{m}{n}-1} \times \text{the fraction.}$$

$$1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} + \&c, \text{ continued to } m \text{ terms,}$$

$$1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} + \&c, \text{ continued to } n \text{ terms.}$$

Therefore, the fraction $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$, (which is shewn in the foregoing Lemma to be equal to the fraction

$$\frac{p^{m-1} + p^{m-2}q + p^{m-3}q^2 + p^{m-4}q^3 + p^{m-5}q^4 + \&c, \text{ continued to } m \text{ terms,}}{p^{\frac{m}{n}-1} + p^{\frac{m}{n}-2}q + p^{\frac{m}{n}-3}q^2 + p^{\frac{m}{n}-4}q^3 + p^{\frac{m}{n}-5}q^4 + \&c, \text{ continued to } n \text{ terms,}}$$

$$p^{\frac{m}{n}-1} + p^{\frac{m}{n}-2}q + p^{\frac{m}{n}-3}q^2 + p^{\frac{m}{n}-4}q^3 + p^{\frac{m}{n}-5}q^4 + \&c, \text{ continued to } n \text{ terms,}$$

will be equal to $p^{\frac{m}{n}-1} \times$ the fraction

$$1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} + \&c, \text{ continued to } m \text{ terms,}$$

$$1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} + \&c, \text{ continued to } n \text{ terms.}$$

Q. E. D.

These things being premised, Mr. Landen's Demonstration of the Binomial Theorem will be as follows :

7. In the first place, Mr. Landen supposes that the quantity $\sqrt[n]{1+x}$, or the $\frac{m}{n}$ -th power of the binomial quantity $1+x$, will be equal to a certain series of quantities, of which the first term will be 1, and the following terms will involve the several powers of x in their natural order, to wit, $x, x^2, x^3, x^4, x^5, \&c$, multiplied by certain fixt numbers, as co-efficients, which may be denoted by the letters B, C, D, E, F, G, H, &c, or that $\sqrt[n]{1+x}$ is = the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, in which the co-efficients B, C, D, E, F, &c, are hitherto unknown, and are now to be investigated. And this he supposes to be true, of whatever magnitude, not greater than 1, the quantity x be taken.

8. Now, if $\sqrt[n]{1+x}$ is always equal to the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, whatever be the magnitude of x , (so long as it is not greater than 1,) it follows that, if we suppose x to decrease from its first magnitude (which we will suppose to be denoted by x), to a magnitude somewhat less than

x , which we will call y , we shall have $\sqrt[n]{1+y}$ equal to a series containing the same powers of y , combined with the same co-efficients B, C, D, E, F, &c, respectively, as there were powers of x in the former series which was equal to

$\sqrt[n]{1+x}$; or $\sqrt[n]{1+y}$ will be equal to the series $1 + By + Cy^2 + Dy^3 + Ey^4 + Fy^5 + \&c$. Therefore, if we subtract this latter equation from the former, we

we shall have $\frac{1+x^m}{1+x} - \frac{1+y^m}{1+y} =$ the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$ — the series $1 + By + Cy^2 + Dy^3 + Ey^4 + Fy^5 + \&c =$
 $B \times \overline{x-y} + C \times \overline{x^2-y^2} + D \times \overline{x^3-y^3} + E \times \overline{x^4-y^4} + F \times \overline{x^5-y^5} + \&c$; and consequently, if we divide both sides by $x-y$, we shall have

$$\frac{\frac{1+x^m}{1+x} - \frac{1+y^m}{1+y}}{x-y} = B + C \times \frac{x^2-y^2}{x-y} + D \times \frac{x^3-y^3}{x-y} + E \times \frac{x^4-y^4}{x-y} + F \times \frac{x^5-y^5}{x-y} + \&c =$$

(by the first Lemma) $B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3} + F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4} + \&c.$

But $x-y$ is $= 1+x-\overline{1+y}$. Therefore $\frac{\frac{1+x^m}{1+x} - \frac{1+y^m}{1+y}}{1+x-\overline{1+y}}$ will be $=$
 $\frac{B + C \times \overline{x+y} + D \times \overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3} + F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4} + \&c.}{1+x-\overline{1+y}}$. Now it has been shewn in the Corollary

to Lemma 2, that $\frac{p^{\frac{m}{n}} - q^{\frac{m}{n}}}{p - q}$ is $= p^{\frac{m}{n}-1} \times$ the fraction

$$1 + \frac{q}{p} + \frac{q^2}{p^2} + \frac{q^3}{p^3} + \frac{q^4}{p^4} + \&c, \text{ continued to } m \text{ terms,}$$

$$1 + \frac{q^{\frac{m}{n}}}{p^{\frac{m}{n}}} + \frac{q^{\frac{2m}{n}}}{p^{\frac{2m}{n}}} + \frac{q^{\frac{3m}{n}}}{p^{\frac{3m}{n}}} + \frac{q^{\frac{4m}{n}}}{p^{\frac{4m}{n}}} + \&c, \text{ continued to } n \text{ terms.}$$

Therefore, if we substitute $1+x$ for p , and $1+y$ for q , it will follow that

$$\frac{\frac{1+x^m}{1+x} - \frac{1+y^m}{1+y}}{1+x-\overline{1+y}} \text{ will be } = \frac{1+x^{\frac{m}{n}}}{1+x} \times \text{the fraction}$$

$$1 + \frac{1+y}{1+x} + \frac{1+y^2}{1+x^2} + \frac{1+y^3}{1+x^3} + \&c, \text{ continued to } m \text{ terms,}$$

$$1 + \frac{1+y^{\frac{m}{n}}}{1+x^{\frac{m}{n}}} + \frac{1+y^{\frac{2m}{n}}}{1+x^{\frac{2m}{n}}} + \frac{1+y^{\frac{3m}{n}}}{1+x^{\frac{3m}{n}}} + \&c, \text{ continued to } n \text{ terms.}$$

But it has been shewn that $\frac{\frac{1+x^m}{1+x} - \frac{1+y^m}{1+y}}{1+x-\overline{1+y}}$ is $= B + C \times \overline{x+y} + D \times$
 $\overline{x^2+xy+y^2} + E \times \overline{x^3+x^2y+xy^2+y^3} + F \times \overline{x^4+x^3y+x^2y^2+xy^3+y^4}$
 $+ \&c.$ Therefore $\frac{1+x^{\frac{m}{n}}}{1+x} \times$ the fraction

$$1 + \frac{1+y}{1+x} + \frac{1+y^2}{1+x^2} + \frac{1+y^3}{1+x^3} + \&c, \text{ continued to } m \text{ terms,}$$

$$1 + \frac{1+y^{\frac{m}{n}}}{1+x^{\frac{m}{n}}} + \frac{1+y^{\frac{2m}{n}}}{1+x^{\frac{2m}{n}}} + \frac{1+y^{\frac{3m}{n}}}{1+x^{\frac{3m}{n}}} + \&c, \text{ continued to } n \text{ terms,}$$

will

will be $= B + C \times \overline{x + y} + D \times \overline{x^2 + xy + y^2} + E \times \overline{x^3 + x^2y + xy^2 + y^3} + F \times \overline{x^4 + x^3y + x^2y^2 + xy^3 + y^4} + \&c.$

9. Now let y (which was supposed to be less than x ,) be supposed to increase gradually till it becomes equal to x .

Then, since the last equation is always true, however little y may differ from x , it will also be true when y becomes absolutely equal to x ; as might easily be proved by a demonstration *ex absurdo*, if the matter were not too evident to make such a demonstration necessary. But, when y is equal to x , we shall have $x + y = 2x$, and $x^2 + xy + y^2 = 3x^2$, and $x^3 + x^2y + xy^2 + y^3 = 4x^3$, and $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 5x^4$; and, in like manner, in the following terms of the series $B + C \times \overline{x + y} + D \times \overline{x^2 + xy + y^2} + E \times \overline{x^3 + x^2y + xy^2 + y^3} + F \times \overline{x^4 + x^3y + x^2y^2 + xy^3 + y^4} + \&c$ we shall have $6x^5$, $7x^6$, $8x^7$, $\&c.$

And $\frac{1+y}{1+x}$ will, in this case, be $= \frac{1+x}{1+x} = 1$, and consequently, $\left(\frac{1+y}{1+x}\right)^2$ and $\left(\frac{1+y}{1+x}\right)^3$ and $\left(\frac{1+y}{1+x}\right)^4$, and all the following integral powers of $\frac{1+y}{1+x}$, and likewise

$\left(\frac{1+y}{1+x}\right)^{\frac{m}{n}}$ and $\left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}}$ and $\left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}}$, and all the following fractional powers of

$\frac{1+y}{1+x}$, will become equal to 1. Therefore the last equation, to wit, $\frac{1+y}{1+x}^{\frac{m}{n}-1} \times$ the fraction

$1 + \frac{1+y}{1+x} + \left(\frac{1+y}{1+x}\right)^2 + \left(\frac{1+y}{1+x}\right)^3 + \&c$, continued to m terms,

$1 + \left(\frac{1+y}{1+x}\right)^{\frac{m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}} + \&c$, continued to n terms,

$= B + C \times \overline{x + y} + D \times \overline{x^2 + xy + y^2} + E \times \overline{x^3 + x^2y + xy^2 + y^3} + F \times \overline{x^4 + x^3y + x^2y^2 + xy^3 + y^4} + \&c$, will, in this case, be transformed

into the equation $\frac{1+y}{1+x}^{\frac{m}{n}-1} \times$ the fraction

$1 + 1 + 1 + 1 + \&c$, continued to m terms,

$1 + 1 + 1 + 1 + \&c$, continued to n terms,

$+ E \times 4x^3 + F \times 5x^4 + \&c$, or $\frac{1+y}{1+x}^{\frac{m}{n}-1} \times \frac{m}{n} = B + C \times 2x + D \times$

$3x^2 + E \times 4x^3 + F \times 5x^4 + \&c$, or $\frac{m}{n} \times \frac{1+y}{1+x}^{\frac{m}{n}-1} = B + C \times 2x +$

$D \times 3x^2 + E \times 4x^3 + F \times 5x^4 + \&c.$ Therefore, (multiplying both sides

of this last equation by $1 + x$,) we shall have $\frac{m}{n} \times \frac{1+y}{1+x}^{\frac{m}{n}} =$

$B + C \times 2x + D \times 3x^2 + E \times 4x^3 + F \times 5x^4 + \&c$

$+ B \times x + C \times 2x^2 + D \times 3x^3 + E \times 4x^4 + \&c$

$= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c$

$+ Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c.$

But $1 + x^{\frac{m}{n}}$ is = the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$,
 and consequently $\frac{m}{n} \times 1 + x^{\frac{m}{n}}$ is = $\frac{m}{n} \times$ the series $1 + Bx + Cx^2 + Dx^3 +$
 $Ex^4 + Fx^5 + \&c$ = the series $\frac{m}{n} + \frac{m}{n} Bx + \frac{m}{n} Cx^2 + \frac{m}{n} Dx^3 + \frac{m}{n} Ex^4 +$
 $\frac{m}{n} Fx^5 + \&c$. Therefore the series $\frac{m}{n} + \frac{m}{n} Bx + \frac{m}{n} Cx^2 + \frac{m}{n} Dx^3 + \frac{m}{n}$
 $Ex^4 + \frac{m}{n} Fx^5 + \&c$ will be =

$$\begin{aligned} & B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ & + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c. \end{aligned}$$

10. By the help of this equation $\frac{m}{n} + \frac{m}{n} Bx + \frac{m}{n} Cx^2 + \frac{m}{n} Dx^3 + \frac{m}{n}$
 $Ex^4 + \frac{m}{n} Fx^5 + \&c$ =

$$\begin{aligned} & B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ & + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c \end{aligned}$$

we may determine the values of B, C, D, E, F, &c, or the co-efficients of the
 powers of x in the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$

(which is = $1 + x^{\frac{m}{n}}$), by proceeding in the following manner:

11. This equation is always true, how small soever we suppose the magnitude
 of x to be: and therefore, it will also be true when x is = 0. But, when x
 is = 0, all the terms on both sides of the equation $\frac{m}{n} + \frac{m}{n} Bx + \frac{m}{n} Cx^2 + \frac{m}{n}$

$$Dx^3 + \frac{m}{n} Ex^4 + \frac{m}{n} Fx^5 + \&c =$$

$$\begin{aligned} & B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ & + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c \end{aligned}$$

that involve any power of x , will become equal to 0 likewise, that is, all the
 terms, except $\frac{m}{n}$ on the left side of the equation, and B on the right side of it,

will become equal to 0; and consequently, $\frac{m}{n}$ and B will be the only remain-

ing terms of the equation. Therefore B will be = $\frac{m}{n}$; that is, the co-efficient
 of x in Bx , the second term of the assumed series $1 + Bx + Cx^2 + Dx^3 +$

$Ex^4 + Fx^5 + \&c$, which is = $1 + x^{\frac{m}{n}}$, will be = $\frac{m}{n}$.

Q. E. I.

12. To find the value of C, the co-efficient of x^2 in the third term Cx^2 of
 the said assumed series, we must proceed as follows:

Since $\frac{m}{n} + \frac{m}{n} Bx + \frac{m}{n} Cx^2 + \frac{m}{n} Dx^3 + \frac{m}{n} Ex^4 + \&c$, is =

$$\begin{aligned} & B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ & + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c, \end{aligned}$$

and B has been shewn to be equal to $\frac{m}{n}$, it follows that, if we subtract $\frac{m}{n}$ and B

on

on the opposite sides of the equation, the remaining quantities will be equal to each other; that is, the series $\frac{m}{n} Bx + \frac{m}{n} Cx^2 + \frac{m}{n} Dx^3 + \frac{m}{n} Ex^4 + \&c$ will be equal to the series

$$2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 + \&c \\ + Bx + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c.$$

Therefore (dividing all the terms by x), we shall have $\frac{m}{n} B + \frac{m}{n} Cx + \frac{m}{n} Dx^2 + \frac{m}{n} Ex^3 + \&c =$

$$2C + 3Dx + 4Ex^2 + 5Fx^3 + \&c \\ + B + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c,$$

and (supposing x to become $= 0$) $\frac{m}{n} B = 2C + B$. Therefore $2C$ will be $=$

$$\frac{m}{n} B - B = \frac{m}{n} - 1 \times B = \frac{m-n}{n} \times B; \text{ and } C \text{ will be } = \frac{m-n}{2n} B = \frac{m-n}{2n} \times$$

$\frac{m}{n}$, or $\frac{m}{n} \times \frac{m-n}{2n}$; that is, the co-efficient of x^2 in the third term Cx^2 of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, (which is $=$

$$1 + x^{\frac{m}{n}}), \text{ will be } \frac{m-n}{2n} \times B, \text{ or } \frac{m}{n} \times \frac{m-n}{2n}.$$

Q. E. I.

13. To find the value of D , the co-efficient of x^3 in the fourth term of the said assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, we must proceed as follows:

It has been seen in the last article that $\frac{m}{n} B + \frac{m}{n} Cx + \frac{m}{n} Dx^2 + \frac{m}{n} Ex^3 + \&c$ is $= \left\{ \begin{array}{l} 2C + 3Dx + 4Ex^2 + 5Fx^3 + \&c \\ + B + 2Cx^2 + 3Dx^3 + 4Ex^4 + \&c \end{array} \right\}$, and likewise that $\frac{m}{n} B$ is $= 2C + B$. Therefore, if we subtract this last equation from the former, we shall have $\frac{m}{n} Cx + \frac{m}{n} Dx^2 + \frac{m}{n} Ex^3 + \&c =$

$$\left\{ \begin{array}{l} 3Dx + 4Ex^2 + 5Fx^3 + \&c \\ + 2Cx + 3Dx^2 + 4Ex^3 + \&c \end{array} \right\}, \text{ and consequently (dividing all the terms by } x), \frac{m}{n} C + \frac{m}{n} Dx + \frac{m}{n} Ex^2 + \&c =$$

$$\left\{ \begin{array}{l} 3D + 4Ex + 5Fx^2 + \&c \\ + 2C + 3Dx + 4Ex^2 + \&c \end{array} \right\}, \text{ and (supposing } x \text{ to become } = 0), \frac{m}{n} C$$

$$= 3D + 2C. \text{ Therefore } 3D \text{ will be } = \frac{m}{n} C - 2C = \frac{m-2n}{n} \times C = \frac{m-2n}{n} \times C,$$

$$\text{and } D \text{ will be } = \frac{m-2n}{3n} \times C, \text{ or } \frac{m-2n}{3n} + \frac{m-n}{2n} \times \frac{m}{n}, \text{ or } \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n};$$

$$\text{that is, the co-efficient of } x^3 \text{ in the fourth term of the series } 1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c \text{ (which is } = 1 + x^{\frac{m}{n}}), \text{ will be } = \frac{m-2n}{3n} \times C,$$

$$\text{or } \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}.$$

Q. E. I.

14. To find the value of E, the co-efficient of x^4 in the fifth term of the series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$ (which is $= \frac{m}{1+x^n}$), we must proceed as follows:

It has been seen in the last article that $\frac{m}{n} C + \frac{m}{n} Dx + \frac{m}{n} Ex^2 + \&c$ is $= \frac{3D + 4Ex + 5Fx^2 + \&c}{+ 2C + 3Dx + 4Ex^2 + \&c}$, and likewise that $\frac{m}{n} C$ is $= 3D + 2C$. Therefore, if we subtract this last equation from the former, we shall have $\frac{m}{n} Dx + \frac{m}{n} Ex^2 + \&c = \left\{ \frac{4Ex + 5Fx^2 + \&c}{+ 3Dx + 4Ex^2 + \&c} \right\}$, and consequently (dividing all the terms by x), $\frac{m}{n} D + \frac{m}{n} Ex + \&c = \left\{ \frac{4E + 5Fx + \&c}{+ 3D + 4Ex + \&c} \right\}$, and (supposing x to become $= 0$), $\frac{m}{n} D = 4E + 3D$. Therefore $4E$ will be $= \frac{m}{n} D - 3D = \frac{m-3n}{n} \times D = \frac{m-3n}{n} \times D$; and consequently E will be $= \frac{m-3n}{4n} \times D$; that is, the co-efficient of x^4 in the fifth term of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, (which is $= \frac{m}{1+x^n}$), will be $\frac{m-3n}{4n} \times D$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}$.
Q. E. I.

15. Lastly, to find the value of F, the co-efficient of x^5 in the sixth term of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$, we must proceed as follows:

It has been seen in the last article that $\frac{m}{n} D + \frac{m}{n} Ex + \&c$ is $= \frac{4E + 5Fx + \&c}{+ 3D + 4Ex + \&c}$, and likewise that $\frac{m}{n} D$ is $= 4E + 3D$. Therefore, if we subtract this last equation from the former, we shall have $\frac{m}{n} Ex + \frac{m}{n} Fx^2 + \frac{m}{n} Gx^3 + \&c = \left\{ \frac{5Fx + 6Gx^2 + 7Hx^3 + \&c}{+ 4Ex + 5Fx^2 + 6Gx^3 + \&c} \right\}$, and consequently (dividing all the terms by x), $\frac{m}{n} E + \frac{m}{n} Fx + \frac{m}{n} Gx^2 + \&c = \left\{ \frac{5F + 6Gx + 7Hx^2 + \&c}{+ 4E + 5Fx + 6Gx^2 + \&c} \right\}$, and (supposing x to become $= 0$), $\frac{m}{n} E = 5F + 4E$. Therefore $5F$ will be $= \frac{m}{n} E - 4E = \frac{m-4n}{n} \times E = \frac{m-4n}{n} \times E$, and consequently F will be $= \frac{m-4n}{5n} \times E$; that is, the co-efficient of x^5 in the sixth term of the assumed series $1 + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$ (which is $= \frac{m}{1+x^n}$), will be $\frac{m-4n}{5n} \times E$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}$.
Q. E. I.

16. And,

16. And, in like manner, we shall have for the determination of the values of the following co-efficients G, H, I, K, L, &c, the following equations, to wit,

$$\frac{m}{n} F = 6 G + 5 F,$$

$$\frac{m}{n} G = 7 H + 6 G,$$

$$\frac{m}{n} H = 8 I + 7 H,$$

$$\frac{m}{n} I = 9 K + 8 I,$$

$$\text{and } \frac{m}{n} K = 10 L + 9 K,$$

&c;

And consequently

$$6 G = \frac{m}{n} F - 5 F = \frac{m-5n}{n} \times F = \frac{m-5n}{n} \times F,$$

$$\text{and } 7 H = \frac{m}{n} G - 6 G = \frac{m-6n}{n} \times G = \frac{m-6n}{n} \times G,$$

$$\text{and } 8 I = \frac{m}{n} H - 7 H = \frac{m-7n}{n} \times H = \frac{m-7n}{n} \times H,$$

$$\text{and } 9 K = \frac{m}{n} I - 8 I = \frac{m-8n}{n} \times I = \frac{m-8n}{n} \times I,$$

$$\text{and } 10 L = \frac{m}{n} K - 9 K = \frac{m-9n}{n} \times K = \frac{m-9n}{n} \times K,$$

&c; and consequently

$$G = \frac{m-5n}{6n} \times F,$$

$$\text{and } H = \frac{m-6n}{7n} \times G,$$

$$\text{and } I = \frac{m-7n}{8n} \times H,$$

$$\text{and } K = \frac{m-8n}{9n} \times I,$$

$$\text{and } L = \frac{m-9n}{10n} \times K.$$

17. And in the same manner we may determine as many more of these co-efficients as we think proper, the law of their generation, or continuation, from each other being manifest from the manner in which the final equation $\frac{m}{n} +$

$$\frac{m}{n} B x + \frac{m}{n} C x^2 + \frac{m}{n} D x^3 + \frac{m}{n} E x^4 + \frac{m}{n} F x^5 + \&c =$$

$\left. \begin{array}{l} B + 2 C x + 3 D x^2 + 4 E x^3 + 5 F x^4 + \&c \\ + B x + 2 C x^2 + 3 D x^3 + 4 E x^4 + \&c \end{array} \right\}$ in Art. 9, (by means of which they are investigated,) was obtained.

18. We may therefore conclude that, if m and n be any whole numbers

whatsoever, $1 + x$ will be equal to the series $1 + \frac{m}{n} x + \frac{m}{n} \times \frac{m-n}{2n} x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times$

$\frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5 + \&c$, or (putting $A = 1$, $B = \frac{m}{n} A$, $C = \frac{m-2n}{2n}$, B , $D = \frac{m-2n}{3n} C$, $E = \frac{m-3n}{4n} D$, $F = \frac{m-4n}{5n} E$, and $G, H, I, K, L, \&c$, for the co-efficients of the following powers of x , to wit, $x^6, x^7, x^8, x^9, x^{10}, \&c$, derived from the former co-efficients by the same law of generation or continuation,) to the series $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 + \frac{m-2n}{3n} C x^3 + \frac{m-3n}{4n} D x^4 + \frac{m-4n}{5n} E x^5 + \frac{m-5n}{6n} F x^6 + \frac{m-6n}{7n} G x^7 + \frac{m-7n}{8n} H x^8 + \frac{m-8n}{9n} I x^9 + \frac{m-9n}{10n} K x^{10} + \&c$. Q. E. D.

19. Coroll. 1. If m is less than n , (as, for example, if m is 6 and n is 17,) $m - n$ will be a negative quantity, and therefore the third term of the series, to wit, $\frac{m}{n} \times \frac{m-n}{2n} x^2$, will be a negative quantity likewise, and will be equal to $-\frac{m}{n} \times \frac{n-m}{2n} x^2$, and consequently must be subtracted from the two first terms 1 and $\frac{m}{n} x$, instead of being added to them. And, in like manner, $\frac{m-2n}{3n}$ and $\frac{m-3n}{4n}$ and $\frac{m-4n}{5n}$, and all the following factors in the several co-efficients of the powers of x in the following terms of the series, will also be negative quantities, on account of the excess of $2n, 3n, 4n, \&c$, above m : and consequently, those terms in which these negative factors occur an odd number of times, will be negative, and must be subtracted from 1 and $\frac{m}{n} x$, the two first terms of the series; and those terms in which the said negative factors occur an even number of times, will be positive, and must be added to the said two first terms; that is, the third, fifth, seventh, and other following odd terms of the series will be negative, and must be subtracted from 1 and $\frac{m}{n} x$; and the fourth, sixth, eighth, and other following even terms of the series will be positive, and must be added to the said two first terms. Therefore, when m is less than n , the foregoing series, when its terms are set down correctly, with the proper signs prefixed to them, will be as follows, to wit, $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n} x^5 - \&c$, or (if A be put $= 1$, $B = \frac{m}{n}$, $C = \frac{n-m}{2n} B$, $D = \frac{2n-m}{3n} C$, $E = \frac{3n-m}{4n} D$, $F = \frac{4n-m}{5n} E$, and $G, H, I, K, L, \&c$, to $\frac{5n-m}{6n} F$, $\frac{6n-m}{7n} G$, $\frac{7n-m}{8n} H$, $\frac{8n-m}{9n} I$, $\frac{9n-m}{10n} K$, $\&c$, respectively, $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 - \frac{5n-m}{6n} F x^6 + \frac{6n-m}{7n} G x^7 - \frac{7n-m}{8n} H x^8 + \frac{8n-m}{9n} I x^9 - \frac{9n-m}{10n} K x^{10} + \&c$.

20. Coroll. 2. If m is an exact aliquot part of n , and is contained in it p times, or, if n is $= pm$, p being put for some whole number, the foregoing series will exhibit the p th root of the binomial quantity $1 + x$.

For we shall then have $\frac{m}{n} = \frac{1}{p}$, and $\frac{n-m}{2n} = \frac{p-m}{2pm} = \frac{p-1}{2p}$, and $\frac{2n-m}{3n} = \frac{2pm-m}{3pm} = \frac{2p-1}{3p}$, and $\frac{3n-m}{4n} = \frac{3pm-m}{4pm} = \frac{3p-1}{4p}$, and $\frac{4n-m}{5n} = \frac{4pm-m}{5pm} = \frac{4p-1}{5p}$, and, in like manner, $\frac{5n-m}{6n}$, and $\frac{6n-m}{7n}$, and $\frac{7n-m}{8n}$, and $\frac{8n-m}{9n}$, and $\frac{9n-m}{10n}$, &c, equal to $\frac{5p-1}{6p}$, $\frac{6p-1}{7p}$, $\frac{7p-1}{8p}$, $\frac{8p-1}{9p}$, and $\frac{9p-1}{10p}$, &c, respectively; and

consequently $\sqrt[n]{1+x}$ will be $= \sqrt[p]{1+x}$, and the series $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n} x^5 - \&c$ will be $= 1 + \frac{1}{p}x - \frac{1}{p} \times \frac{p-1}{2p} x^2 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} x^3 - \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} x^4 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} \times \frac{4p-1}{5p} x^5 - \&c$. Therefore $\sqrt[p]{1+x}$, or the p th root of the binomial quantity $1+x$, will be $=$ the series $1 + \frac{x}{p} - \frac{1}{p} \times \frac{p-1}{2p} x^2 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} x^3 - \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} x^4 + \frac{1}{p} \times \frac{p-1}{2p} \times \frac{2p-1}{3p} \times \frac{3p-1}{4p} \times \frac{4p-1}{5p} x^5 - \&c$, or $1 + \frac{1}{p} Ax - \frac{p-1}{2p} Bx^2 + \frac{2p-1}{3p} Cx^3 - \frac{3p-1}{4p} Dx^4 + \frac{4p-1}{5p} Ex^5 - \&c$. Q. E. I.

An Example to the foregoing Corollary.

21. Let it be required to find by means of the last-mentioned series the cube root of the binomial quantity $1+x$, or the value of $\sqrt[3]{1+x}$.

Here p is $= 3$. We shall therefore have

$$B = \frac{1}{p} \times A = \frac{1}{3} A,$$

$$\text{and } C = \frac{p-1}{2p} B = \frac{3-1}{2 \times 3} B = \frac{2}{6} B,$$

$$\text{and } D = \frac{2p-1}{3p} C = \frac{6-1}{9} C = \frac{5}{9} C,$$

$$\text{and } E = \frac{3p-1}{4p} D = \frac{9-1}{12} D = \frac{8}{12} D,$$

$$\text{and } F = \frac{4p-1}{5p} E = \frac{12-1}{15} E = \frac{11}{15} E,$$

$$\text{and } G = \frac{5p-1}{6p} F = \frac{15-1}{18} F = \frac{14}{18} F,$$

$$\text{and } H = \frac{6p-1}{7p} G = \frac{18-1}{21} G = \frac{17}{21} G,$$

$$\text{and } I = \frac{7p-1}{8p} H = \frac{21-1}{24} H = \frac{20}{24} H,$$

$$\text{and } K = \frac{8p-1}{9p} I = \frac{24-1}{27} I = \frac{23}{27} I,$$

$$\text{and } L = \frac{9p-1}{10p} K = \frac{27-1}{30} K = \frac{26}{30} K,$$

$$\text{and } M = \frac{10p-1}{11p} L = \frac{30-1}{33} L = \frac{29}{33} L,$$

$$\text{and } N = \frac{11p-1}{12p} M = \frac{33-1}{36} M = \frac{32}{36} M,$$

$$\text{and } O = \frac{12p-1}{13p} N = \frac{36-1}{39} N = \frac{35}{39} N,$$

$$\text{and } P = \frac{13p-1}{14p} O = \frac{39-1}{42} O = \frac{38}{42} O,$$

$$\text{and } Q = \frac{14p-1}{15p} P = \frac{42-1}{45} P = \frac{41}{45} P,$$

$$\text{and } R = \frac{15p-1}{16p} Q = \frac{45-1}{48} Q = \frac{44}{48} Q,$$

$$\text{and } S = \frac{16p-1}{17p} R = \frac{48-1}{51} R = \frac{47}{51} R,$$

$$\text{and } T = \frac{17p-1}{18p} S = \frac{51-1}{54} S = \frac{50}{54} S.$$

Therefore the series $1 + \frac{1}{p} Ax - \frac{p-1}{2p} Bx^2 + \frac{2p-1}{3p} Cx^3 - \frac{3p-1}{4p} Dx^4 + \frac{4p-1}{5p} Ex^5 - \frac{5p-1}{6p} Fx^6 + \&c$ will be $= 1 + \frac{1}{3} Ax - \frac{2}{6} Bx^2 + \frac{5}{9} Cx^3 - \frac{8}{12} Dx^4 + \frac{11}{15} Ex^5 - \frac{14}{18} Fx^6 + \frac{17}{21} Gx^7 - \frac{20}{24} Hx^8 + \frac{23}{27} Ix^9 - \frac{26}{30} Kx^{10} + \frac{29}{33} Lx^{11} - \frac{32}{36} Mx^{12} + \frac{35}{39} Nx^{13} - \frac{38}{42} Ox^{14} + \frac{41}{45} Px^{15} - \frac{44}{48} Qx^{16} + \frac{47}{51} Rx^{17} - \frac{50}{54} Sx^{18} + \&c = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19683} - \frac{935x^8}{59049} + \frac{21505x^9}{1,594,323} - \frac{55,913x^{10}}{4,782,969} + \frac{147,407x^{11}}{14,348,907} - \frac{1,179,256x^{12}}{129,140,163} + \frac{41,273,960x^{13}}{5,036,466,357} - \frac{112,029,320x^{14}}{15,109,399,071} + \frac{918,640,424x^{15}}{135,984,591,639} - \frac{2,526,261,166x^{16}}{407,953,774,917} + \frac{118,734,274,802x^{17}}{20,805,642,520,767} - \frac{2,968,356,870,050x^{18}}{561,752,348,060,709} + \&c$, which therefore is $= \sqrt[3]{1+x}$, or the cube root of the binomial quantity $1+x$. Q. E. I.

22. Coroll. 3. The foregoing proposition demonstrated by Mr. Landen, to wit, that, if m and n be any whole numbers whatsoever, the quantity $\sqrt[n]{1+x}$, or the $\frac{m}{n}$ -th power of the binomial quantity $1+x$, is equal to the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n} x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5 + \&c$, includes within it the case of the integral powers of $1+x$ which was the subject of the preceeding discourse. For we need only suppose m to be greater than n , and to contain it p times,

p times, or to be equal to pn , and we shall have $\overline{1+x}^{\frac{m}{n}} = \overline{1+x}^p$. Therefore in this case the foregoing series will be $= \overline{1+x}^p$. But, when $\frac{m}{n}$ is $= p$, and m is $= pn$, we shall have

$$\begin{aligned}
 \frac{m-n}{2n} &= \frac{pn-n}{2n} = \frac{p-1}{2}, \\
 \text{and } \frac{m-2n}{3n} &= \frac{pn-2n}{3n} = \frac{p-2}{3}, \\
 \text{and } \frac{m-3n}{4n} &= \frac{pn-3n}{4n} = \frac{p-3}{4}, \\
 \text{and } \frac{m-4n}{5n} &= \frac{pn-4n}{5n} = \frac{p-4}{5},
 \end{aligned}$$

and, in like manner, $\frac{m-5n}{6n}$, $\frac{m-6n}{7n}$, $\frac{m-7n}{8n}$, $\frac{m-8n}{9n}$, and $\frac{m-9n}{10n}$, &c. $=$ to $\frac{pn-5n}{6n}$, $\frac{pn-6n}{7n}$, $\frac{pn-7n}{8n}$, $\frac{pn-8n}{9n}$, and $\frac{pn-9n}{10n}$, &c, respectively, or to $\frac{p-5}{6}$, $\frac{p-6}{7}$, $\frac{p-7}{8}$, $\frac{p-8}{9}$, and $\frac{p-9}{10}$, &c, respectively; and consequently the series $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n} x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} x^4 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5 + \&c$, will be $=$ the series $1 + px + p \times \frac{p-1}{2} x^2 + p \times \frac{p-1}{2} \times \frac{p-2}{3} x^3 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} x^4 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5} x^5 + \&c$. Therefore $\overline{1+x}^p$ will be $=$ the series $1 + px + p \times \frac{p-1}{2} x^2 + p \times \frac{p-1}{2} \times \frac{p-2}{3} x^3 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} x^4 + p \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5} x^5 + \&c$; which is the binomial theorem in the case of integral powers, which was demonstrated in the foregoing discourse.

A
D I S C O U R S E

C O N C E R N I N G T H E

B I N O M I A L T H E O R E M,

In the Case of fractional Powers, or Powers of which the Indexes are Fractions.

B Y *FRANCIS MASERES*, E s q. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

A R T I C L E I.

THE Proposition known by the name of the *Binomial Theorem*, to wit,
“ that $1 + x^m$, or the m th power of the binomial quantity $1 + x$, is equal
to the series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times$
 $\frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c,$ ”
is found to be true, not only in the case of the integral powers of $1 + x$, or
when the index m of the power to which the said quantity is to be raised, is a
whole number, but also in the case of its roots, or when the index m is a
fraction, of which 1 is the numerator, and any whole number whatsoever is the
denominator, such as the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ &c; and likewise in
the case of the powers of any roots of the said binomial quantity $1 + x$, or the
roots of any powers of it, (which comes to the same thing,) or when m is
equal to a fraction of which any whole number whatsoever is the numerator,
and any other whole number whatsoever is the denominator, such as the fractions
 $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$, &c, or $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}$, &c. The truth of this theo-
rem in the former case, or when the index m is a whole number, has been
already shewn in the last tract but one in this collection, from page 153 to page
169;

169; and therefore will here be taken for granted, and made use of as a step in the reasonings by which we shall endeavour in the present Discourse to establish the truth of it in the other cases. But, before we attempt to demonstrate the said theorem in those other cases, it will be proper to set it forth in the expressions and forms in which it will appear in those cases, upon a supposition that the said theorem extends to the said cases, or to set down those expressions of the first theorem above-mentioned, which will arise from a supposition that it continues to be true when the index of the power to which the binomial quantity $1 + x$ is to be raised, is a fraction, as well as when it is a whole number. And, in doing this, it will be convenient to divide the subject into three cases, according as the index of the power, to which $1 + x$ is to be raised, shall be a fraction of which 1 is the numerator, and any whole number whatsoever is the denominator, or a fraction of which any whole number whatsoever is the numerator, and any *greater* whole number is the denominator, or a fraction of which any whole number whatsoever is the numerator, and any *lesser* whole number is the denominator. The expressions that will be derived from the original series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$, according to these different suppositions, will be as follows:

Of the Binomial Theorem in the Case of Roots, as derived from the same Theorem in the Case of Integral Powers.

2. When the index m of the power to which the binomial quantity $1 + x$ is to be raised, is equal to a fraction of which 1 is the numerator and a whole number is the denominator, let n be put for the said denominator, or let m be supposed to be $= \frac{1}{n}$. Then, by substituting $\frac{1}{n}$ instead of m in the terms of the

foregoing theorem, $\overline{1 + x}^m = 1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$, we shall (if the said theorem extends to the case of roots,) have

$\overline{1 + x}^{\frac{1}{n}} =$ to the series

$$\begin{aligned} &1 + \frac{\frac{1}{n}}{1}x + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2}x^2 + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3}x^3 \\ &+ \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3} \times \frac{\frac{1}{n}-3}{4}x^4 \\ &+ \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3} \times \frac{\frac{1}{n}-3}{4} \times \frac{\frac{1}{n}-4}{5}x^5 + \&c \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{1}{n}x + \frac{1}{n} \times \frac{1-n}{2} x^2 + \frac{1}{n} \times \frac{1-n}{2} \times \frac{1-2n}{3} x^3 \\
&\quad + \frac{1}{n} \times \frac{1-n}{2} \times \frac{1-2n}{3} \times \frac{1-3n}{4} x^4 \\
&\quad + \frac{1}{n} \times \frac{1-n}{2} \times \frac{1-2n}{3} \times \frac{1-3n}{4} \times \frac{1-4n}{5} x^5 + \&c \\
&= 1 + \frac{1}{n}x + \frac{1}{n} \times \frac{1-n}{2n} x^2 + \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} x^3 \\
&\quad + \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} \times \frac{1-3n}{4n} x^4 \\
&\quad + \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} \times \frac{1-3n}{4n} \times \frac{1-4n}{5n} x^5 + \&c \\
&= 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 \\
&\quad - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 \\
&\quad + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5 - \&c;
\end{aligned}$$

that is, $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$, will (if the binomial theorem extends to the case of roots,) be equal to the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5 + \&c$, or (if, for the sake of brevity, we put $A = 1$, and $B = \frac{1}{n}A$, and $C = \frac{n-1}{2n}B$, and $D = \frac{2n-1}{3n}C$, and $E = \frac{3n-1}{4n}D$, and $F = \frac{4n-1}{5n}E$, and $G, H, I, K, L, \&c = \frac{5n-1}{6n}F, \frac{6n-1}{7n}G, \frac{7n-1}{8n}H, \frac{8n-1}{9n}I, \frac{9n-1}{10n}K, \&c$, respectively,) to the series $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \frac{5n-1}{6n}Fx^6 + \frac{6n-1}{7n}Gx^7 - \frac{7n-1}{8n}Hx^8 + \frac{8n-1}{9n}Ix^9 - \frac{9n-1}{10n}Kx^{10} + \&c$; in which the second term $\frac{1}{n}Ax$, or $\frac{1}{n}x$, is marked with the sign $+$, or is to be added to the first term 1 , but all the following terms are marked alternately with the signs $-$ and $+$, or are to be subtracted from, and added to, the said first term 1 , or the two first terms $1 + \frac{1}{n}x$, alternately.

This is, as I apprehend, the clearest and most convenient way of expressing the binomial theorem in the case of roots, or of setting down the series which

is equal to $\sqrt[n]{1+x}$, or to $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$.

Of the Binomial Theorem in the Case of the Powers of Roots, when the Index of the Power of the Root is less than the Index of the Root.

3. When the index m stands for a fraction, of which a whole number is the numerator, and another whole number, greater than the former, is the denominator, we must substitute $\frac{m}{n}$ instead of m in the theorem stated in Art. 1, to

wit, the theorem, $\sqrt[n]{1+x}^m = 1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$; and then (if the binomial theorem extends to this case,) we shall

have $\sqrt[n]{1+x}^{\frac{m}{n}} =$ the series $1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$

$= 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$

$= 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$, or (if, for

the sake of brevity, we put $A = 1$, and $B = \frac{m}{n}$, or $\frac{m}{n}A$, and $C = \frac{n-m}{2n}B$, and D

$= \frac{2n-m}{3n}C$, and $E = \frac{3n-m}{4n}D$, and $F = \frac{4n-m}{5n}E$, and $G, H, I, K, L, \&c =$

$\frac{5n-m}{6n}F, \frac{6n-m}{7n}G, \frac{7n-m}{8n}H, \frac{8n-m}{9n}I, \frac{9n-m}{10n}K, \&c$, respectively, to the series

$1 + \frac{m}{n}Ax - \frac{n-m}{2n}Bx^2 + \frac{2n-m}{3n}Cx^3 - \frac{3n-m}{4n}Dx^4 + \frac{4n-m}{5n}Ex^5 - \frac{5n-m}{6n}$

$Fx^6 + \frac{6n-m}{7n}Gx^7 - \frac{7n-m}{8n}Hx^8 + \frac{8n-m}{9n}Ix^9 - \frac{9n-m}{10n}Kx^{10} + \&c$; in which

the second term $\frac{m}{n}Ax$, or $\frac{m}{n}x$, is always to be marked with the sign $+$, or to be

added to the first term 1 , but the third, and fourth, and fifth, and other following

terms, are marked alternately with the sign $-$ and the sign $+$, or are to be

subtracted from, and added to, the said first term 1 , or the two first terms

$1 + \frac{m}{n}x$, alternately.

This

This is, as I apprehend, the clearest and most convenient way of expressing the binomial theorem in this case of the power of a root of the binomial quantity $1 + x$, when m , or the index of the power to which the root of $1 + x$ is to be raised is less than n , or the index of the root.

Of the Binomial Theorem in the Case of the Powers of Roots, when the Index of the Power of the Root is greater than the Index of the Root.

4. When the index m stands for a fraction, of which a whole number is the numerator, and another whole number, less than the former, is the denominator, we must, as in the last case, substitute $\frac{m}{n}$ instead of m in the theorem stated

Art. 1, to wit, the theorem, $\overline{1 + x}^m = 1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$; and then (if the binomial theorem extends to this

case,) we shall have $\overline{1 + x}^{\frac{m}{n}} = 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c = 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c = 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c = 1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c.$

5. Now, since m is in this case greater than n , it will be possible to subtract n from m , and consequently the term $\frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2$ will be a positive, or affirmative, term, and must be added to the first term 1; and therefore the three first terms of this series, to wit, 1, $\frac{\frac{m}{n}}{1}x$, and $\frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2$, will in this case be added together, or connected by the sign $+$. And, if m is not only greater than n , but likewise than $2n$, the fourth term $\frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3$, will also be a positive or affirmative quantity, and must be marked with the sign $+$, and added to the three former terms. And, in like manner, if m is greater than $3n$, but less than $4n$, the five first terms of the series will be positive, or added to each other; and, if m is greater than $4n$, but less than $5n$, the first six terms of the series will be positive, or added to each other; and, if m is greater than $5n$, but less than $6n$, or greater than $6n$, but less than $7n$, or greater than $7n$, but less than

than $8n$, the first seven, or eight, or nine, terms of the series will, respectively, be positive, or added to each other. But, however great we may suppose m to be in comparison to n , it is evident that there will always be some term or other of the series in which the multiple of n that is to be subtracted from m , will become equal to, or greater than m . Let pn denote the greatest multiple of n , that is less than m . Then it is evident that the next multiple of n , namely, $p + 1 \times n$, or $pn + n$, must either be equal to, or greater than m . If it is equal to m , $m - pn - n$ will be $= m - m = 0$, and consequently that term of the series in the numerator of which the quantity $m - pn - n$ occurs, will be $= 0$ likewise; and so will all the subsequent terms of the series, because they are derived from the said term by multiplication. Therefore, the series will in this case consist of a finite number of terms, of which the term in which the quantity $m - pn$ first occurs will be the last. But, if $pn + n$ is greater than m , it cannot be subtracted from m , but m must be subtracted from it, and the term in which $m - pn - n$ occurs must, (instead of being added to the first term 1, as the foregoing terms were,) be subtracted from it; after which all the following terms of the series will be alternately added to, and subtracted from, the said first term 1, just as in the last case, (which is set down in Art. 3) all the terms after the second term $\frac{m}{n}x$ are so alternately subtracted and added. Therefore (if, for the sake of brevity, we put $A = 1$, and $B = \frac{m}{n}$, and $C = \frac{m-n}{2n}B$, and $D = \frac{m-2n}{3n}C$, or $\frac{2n-m}{3n}$, according as m is greater, or less, than $2n$; and $E = \frac{m-3n}{4n}D$, or $\frac{3n-m}{4n}D$, according as m is greater, or less, than $3n$; and $F = \frac{m-4n}{5n}E$, or $\frac{4n-m}{5n}E$, according as m is greater, or less, than $4n$; and, in like manner, G, H, I, K, L , &c. =, respectively, to $\frac{m-5n}{6n}F$, $\frac{m-6n}{7n}G$, $\frac{m-7n}{8n}H$, $\frac{m-8n}{9n}I$, $\frac{m-9n}{10n}K$, &c, or to $\frac{5n-m}{6n}F$, $\frac{6n-m}{7n}G$, $\frac{7n-m}{8n}H$, $\frac{8n-m}{9n}I$, $\frac{9n-m}{10n}K$, &c,) we shall have $1 + x \sqrt[n]{\frac{m}{n}} = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 + \&c$, till we come to the term in which the generating fraction is $\frac{m-pn}{p+1 \times n}$, or $\frac{m-pn}{pn+n}$, after which the terms will be marked with the signs $-$ and $+$ alternately, or will be alternately subtracted from, and added to, the first term 1, as in the former case set forth in Art. 3; and the numerators of the generating fractions in the following terms will be the excesses of the successive multiples of n , to wit, $pn + n$, $pn + 2n$, $pn + 3n$, $pn + 4n$, $pn + 5n$, &c, above m , as they before were the excesses of m above the preceeding lower multiples of n , to wit, $2n$, $3n$, $4n$, $5n$, $6n$, &c.

6. This law of the continuation of the co-efficients $C, D, E, F, G, H, I, K, L$, &c, in the terms of the series that is equal to $1 + x \sqrt[n]{\frac{m}{n}}$, when m is greater than n , will be more clearly understood by setting down a few examples of this series according to different relative magnitudes of m and n . This may be done as follows:

If

If m is greater than n , but less than $2n$, we shall have $\overline{1+x}^{\frac{m}{n}} =$ the series
 $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4$
 $- \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5 + \&c = 1 + \frac{m}{n}Ax$
 $+ \frac{m-n}{2n}Bx^2 - \frac{2n-m}{3n}Cx^3 + \frac{3n-m}{4n}Dx^4 - \frac{4n-m}{5n}Ex^5 + \frac{5n-m}{6n}Fx^6 - \frac{6n-m}{7n}Gx^7$
 $+ \frac{7n-m}{8n}Hx^8 - \frac{8n-m}{9n}Ix^9 + \frac{9n-m}{10n}Kx^{10} - \&c.$

If m is greater than $2n$, but less than $3n$, we shall have $\overline{1+x}^{\frac{m}{n}} =$ the series
 $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{3n-m}{4n}x^4$
 $+ \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5 - \&c = 1 + \frac{m}{n}Ax$
 $+ \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 - \frac{3n-m}{4n}Dx^4 + \frac{4n-m}{5n}Ex^5 - \frac{5n-m}{6n}Fx^6 +$
 $\frac{6n-m}{7n}Gx^7 - \frac{7n-m}{8n}Hx^8 + \frac{8n-m}{9n}Ix^9 - \frac{9n-m}{10n}Kx^{10} + \&c.$

If m is greater than $3n$, but less than $4n$, we shall have $\overline{1+x}^{\frac{m}{n}} =$ the series
 $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4$
 $- \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{4n-m}{5n}x^5 + \&c = 1 + \frac{m}{n}Ax$
 $+ \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 - \frac{4n-m}{5n}Ex^5 + \frac{5n-m}{6n}Fx^6 -$
 $\frac{6n-m}{7n}Gx^7 + \frac{7n-m}{8n}Hx^8 - \frac{8n-m}{9n}Ix^9 + \frac{9n-m}{10n}Kx^{10} - \&c.$

And, if m is greater than $4n$, but less than $5n$, we shall have $\overline{1+x}^{\frac{m}{n}} =$
 $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}x^4$
 $+ \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n}x^5 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} \times \frac{m-4n}{5n} \times \frac{5n-m}{6n}x^6$
 $+ \&c = 1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 + \frac{m-2n}{3n}Cx^3 + \frac{m-3n}{4n}Dx^4 + \frac{m-4n}{5n}Ex^5 - \frac{5n-m}{6n}Fx^6$
 $+ \frac{6n-m}{7n}Gx^7 - \frac{7n-m}{8n}Hx^8 + \frac{8n-m}{9n}Ix^9 - \frac{9n-m}{10n}Kx^{10} + \&c.$

These examples will, I presume, be sufficient to illustrate the course of the co-efficients C, D, E, F, G, H, I, K, L, &c, in all the different relative magnitudes of m and n that can be supposed.

7. Before we enter upon the investigations of these important theorems, it may not be amiss to apply the three foregoing serieses, (which have been derived from the original series set down in Art. 1, by substituting $\frac{1}{n}$ and $\frac{m}{n}$ in its

its terms instead of m ,) to the determination of the values of $\sqrt[n]{1+x}$ and $\sqrt[n]{1-x}$ in a few particular instances: after which we shall prove, by raising the particular serieses thereby obtained to the proper powers, that in those instances, at least, the several theorems are true; which will be a good introduction to the general investigations of the said theorems in all other values of m and n , which will be presented to the reader's consideration in the subsequent part of this Discourse.

Examples of the Extraction of some particular Roots of the Binomial Quantity $1+x$ by means of the Series given above in Art. 2.

8. In the first place we will extract the square root of the binomial quantity $1+x$ by means of the series given in Art. 2, to wit, the series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c.$

Now, in this case, $\sqrt[n]{1+x}$ is $= \sqrt{1+x}$, or n is $= 2$. Therefore $2n$ is $(= 2 \times 2) = 4$, and $3n$ is $(= 3 \times 2) = 6$, and $4n$ is $(= 4 \times 2) = 8$, and $5n$ is $(= 5 \times 2) = 10$, and consequently $n-1$ is $(= 2-1) = 1$, and $2n-1$ is $(= 4-1) = 3$, and $3n-1$ is $(= 6-1) = 5$, and $4n-1$ is $(= 8-1) = 7$. We shall therefore have $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c = 1 + \frac{1}{2} Ax - \frac{1}{4} Bx^2 + \frac{3}{6} Cx^3 - \frac{5}{8} Dx^4 + \frac{7}{10} Ex^5 - \&c = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c.$ Therefore, if the binomial

theorem is true in the case of roots, $\sqrt{1+x}$, or the square-root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c.$

9. Now that this series is really equal to the square-root of $1+x$, will appear by multiplying it into itself. For we shall find that the product of the said multiplication will be $1+x$. This may be done in the manner following:

$$\begin{array}{r}
 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c \\
 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c \\
 \hline
 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c \\
 + \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{16} + \frac{x^4}{32} - \frac{5x^5}{256} + \&c \\
 - \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{64} - \frac{x^5}{128} + \&c \\
 + \frac{x^3}{16} + \frac{x^4}{32} - \frac{x^5}{128} + \&c \\
 - \frac{5x^4}{128} - \frac{5x^5}{256} + \&c \\
 + \frac{7x^5}{256} + \&c \\
 \hline
 1 + x \quad * \quad * \quad * \quad * \quad * \quad \&c.
 \end{array}$$

It appears therefore that the series $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c$ is really and truly, as far as relates to the said six first terms of it, the square-root of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of square-roots.

10. In the next place we will investigate the value of $\sqrt[n]{1+x}$, or the cube-root of the binomial quantity $1 + x$, by means of the same series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$.

Now in this case n is $= 3$, and consequently we shall have $2n = 6$, $3n = 9$, $4n = 12$, and $5n = 15$, and $n - 1 = 2$, $2n - 1 = 5$, $3n - 1 = 8$, and $4n - 1 = 11$. And therefore the series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$ will, in this case, be $= 1 + \frac{1}{3} Ax - \frac{2}{6} Bx^2 + \frac{5}{9} Cx^3 - \frac{8}{12} Dx^4 + \frac{11}{15} Ex^5 - \&c = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c$.

11. Now that this series is really equal to the cube-root of $1+x$, will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be equal to $1 + x$. These multiplications will be as follows:

$$\begin{array}{r}
 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c \\
 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c \\
 \hline
 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c \\
 + \frac{x}{3} + \frac{x^2}{9} - \frac{x^3}{27} + \frac{5x^4}{243} - \frac{10x^5}{729} + \&c \\
 - \frac{x^2}{9} - \frac{x^3}{27} + \frac{x^4}{81} - \frac{5x^5}{729} + \&c \\
 + \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{5x^5}{729} + \&c \\
 - \frac{10x^4}{243} - \frac{10x^5}{729} - \&c \\
 + \frac{22x^5}{729} - \&c \\
 \hline
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c
 \end{array}$$

$$\begin{array}{r}
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c \\
 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c \\
 \hline
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c \\
 + \frac{x}{3} + \frac{2x^2}{9} - \frac{x^3}{27} + \frac{4x^4}{243} - \frac{7x^5}{729} + \&c \\
 - \frac{x^2}{9} - \frac{2x^3}{27} + \frac{x^4}{81} - \frac{4x^5}{729} + \&c \\
 + \frac{5x^3}{81} + \frac{10x^4}{243} - \frac{5x^5}{729} + \&c \\
 - \frac{10x^4}{243} - \frac{20x^5}{729} + \&c \\
 + \frac{22x^5}{729} + \&c \\
 \hline
 1 + x \quad * \quad * \quad * \quad * \quad \&c.
 \end{array}$$

It appears therefore that the series $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c$ is really and truly, as far as relates to the first six terms of it, the cube-root of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of cube-roots as well as in that of square-roots.

Examples of the Extraction of the Roots of some particular Powers of the Binomial Quantity $1 + x$ by means of the Series given above in Art. 3; in which the Numerator of the Fractional Index $\frac{m}{n}$ is supposed to be less than its Denominator.

12. In the next place we will investigate the value of $\sqrt[3]{1+x}$, or of the cube-root of the square of the binomial quantity $1 + x$, by means of the series $1 + \frac{m}{n} Ax - \frac{n-m}{2n} Bx^2 + \frac{2n-m}{3n} Cx^3 - \frac{3n-m}{4n} Dx^4 + \frac{4n-m}{5n} Ex^5 - \&c$ given in Art. 3 as the value of $\sqrt[n]{1+x}$.

Now in this case $m = 2$, and $n = 3$. And therefore we shall have $2n = 6$, $3n = 9$, $4n = 12$, and $5n = 15$,

$$\text{and } n - m (= 3 - 2) = 1,$$

$$2n - m (= 6 - 2) = 4,$$

$$3n - m (= 9 - 2) = 7,$$

$$\text{and } 4n - m (= 12 - 2) = 10,$$

and consequently $1 + \frac{m}{n} Ax - \frac{n-m}{2n} Bx^2 + \frac{2n-m}{3n} Cx^3 - \frac{3n-m}{4n} Dx^4 + \frac{4n-m}{5n} Ex^5 - \&c = 1 + \frac{2}{3} Ax - \frac{1}{6} Bx^2 + \frac{4}{9} Cx^3 - \frac{7}{12} Dx^4 + \frac{10}{15} Ex^5 - \&c = 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c$. Therefore, if the binomial theorem is true in the case of the powers of roots, or the roots of powers,

$\sqrt[2]{1+x^3}$, or the cube-root of the square of the binomial quantity $1+x$, will be equal to the series $1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c.$

13. Now that this series is really equal to $\sqrt[2]{1+x^3}$, or to the cube-root of the square of the binomial quantity $1+x$, or to the cube-root of the trinomial quantity $1+2x+xx$, will appear by multiplying the said series twice into itself. For we shall find that the product of these multiplications will be the said trinomial quantity. These multiplications will be as follows:

$$\begin{array}{r}
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c \\
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c \\
 \hline
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c \\
 + \frac{2x}{3} + \frac{4x^2}{9} - \frac{2x^3}{27} + \frac{8x^4}{243} - \frac{14x^5}{729} + \&c \\
 - \frac{x^2}{9} - \frac{2x^3}{27} + \frac{x^4}{81} - \frac{4x^5}{729} + \&c \\
 + \frac{4x^3}{81} + \frac{8x^4}{243} - \frac{4x^5}{729} + \&c \\
 - \frac{7x^4}{243} - \frac{14x^5}{729} + \&c \\
 + \frac{14x^5}{729} + \&c \\
 \hline
 1 + \frac{4x}{3} + \frac{2x^2}{9} - \frac{4x^3}{81} + \frac{5x^4}{243} - \frac{8x^5}{729} \&c.
 \end{array}$$

$$\begin{array}{r}
 1 + \frac{4x}{3} + \frac{2x^2}{9} - \frac{4x^3}{81} + \frac{5x^4}{243} - \frac{8x^5}{729} + \&c \\
 1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c \\
 \hline
 1 + \frac{4x}{3} + \frac{2x^2}{9} - \frac{4x^3}{81} + \frac{5x^4}{243} - \frac{8x^5}{729} + \&c \\
 + \frac{2x}{3} + \frac{8x^2}{9} + \frac{4x^3}{27} - \frac{8x^4}{243} + \frac{10x^5}{729} - \&c \\
 - \frac{x^2}{9} - \frac{4x^3}{27} - \frac{2x^4}{81} + \frac{4x^5}{729} - \&c \\
 + \frac{4x^3}{81} + \frac{16x^4}{243} + \frac{8x^5}{729} - \&c \\
 - \frac{7x^4}{243} - \frac{28x^5}{729} - \&c \\
 + \frac{14x^5}{729} + \&c \\
 \hline
 1 + 2x + xx \quad * \quad * \quad * \quad \&c.
 \end{array}$$

It appears therefore that the series $1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c$ is really and truly the cube-root of the trinomial quantity $1+2x+xx$, or of the square of the binomial quantity $1+x$, and consequently that the binomial theorem

theorem is true in the case of the cube-root of the square of a binomial quantity, or when m , the numerator of the fraction $\frac{m}{n}$, which is the index of the power of $1 + x$, is $= 2$, and n , the denominator of the said fraction, is $= 3$.

14. As another example of the investigation of the value of $\sqrt[n]{1+x}^{\frac{m}{n}}$ by means of the foregoing series $1 + \frac{m}{n} Ax - \frac{n-m}{2n} Bx^2 + \frac{2n-m}{3n} Cx^3 - \frac{3n-m}{4n} Dx^4 + \frac{4n-m}{5n} Ex^5 - \&c$, we will suppose m to be $= 3$, and n to be $= 5$, or

$\sqrt[n]{1+x}^{\frac{m}{n}}$ to be equal to $\sqrt[5]{1+x}^{\frac{3}{5}}$, or to the fifth root of the cube of the binomial quantity $1 + x$, or to the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$.

In this case we shall have $2n = 10$, $3n = 15$, $4n = 20$, and $5n = 25$, and consequently

$$\begin{aligned} n - m & (= 5 - 3) = 2, \\ \text{and } 2n - m & (= 10 - 3) = 7, \\ \text{and } 3n - m & (= 15 - 3) = 12, \\ \text{and } 4n - m & (= 20 - 3) = 17, \end{aligned}$$

$$\begin{aligned} \text{and } 1 + \frac{m}{n} Ax - \frac{n-m}{2n} Bx^2 + \frac{2n-m}{3n} Cx^3 - \frac{3n-m}{4n} Dx^4 + \frac{4n-m}{5n} Ex^5 - \&c \\ = 1 + \frac{3}{5} Ax - \frac{2}{10} Bx^2 + \frac{7}{15} Cx^3 - \frac{12}{20} Dx^4 + \frac{17}{25} Ex^5 - \&c = 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c. \end{aligned}$$

Therefore, if the binomial theorem is true in the case of the powers of roots, or the roots of powers, the quantity $\sqrt[5]{1+x}^{\frac{3}{5}}$, or the fifth root of the cube of $1 + x$, or the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will be equal to the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c$.

15. Now that this series is really equal to $\sqrt[5]{1+x}^{\frac{3}{5}}$, or to the fifth root of the cube of the binomial quantity $1 + x$, or to the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will appear by raising the said series to the fifth power by multiplying it first into itself, whereby we shall obtain its square, and afterwards multiplying the said square into itself, whereby we shall obtain its fourth power, and, lastly, multiplying the said fourth power of it into the series itself, whereby we shall obtain its fifth power. For we shall find that the product of these three multiplications will be the said quadrinomial quantity $1 + 3x + 3x^2 + x^3$. These multiplications will be as follows:

$$\begin{array}{r} 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c \\ 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c \\ \hline \end{array}$$

$$\begin{array}{r}
 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c \\
 + \frac{3x}{5} + \frac{9x^2}{25} - \frac{9x^3}{125} + \frac{21x^4}{625} - \frac{63x^5}{3125} + \&c \\
 - \frac{3x^2}{25} - \frac{9x^3}{125} + \frac{9x^4}{625} - \frac{21x^5}{3125} + \&c \\
 + \frac{7x^3}{125} + \frac{21x^4}{625} - \frac{3125}{21x^5} + \&c \\
 - \frac{21x^4}{625} - \frac{63x^5}{3125} + \&c \\
 + \frac{357x^5}{15625} + \&c
 \end{array}$$

$$1 + \frac{6x}{5} + \frac{3x^2}{25} - \frac{4x^3}{125} + \frac{9x^4}{625} - \frac{126x^5}{15625} + \&c.$$

This is the square of the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c.$

$$\begin{array}{r}
 1 + \frac{6x}{5} + \frac{3x^2}{25} - \frac{4x^3}{125} + \frac{9x^4}{625} - \frac{126x^5}{15625} + \&c \\
 1 + \frac{6x}{5} + \frac{3x^2}{25} - \frac{4x^3}{125} + \frac{9x^4}{625} - \frac{126x^5}{15625} + \&c \\
 \hline
 1 + \frac{6x}{5} + \frac{3x^2}{25} - \frac{4x^3}{125} + \frac{9x^4}{625} - \frac{126x^5}{15625} + \&c \\
 + \frac{6x}{5} + \frac{36x^2}{25} + \frac{18x^3}{125} - \frac{24x^4}{625} + \frac{54x^5}{3125} + \&c \\
 + \frac{3x^2}{25} + \frac{18x^3}{125} + \frac{9x^4}{625} - \frac{3125}{12x^5} + \&c \\
 - \frac{4x^3}{125} - \frac{24x^4}{625} - \frac{12x^5}{3125} + \&c \\
 + \frac{9x^4}{625} + \frac{54x^5}{3125} + \&c \\
 - \frac{126x^5}{15625} - \&c
 \end{array}$$

$$1 + \frac{12x}{5} + \frac{42x^2}{25} + \frac{28x^3}{125} - \frac{21x^4}{625} + \frac{168x^5}{15625} + \&c.$$

This is the fourth power of the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c.$

$$\begin{array}{r}
 1 + \frac{12x}{5} + \frac{42x^2}{25} + \frac{28x^3}{125} - \frac{21x^4}{625} + \frac{168x^5}{15625} - \&c \\
 1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c \\
 \hline
 1 + \frac{12x}{5} + \frac{42x^2}{25} + \frac{28x^3}{125} - \frac{21x^4}{625} + \frac{168x^5}{15625} - \&c \\
 + \frac{3x}{5} + \frac{36x^2}{25} + \frac{126x^3}{125} + \frac{84x^4}{625} - \frac{63x^5}{3125} + \&c \\
 - \frac{3x^2}{25} - \frac{36x^3}{125} - \frac{126x^4}{625} - \frac{84x^5}{3125} + \&c \\
 + \frac{7x^3}{125} + \frac{84x^4}{625} + \frac{294x^5}{3125} + \&c \\
 - \frac{21x^4}{625} - \frac{252x^5}{3125} - \&c \\
 + \frac{357x^5}{15625} - \&c
 \end{array}$$

$$1 + 3x + 3x^2 + x^3 \quad * \quad * \quad \&c.$$

It appears therefore that the series $1 + \frac{3x}{5} - \frac{3x^2}{25} + \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c$, is really and truly the fifth root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, or of the cube of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of the fifth root of the cube of a binomial quantity, or when m , the numerator of the fraction $\frac{m}{n}$, which is the index of the power of $1 + x$, is $= 3$, and n , the denominator of the said fraction, is $= 5$.

Examples of the Extraction of the Roots of some particular Powers of the Binomial Quantity $1 + x$ by means of the Series given above in Art. 4 and 5, in which the Numerator of the Fractional Index $\frac{m}{n}$ is supposed to be greater than its Denominator.

16. In the next place we will investigate the value of $\sqrt[3]{1 + x}$, or of the square-root of the cube of the binomial quantity $1 + x$ by means of the series set down in Art. 4 and 5. Now in this case m is $= 3$, and n is $= 2$, and consequently m is greater than n , but less than $2n$. Therefore, by Art. 6, the series

that is equal to $\sqrt[3]{1 + x}$ will, in this case, be $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 - \frac{2n-m}{3n} Cx^3 + \frac{3n-m}{4n} Dx^4 - \frac{4n-m}{5n} Ex^5 + \&c$.

Now, since m is $= 3$, and n is $= 2$, we shall have $m - n (= 3 - 2) = 1$, and $2n = 4$, $3n = 6$, $4n = 8$, and $5n = 10$, and

$$2n - m (= 4 - 3) = 1,$$

$$3n - m (= 6 - 3) = 3,$$

$$4n - m (= 8 - 3) = 5,$$

and the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 - \frac{2n-m}{3n} Cx^3 + \frac{3n-m}{4n} Dx^4 - \frac{4n-m}{5n} Ex^5 + \&c$

$= 1 + \frac{3}{2} Ax + \frac{1}{4} Bx^2 - \frac{1}{6} Cx^3 + \frac{3}{8} Dx^4 - \frac{5}{10} Ex^5 + \&c = 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c$. Therefore, if the binomial theorem

is true in the case of the roots of powers, the quantity $\sqrt[3]{1 + x}$, or the square-root of the cube of the binomial quantity $1 + x$, or the square-root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will be equal to the series $1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c$.

17. Now that this series is really equal to the square-root of the cube of $1 + x$, or to the square-root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, will appear by multiplying the said series into itself. For we shall find that the product of the said multiplication will be the said quadrinomial quantity. This multiplication will be as follows :

$$\begin{array}{r}
 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c \\
 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c \\
 \hline
 1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c \\
 + \frac{3x}{2} + \frac{9x^2}{4} + \frac{9x^3}{16} - \frac{3x^4}{32} + \frac{9x^5}{256} - \&c \\
 + \frac{3x^2}{8} + \frac{9x^3}{16} + \frac{9x^4}{64} - \frac{3x^5}{128} + \&c \\
 - \frac{x^3}{16} - \frac{3x^4}{32} - \frac{3x^5}{128} + \&c \\
 + \frac{3x^4}{128} + \frac{9x^5}{256} + \&c \\
 - \frac{3x^5}{256} - \&c \\
 \hline
 1 + 3x + 3x^2 + x^3 \quad * \quad *
 \end{array}$$

It appears therefore that the series $1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c$, is really and truly the square root of the quadrinomial quantity $1 + 3x + 3x^2 + x^3$, or of the cube of the binomial quantity $1 + x$, and consequently that the binomial theorem is true in the case of the square-root of the cube of a binomial quantity, or when m is $= 3$, and n is $= 2$.

18. I shall add one more example of the series given in Art. 4 and 5, in which it is supposed that the numerator m is greater than the denominator n .

Let it be required to find, by means of the said series, the value of $\sqrt[5]{1+x}$, or of the cube-root of the fifth power of the binomial quantity $1+x$, or of the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$.

In this case m is $= 5$, and n is $= 3$. Therefore m (though greater than n),

is less than $2n$, and consequently, (by Art. 6,) $\sqrt[5]{1+x}$ will be equal to the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 - \frac{2n-m}{3n} Cx^3 + \frac{3n-m}{4n} Dx^4 - \frac{4n-m}{5n} Ex^5 + \&c$.

Now, since m is $= 5$, and n is $= 3$, we shall have $m-n (= 5-3) = 2$, and $2n = 6$, $3n = 9$, $4n = 12$, $5n = 15$, and

$$\begin{array}{l}
 2n-m (= 6-5) = 1, \\
 3n-m (= 9-5) = 4, \\
 \text{and } 4n-m (= 12-5) = 7,
 \end{array}$$

and consequently the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 - \frac{2n-m}{3n} Cx^3 + \frac{3n-m}{4n} Dx^4 - \frac{4n-m}{5n} Ex^5 + \&c$

$$\begin{aligned}
 &= 1 + \frac{5}{3} Ax + \frac{2}{6} Bx^2 - \frac{1}{9} Cx^3 + \frac{4}{12} Dx^4 - \frac{7}{15} Ex^5 + \&c \\
 &= 1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c.
 \end{aligned}$$

Therefore, if the binomial theorem is true in the case of the roots of powers, the quantity

$\sqrt[5]{1+x}$, or the cube-root of the fifth power of the binomial quantity $1+x$, or the

the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, will be equal to the series $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c.$

19. Now that this series is really equal to the cube-root of the fifth power of $1 + x$, or to the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be the said sextinomial quantity. These multiplications will be as follows:

$$\begin{array}{r}
 1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c \\
 1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c \\
 \hline
 1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c \\
 + \frac{5x}{3} + \frac{25x^2}{9} + \frac{25x^3}{27} - \frac{25x^4}{243} + \frac{25x^5}{729} - \&c \\
 + \frac{5x^2}{9} + \frac{25x^3}{27} + \frac{25x^4}{81} - \frac{25x^5}{729} + \&c \\
 - \frac{5x^3}{81} - \frac{25x^4}{243} - \frac{25x^5}{729} + \&c \\
 + \frac{5x^4}{243} + \frac{25x^5}{729} + \&c \\
 - \frac{7x^5}{729} - \&c \\
 \hline
 1 + \frac{10x}{3} + \frac{35x^2}{9} + \frac{140x^3}{81} + \frac{35x^4}{243} - \frac{14x^5}{729} + \&c. \\
 1 + \frac{10x}{3} + \frac{35x^2}{9} + \frac{140x^3}{81} + \frac{35x^4}{243} - \frac{14x^5}{729} + \&c \\
 1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c \\
 \hline
 1 + \frac{10x}{3} + \frac{35x^2}{9} + \frac{140x^3}{81} + \frac{35x^4}{243} - \frac{14x^5}{729} + \&c \\
 + \frac{5x}{3} + \frac{50x^2}{9} + \frac{175x^3}{27} + \frac{700x^4}{243} + \frac{175x^5}{729} - \&c \\
 + \frac{5x^2}{9} + \frac{50x^3}{27} + \frac{175x^4}{81} + \frac{700x^5}{729} - \&c \\
 - \frac{5x^3}{81} - \frac{50x^4}{243} - \frac{175x^5}{729} + \&c \\
 + \frac{5x^4}{243} + \frac{50x^5}{729} + \&c \\
 - \frac{7x^5}{729} - \&c \\
 \hline
 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 + \&c.
 \end{array}$$

It appears therefore that the series $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c$ is really and truly the cube-root of the fifth power of the binomial quantity $1 + x$, or the cube-root of the sextinomial quantity $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$, and consequently that the binomial theorem is true in the case of the cube-root of the fifth power of a binomial quantity, or when m is $= 5$, and n is $= 3$.

20. These six examples of the application of the binomial theorem to the investigation of the square and cube-roots of the binomial quantity $1 + x$, and of the cube-root of the square of $1 + x$, and of the fifth root of the cube of $1 + x$, and of the square-root of its cube, and the cube-root of its fifth power, may suffice for the purpose of illustrating the meaning of that celebrated theorem, and the manner of applying it to particular cases, when the index of the power of $1 + x$ is either the fraction $\frac{1}{n}$ or the fraction $\frac{m}{n}$. And the proofs that have been given of the truth of the serieses obtained by means of the binomial theorem in these examples, by raising the serieses so obtained to the proper powers, and shewing that the powers thereby produced are exactly what they ought to be, afford a very strong presumption, "that, since the binomial theorem is true in these particular values of m and n , it must also be true in all other values of them whatsoever." But in the mathematical sciences we ought to aim at something more than this high degree of probability, and to endeavour, if possible, to demonstrate the propositions we advance. And this is what I shall now proceed to do with respect to this theorem in the cases mentioned in Art. 2, 3, 4, and 5. And, that we may not fall into confusion by making our inquiries too extensive, and embracing too many objects at the

same time, I shall first consider the case of $\sqrt[n]{1 + x}$, or the n th root of the binomial quantity $1 + x$, and endeavour to demonstrate, that, whatever be

the whole number denoted by n , the value of $\sqrt[n]{1 + x}$ will be equal to the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5 - \&c$, or $1 + \frac{1}{n} Ax - \frac{\sqrt[n-1]{n-1}}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{\sqrt[3n-1]{3n-1}}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, set forth above in Art. 2, and afterwards shall proceed to give a like demonstration of the truth of the other serieses set forth in Art. 3, 4, and 5, for the values of $\sqrt[n]{1 + x}^{\frac{m}{n}}$.

Before we enter on the investigation of this series, it will be necessary to make the following observations.

Observations preparatory to the Investigation of the Series that is equal to $\sqrt[n]{1 + x}$, or the n th root of the binomial quantity $1 + x$.

21. In the first place, we must observe that the first term of the series that is equal to $\sqrt[n]{1 + x}$ or $\sqrt[n]{1 + x}$, must always be 1.

This follows from the common manner of extracting the roots of a binomial, or other more compound quantity, of which 1 is the first term. For, if we were required to extract the square-root, or the cube-root, or any other root, of

of such compound quantity, according to the common rules laid down in books of arithmetick, or algebra, for such extractions, the first step of such an extraction would be to extract the square-root, or the cube-root, or other higher root, of its first member 1; which square-root, or cube-root, or other higher root, of 1, would be 1; so that 1 would be the first term of the said square-root, or cube-root, or other higher root, of the said binomial, or other more compound, quantity $1 + x$, or $1 + x + \&c.$ Q. E. D.

The truth of this observation may also be deduced from the following consideration, to wit, "that the series that is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, must be equal to it in all magnitudes of x , or while x is of any magnitude less than 1, which is supposed to be the first and greater member of the binomial quantity $1 + x$." For, it follows from hence that the said series must be equal to $\sqrt[n]{1+x}$, or to $\sqrt[n]{1+x}$, when x is $= 0$. But, when x is $= 0$, $\sqrt[n]{1+x}$ is $= \sqrt[n]{1+0} = 1^n = 1$. Therefore, the said series must, in that case, be $= 1$. Therefore, the first term of the said series cannot be involved with any of the powers of x , (by which it would be made to become equal to 0 when x was equal to 0,) but must be $= 1$. Q. E. D.

22. In the next place we must observe, that the second and third, and other following terms, of the series that is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, will contain the several powers of x , to wit, $x, x^2, x^3, x^4, x^5, \&c.$ in their natural order, without interruption, and consequently that the said quantity will be equal to a series of the following form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$; in which the second and third, and other following terms, $Bx, Cx^2, Dx^3, Ex^4, Fx^5$, are to be either added to the first term 1, and in that case to have the sign $+$ prefixed to them, or to be subtracted from the said first term, and in that case to have the sign $-$ prefixed to them. But, as it is not yet certain to which of the said terms the sign $+$ is to be prefixed, and which of them are to be marked with the sign $-$, or which of them are to be added to the said first term 1, and which of them are to be subtracted from it, I have not, on this occasion, (where it was only necessary to mention the form of the series which is equal to $\sqrt[n]{1+x}$;) prefixed either of these signs to any of them, but have only separated the several terms of the said series one from another by placing a comma after each term.

23. Now, "that this observation is true, or that $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, will be equal to a series of the said form, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$," has already been shewn in two instances, to wit, when n is $= 2$, and when n is $= 3$, or in the cases of the square-root and the cube-root of $1 + x$. For it has been shewn by Art. 9, that the series $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \&c$ is equal

2 E 2

to

to the square root of $1 + x$; and it has been shewn in Art. 11, that the series $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \&c$ is equal to the cube-root of $1 + x$. And both these series are of the same form as the general series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ above-mentioned. And, "that all other roots of the binomial quantity $1 + x$ must likewise be equal to serieses of the same form," will appear from considering the manner of extracting any root of such a binomial quantity. For the first step of such an extraction gives us the first term of the root sought; which first term (as we have already observed) is always 1: and the second step of such an extraction is to square the said first term, 1, of the root sought, if we are performing the extraction of the square-root of $1 + x$; or to cube the said first term, 1, of the root sought, if we are performing the extraction of the cube-root of $1 + x$; or, in general, to raise the said first term, 1, of the root sought to the n th power, if we are performing the extraction of the n th root of $1 + x$; and then to subtract such square, cube, or n th power of the said first term, 1, of the root sought (which will always be 1, because all the powers of 1 are equal to 1), from $1 + x$, or the original quantity of which the root is sought: in consequence of which subtraction there will remain the quantity x (which is the second and lesser member of the binomial quantity $1 + x$), for the groundwork of the gradual evolution of the second and third and other following terms of the series, or root, sought. And the third step, or process, of such extraction (whereby we shall obtain the second term of the series that is equal to the root sought) is to double the first term of it already found, to wit, the term 1, in the case of the extraction of the square root; and to treble the said first term 1, in the case of the extraction of the cube-root; and, in general, to multiply the said first term, 1, by n in the case of the extraction of the n th root; and then to divide the said remainder x by the said product; to wit, by 2×1 , or 2, in the case of the square-root; and by 3×1 , or 3, in the case of the cube-root; and by $n \times 1$, or n , in the case of the n th root: which will give us $\frac{x}{2}$ in the case of the square-root, and $\frac{x}{3}$ in the case of the cube-root, and $\frac{x}{n}$ in the case of the n th root, for

the second term of the series that is equal to $\sqrt[1]{1+x}$, $\sqrt[3]{1+x}$, or $\sqrt[n]{1+x}$: so that the two first terms of the series that is equal to $\sqrt[2]{1+x}$ will be $1 + \frac{x}{2}$, and the two first terms of the series that is equal to $\sqrt[3]{1+x}$ will be $1 + \frac{x}{3}$, and in general, the two first terms of the series that is equal to $\sqrt[n]{1+x}$ will be $1 + \frac{x}{n}$.

And the next operations of these extractions will be to raise $1 + \frac{x}{2}$ to the second power in the extraction of the square-root, and to raise $1 + \frac{x}{3}$ to the third power in the extraction of the cube-root, and, in general, to raise $1 + \frac{x}{n}$ to the n th power in the extraction of the n th root; and then to subtract these powers from the original binomial quantity $1 + x$, of which we are seeking the root; or,

to speak more correctly, (because these powers of $1 + \frac{x}{2}$, $1 + \frac{x}{3}$, and $1 + \frac{x}{n}$ will be greater than $1 + x$,) to subtract $1 + x$ from the said powers: after which the third term of the series that is equal to $\overline{1 + x}^{\frac{1}{2}}$, or $\overline{1 + x}^{\frac{1}{3}}$, or $\overline{1 + x}^{\frac{1}{n}}$, will be obtained by dividing the remainder of the last subtraction by $2 \times \sqrt{1 + \frac{x}{2}}$, or $2 + x$, in the extraction of the square-root; or by $3 \times \sqrt[3]{1 + \frac{x}{3}}$, or $3 + x$, in the extraction of the cube-root; or, in general, by $n \times \sqrt[n]{1 + \frac{x}{n}}$, or $n + x$, in the extraction of the n th root. Now, it is evident, that, by raising the aforesaid powers of the binomial quantities $1 + \frac{x}{2}$, $1 + \frac{x}{3}$, and $1 + \frac{x}{n}$, (in the second member of each of which the simple power of x occurs,) we shall obtain quantities in which the powers of x will occur in their natural order, to wit, x , x^2 , x^3 , x^4 , x^5 , &c, without any interruption. Thus $\overline{1 + \frac{x}{2}}^2$ is $= 1 + x + \frac{xx}{4}$, and $\overline{1 + \frac{x}{3}}^3$ is $(= 1 + 3 \times 1 \times \frac{x}{3} + 3 \times 1 \times \frac{x^2}{9} + \frac{x^3}{27}) = 1 + x + \frac{xx}{3} + \frac{x^3}{27}$, and $\overline{1 + \frac{x}{n}}^n$ is (by the binomial theorem in the case of integral powers, which has been already demonstrated above, in the last tract but one, from page 153 to page 169,) $= 1 + \frac{n}{1} \times \frac{x}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{x^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{x^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{x^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{x^5}{n^5} + \&c = 1 + \frac{n}{1} \times \frac{x}{n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{x^2}{n^2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{x^3}{n^3} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{x^4}{n^4} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{x^5}{n^5} + \&c = 1 + x + \frac{n-1}{2n} x^2 + \frac{n-1}{2n} \times \frac{n-2}{3n} x^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} x^4 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} x^5 + \&c$; in all which powers of the said binomial quantities $1 + \frac{x}{2}$, $1 + \frac{x}{3}$, and $1 + \frac{x}{n}$ the powers of x occur in their natural order, to wit, x , x^2 , x^3 , x^4 , x^5 , &c, without any interruption. Therefore, when $1 + x$ has been subtracted from each of these powers, the remainders, (which will be $\frac{xx}{4}$ in the case of the square-root, and $\frac{xx}{3} + \frac{x^3}{27}$ in the case of the cube-root, and $\frac{n-1}{2n} x^2 + \frac{n-1}{2n} \times \frac{n-2}{3n} x^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} x^4 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} x^5 + \&c$ in the case of the n th root,) will consist of the regular powers of x , beginning with xx , combined with certain numeral co-efficients; and consequently the quotients of the divisions of these remainders by $2 + x$, and $3 + x$, and $n + x$, respectively, will be quantities that involve in them the square of x ; to wit, in the case of the square-root, the said quotient will

will be $\frac{xx}{4} = \frac{xx}{4} - \&c = \frac{xx}{8} - \&c$; and in the case of the cube-root the said quotient will be $\frac{xx}{3} + \frac{x^3}{27} = \frac{xx}{3} \&c = \frac{xx}{9} \&c$; and, in the case of the n th root the said quotient will be $\frac{\frac{n-1}{2n} xx + \frac{n-1}{2n} \times \frac{n-2}{3n} \times x^3 + \&c}{n+x} = \frac{\frac{n-1}{2n} xx}{n} \&c = \frac{n-1}{2nn} xx \&c$; so that the third term of the series that is equal to $\sqrt[n]{1+x}$ will be $\frac{xx}{8}$ and the third term of the series that is equal to $\sqrt[n]{1+x}$ will be $\frac{xx}{9}$ and, in general, the third term of the series that is equal to $\sqrt[n]{1+x}$ will be $\frac{n-1}{2nn} xx$; all which third terms involve in them the square of x . And thus we see that the three first terms of the series that is equal to $\sqrt{1+x}$ will be $1 + \frac{x}{2} - \frac{xx}{8}$, and the three first terms of the series that is equal to $\sqrt[3]{1+x}$ will be $1 + \frac{x}{3} - \frac{xx}{9}$, and the three first terms of the series that is equal to $\sqrt[n]{1+x}$ will be $1 + \frac{x}{n} - \frac{n-1}{2nn} xx$; in all which serieses the powers of x ascend regularly.

And as the following terms of these serieses are derived from the three first terms of them by the repetition of the same processes of multiplication, subtraction, and division, by which the third term is derived from the first and second terms, and the second term is derived from the first, it follows that the fourth and fifth and sixth and other following terms must involve in them the cube and fourth power and fifth power, and other following powers, of x , in their natural order, without interruption, and consequently that the form of the series that is

equal to $\sqrt[n]{1+x}$ will be $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or, rather, (since we now know that the second term of the series is to be added to the first term, 1, and that the third term of it is to be subtracted from them,) $1 + Bx - Cx^2, Dx^3, Ex^4, Fx^5, \&c$.
Q. E. D.

24. The foregoing reasonings may, perhaps, seem rather abstract and difficult, as they are expressed in words in the last article. But the force of them will become more apparent by actually exhibiting to the reader's view an example of the extraction of a root of the binomial quantity $1+x$ in the manner alluded to in them. Now the extraction of the square-root of $1+x$ is performed in the following manner.

The Extraction of the square-root of the binomial quantity $1 + x$.

$$\begin{array}{r}
 1 + x \left(1 + \frac{x}{2} - \frac{xx}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + \&c \right) \\
 2 + \frac{x}{2} \quad \frac{1}{*} + x \\
 \quad + x + \frac{xx}{4} \\
 2 + x - \frac{xx}{8} \quad \frac{xx}{*} - \frac{xx}{4} \\
 \quad - \frac{xx}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
 2 + x - \frac{xx}{4} + \frac{x^3}{16} \quad \frac{x^3}{*} + \frac{x^3}{8} - \frac{x^4}{64} \\
 \quad + \frac{x^3}{8} + \frac{x^4}{16} - \frac{x^5}{64} + \frac{x^6}{256} \\
 2 + x - \frac{xx}{4} + \frac{x^3}{8} - \&c \quad \frac{5x^4}{*} - \frac{5x^4}{64} + \frac{x^5}{64} - \frac{x^6}{256} \\
 \quad - \frac{5x^4}{64} - \frac{5x^5}{128} + \frac{5x^6}{512} + \&c \\
 2 + x - \frac{xx}{4} + \frac{x^3}{8} - \&c \quad \frac{7x^5}{*} + \frac{7x^5}{128} - \frac{7x^6}{512} \&c \\
 \quad + \frac{7x^5}{128} + \frac{7x^6}{256} - \&c \\
 2 + x - \frac{xx}{4} + \frac{x^3}{8} - \&c \quad \frac{21x^6}{*} - \frac{21x^6}{512} \&c \\
 \quad - \frac{21x^6}{512} \&c \\
 \quad *
 \end{array}$$

25. It appears from the foregoing extraction of the square-root of $1 + x$, that the said square-root is equal to a series of quantities of which the first seven terms are $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024}$ in which the powers of x increase regularly from x to x^6 . And it is plain from the manner in which the second of those terms, to wit, $\frac{x}{2}$, and all the terms that follow it, are gradually derived from x , or the remainder of the first subtraction, that the following terms of the series, to whatever number they should be continued, would involve in them the next following powers of x , to wit, $x^7, x^8, x^9, x^{10}, x^{11}, \&c$ in their natural order. And the same thing would appear in the extraction of the cube root of $1 + x$, or of its fourth root, or its fifth root, or any higher root of it whatsoever; though the operations necessary to these extractions would be much more complicated and laborious than those that occur in the foregoing extraction of the square-root. And therefore we may conclude in general from the nature of these extractions, that the n th root of the binomial quantity $1 + x$ will in all cases be equal to a series of the before-mentioned form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, in which the powers of x occur in their natural order, or increase gradually by the continual multiplication of x .

Q. E. D.

26. In

26. In the third place we must observe, that the second term Bx , of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, must be added to the first term 1 , and consequently have the sign $+$ prefixed to it.

This observation has been already shewn to be true in the reasonings used in Art: 23 to demonstrate the second observation. But, as it will be referred to in the following investigation as a necessary preliminary proposition, it will not be amiss to give the following additional demonstration of it before we begin that investigation.

Now, since $1+x$ is greater than 1 , it follows that $\sqrt[n]{1+x}$, or the n th root of $1+x$, or the first of $n-1$ geometrical mean proportionals between 1 and $1+x$, must also be greater than 1 . Consequently the series $1, Bx, Cx^2, Dx^3,$

$Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x}$, must also be greater than 1 . And this must always be true, of how small a magnitude soever we suppose x to be taken, so long as it has any magnitude at all. But, in order that the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, may be always greater than 1 , however small the magnitude of x be taken, it is necessary that Bx should be added to the first term 1 , and not subtracted from it. For, if Bx were subtracted from 1 , it would be possible, by diminishing the magnitude of x , to make all the terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$ that follow Bx , put together become less than Bx , however great we may suppose the magnitudes of the co-efficients $C, D, E, F, \&c$, to be; in which case it is evident that the whole series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ would (even though all the terms after Bx were to be added to the first term 1 , and the series were to be $1 - Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$) be less than the first term 1 : which is contrary to the supposition. Therefore the second term, Bx , cannot be subtracted from the first term 1 , but must be added to it, and consequently must be marked with the sign $+$; and therefore

the series that is equal to $\sqrt[n]{1+x}$, will be of this form, $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$. Q. E. D.

27. By the help of these three preliminary observations we may now proceed to the investigation of the series that is equal to $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$, when n is supposed to stand for any whole number whatsoever. This investigation may be made as follows.

AN ADDITION

T O T H E

Discourse on the Binomial Theorem.

[The Binder is desired to place this and the other sheets with starred signatures and folios after page 216 in the second volume.]

ARTICLE I. The foregoing conclusion contained in articles 22, 23, 24, and 25, to wit, "that $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$, is equal to a series of the before-mentioned form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (in which the powers of x follow each other in their natural order without any interruption, or increase gradually by the continual multiplication of x) or that a series of this form may always exist (to whatever whole number we suppose the letter n to be equal) which shall be equal to the said quantity $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$," may be also shewn in the following manner.

2. Whatever whole number may be supposed to be denoted by n , it is evident that a series of the before-mentioned form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, will be equal to it, if the several numeral co-efficients $B, C, D, E, F, \&c$, of $x, x^2, x^3, x^4, x^5, \&c$, in the second and other following terms of such series, are of such magnitudes, and the said second and other following terms, to wit, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are so connected with the first term 1 , by the signs $+$ and $-$, or by addition and subtraction, that, when the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ is raised to the n th power, or multiplied $n-1$ times into itself, the compound series which, it is evident, will be produced by such multiplication, shall be equal to the binomial quantity $1+x$, or that the two first terms of the said compound series shall be 1 and x , and that the sign prefixed to the said second term x shall be $+$, and that the third, and fourth, and fifth, and sixth, and other following terms of the said compound series shall, each of them, be equal to nothing, or that the members of each of the said terms of the said compound series, after the second term, shall be marked with both the signs $+$ and $-$, that is, some of them with the sign $+$, and others of them with the sign $-$, and that the sum of those of the said members of each term which are marked with the sign $-$ shall be equal to the sum of the other members of the same term which are marked with the sign $+$, so as thereby to counterbalance, or destroy, them, and make the whole of the said compound term be equal to nothing.

3. Thus, for example, if n is $= 2$, the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, will be equal to $\sqrt[1]{1+x}$, or $\sqrt[2]{1+x}$, or the square-root of the binomial quantity $1+x$, if the numeral co-efficients $B, C, D, E, F, \&c$, of $x, x^2, x^3, x^4, x^5, \&c$, in the second and other following terms of the said series, to wit, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are of such magnitudes, and the said second and other following terms of the said series are so connected with the first term 1 by the signs $+$ and $-$, or by addition and subtraction, that, when the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, is raised to the square, or second power, or multiplied $2-1$ times, or once, into itself, the compound series which, it is evident, will be produced by such multiplication, shall be equal to the binomial quantity $1+x$, or that the two first terms of the said compound series shall be 1 and x , and that the sign prefixed to the said second term x shall be $+$, and that the third, and fourth, and fifth, and sixth, and other following terms of the said compound series shall, each of them, be equal to nothing, or that the members of each of the said terms of the said compound series, after the said second term, shall be marked with both the signs $+$ and $-$, that is, some of them with the sign $+$, and the others with the sign $-$, and that the sum of those members of each term which are marked with the sign $-$ shall be equal to the sum of the other members of the same term which are marked with the sign $+$, so as thereby to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing.

4. And, if n is $= 3$, the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, will be equal to $\sqrt[1]{1+x}$, or $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1+x$, if the numeral co-efficients $B, C, D, E, F, \&c$ of $x, x^2, x^3, x^4, x^5, \&c$, in the second and other following terms of the said series, are of such magnitudes, and the said second and other following terms of the said series, to wit, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are so connected with the first term 1 by the signs $+$ and $-$, or by addition and subtraction, that when the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, is raised to the cube, or third power, or multiplied $3-1$ times, or twice, into itself, the compound series which, it is evident, will be produced by the said two multiplications, shall be equal to the binomial quantity $1+x$, or that the two first terms of the said compound series shall be 1 and x , and that the sign prefixed to the said second term x shall be $+$, and that the third, and fourth, and fifth, and sixth, and other following terms of the said compound series shall, each of them, be equal to nothing, or that the members of each of the said terms of the said compound series, after the said second term, shall be marked with both the signs $+$ and $-$, that is, some of them with the sign $+$, and the others with the sign $-$, and that the sum of those members of each term which are marked with the sign $-$ shall be equal to the sum of the other members of the same term which are marked with the sign $+$, so as thereby to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing.

5. We

5. We must therefore endeavour to shew that, if n be equal to 2, or to 3, or to any other whole number whatsoever, it will be possible for a series of the before-mentioned form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, to exist, in which the magnitudes of the numeral co-efficients $B, C, D, E, F, \&c$, of x, x^2, x^3, x^4, x^5 , and the following powers of x , in the second and other following terms of the said series, to wit, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are such, and the said second and other following terms of the said series are so connected with the first term 1 by the signs $+$ and $-$, or by addition and subtraction, that, if the said series be raised to the square or the cube, or the n th power, or be multiplied into itself $2 - 1$, or $3 - 1$ times (that is, once or twice) or $n - 1$ times, the compound series that will arise from such multiplications shall be equal to the binomial quantity $1 + x$, or that the two first terms of the said compound series shall be 1 and x , and that the said second term x shall have the sign $+$ prefixed to it, or shall be added to the first term 1 , and that the third, and fourth, and fifth, and sixth, and all the following terms of the said compound series, shall, each of them, be equal to nothing, or that some of the members of each of the said terms of the said compound series, after the second term, shall be marked with the sign $+$, and the others with the sign $-$, and that the sum of the latter members which are marked with the sign $-$, shall be equal to the sum of the former members, which are marked with the sign $+$, so as thereby to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing.

Of the Square-root of $1 + x$.

6. Now that it is possible for such a series of the before-mentioned form, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, to exist in the case of the square-root of $1 + x$, or when n is equal to 2, may be shewn in the following manner.

Let the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, be multiplied into itself; which may be done as follows.

1,	Bx ,	Cx^2 ,	Dx^3 ,	Ex^4 ,	Fx^5 ,	$\&c$
1,	Bx ,	Cx^2 ,	Dx^3 ,	Ex^4 ,	Fx^5 ,	$\&c$
1,	Bx ,	Cx^2 ,	Dx^3 ,	Ex^4 ,	Fx^5 ,	$\&c$
	Bx ,	B^2x^2 ,	BCx^3 ,	BDx^4 ,	BEx^5 ,	$\&c$
		Cx^2 ,	BCx^3 ,	C^2x^4 ,	CDx^5 ,	$\&c$
			Dx^3 ,	BDx^4 ,	CDx^5 ,	$\&c$
				Ex^4 ,	BEx^5 ,	$\&c$
					Fx^5 ,	$\&c$
1,	$2Bx$,	$2Cx^2$,	$2Dx^3$,	$2Ex^4$,	$2Fx^5$,	$\&c$
		B^2x^2 ,	$2BCx^3$,	$2BDx^4$,	$2BEx^5$,	$\&c$
			C^2x^4 ,	$2CDx^5$,	$\&c$	
			$2Ex^4$	2	$*$	

This

This compound series is the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ Therefore, if we can prove that it is possible that the co-efficients $B, C, D, E, F, \&c.$ may be taken of such magnitudes, and that the second and other following terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ may be so connected with its first term 1 by the signs $+$ and $-$, or by addition and subtraction, as to make the said compound series become equal to $1 + x$, it will follow that the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ will be equal to the square-root of $1 + x$. We must therefore endeavour to shew that it is possible for the numeral co-efficients $B, C, D, E, F, \&c.$ to be of such magnitudes, and for the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ to be so connected with the first term 1 of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ by the signs $+$ and $-$, or by addition and subtraction, that the whole compound series above-mentioned shall be equal to $1 + x$. This may be done in the manner following.

7. In the first place it is evident that the first term of the said compound series is 1 . This is too plain to admit of any proof.

In the next place it is plain that, if we prefix the sign $+$ to the second term Bx of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ so as to convert the said series into the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ the sign $+$ will likewise be prefixed to $2 Bx$, or the second term of the compound series

$\left\{ \begin{array}{l} 1, 2 Bx, 2 Cx^2, \&c \\ B^2x^2, \&c \end{array} \right\}$ above-mentioned, so as to convert it into the series $\left\{ \begin{array}{l} 1 + 2 Bx, 2 Cx^2, \&c \\ B^2x^2, \&c \end{array} \right\}$.

And, in the 3d place, it is evident that, if we suppose $2 B$ to be equal to 1 , or B to be equal to $\frac{1}{2}$, the second term, $2 Bx$, of the said compound series will be equal to $2 \times \frac{1}{2} \times x$, or $\frac{2}{2} \times x$, or x , and consequently the two first terms of the said compound series $\left\{ \begin{array}{l} 1, 2 Bx, 2 Cx^2, \&c \\ B^2x^2, \&c \end{array} \right\}$, or $\left\{ \begin{array}{l} 1 + 2 Bx, 2 Cx^2, \&c \\ B^2x^2, \&c \end{array} \right\}$, will be $1 + x$.

If, therefore, we can prove that it is possible to take the following numeral co-efficients $C, D, E, F, \&c.$ of such magnitudes, and to connect the third and fourth, and other following terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ to wit, the terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ with the two first terms $1, Bx$, or $1 + Bx$, or $1 + \frac{x}{2}$, by the signs $+$ and $-$, or by addition and subtraction, in such a manner, that every following term of the compound series

$$\left\{ \begin{array}{l} 1 + 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c \\ B^2x^2, 2 BCx^3, 2 BDx^4, 2 BEx^5, \&c \\ C^2x^4, 2 CDx^5, \&c \end{array} \right\}$$

or

$$\text{or } \left\{ \begin{array}{l} 1 + x, \quad 2 C x^2, \quad 2 D x^3, \quad 2 E x^4, \quad 2 F x^5, \quad \&c \\ B^2 x^2, \quad 2 B C x^3, \quad 2 B D x^4, \quad 2 B E x^5, \quad \&c \\ C^2 x^4, \quad 2 C D x^5, \quad \&c \end{array} \right\}$$

shall be equal to nothing, or that some of their members (for each of these terms is evidently a compound quantity, or a quantity consisting of more than one member) shall be marked with the sign +, and others of them shall be marked with the sign —, and that the sum of the latter members in each term, which are marked with the sign —, shall be equal to the sum of the former members in the same term, which are marked with the sign +, so as to exactly counterbalance, or destroy, the said former members, and make the whole term become equal to 0, it will then be evident that the whole compound series above mentioned will be equal to its two first terms $1 + x$, and consequently that the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + \frac{x}{2}, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, (of which the said compound series is the square) will be equal to the square-root of $1 + x$. This therefore is what we must now endeavour to prove.

8. In order to this, we must observe, in the first place, that in the first, or upper, horizontal row of terms in the said compound series, to wit, $1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c$, or $1 + x, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c$, the sign + or —, that is to be prefixed to every term of it, must be the same which is prefixed to the corresponding term of the original simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + \frac{x}{2}, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or term which involves the same capital letter, C, or D, or E, or F, &c. Thus, if the sign — is to be prefixed to the term Cx^2 of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, the same sign — must likewise be prefixed to the term $2 Cx^2$, in the upper horizontal row of terms in the compound series

$$\left\{ \begin{array}{l} 1, 2 Bx, 2 Cx^2, 2 Dx^3, \&c \\ B^2 x^2, 2 BC x^3, \&c \end{array} \right\}, \text{ or } \left\{ \begin{array}{l} 1 + x, 2 Cx^2, 2 Dx^3, \&c \\ B^2 x^2, 2 BC x^3, \&c \end{array} \right\}; \text{ because the}$$

said term, $2 Cx^2$, in the said compound series, arises from the addition of the two like terms Cx^2 and Cx^2 to each other, of which the first Cx^2 is the product of the multiplication of Cx^2 by 1, and the other Cx^2 is the product of the multiplication of 1 by Cx^2 , and each of these products must evidently have the same sign + or — prefixed to it as is prefixed to its factor Cx^2 , or the third term of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, its other factor being 1. And for the same reason each of the following terms $2 Dx^3, 2 Ex^4, 2 Fx^5, \&c$, in the said first, or upper, horizontal row of terms must have the same sign + or — prefixed to it as is prefixed to the corresponding term, or term involving the same capital letter D, or E, or F, &c, in the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + \frac{x}{2}, Cx^2, Dx^3, Ex^4, Fx^5, \&c$.

In the second place, it is evident from the last observation that, if we can de
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termine the sign + or —, which ought to be prefixed to any term in the said upper horizontal row of terms $1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c$, or $1 + x, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c$, we shall at the same time determine the sign + or —, which ought to be prefixed to the corresponding term (or term involving the same capital letter, D, or E, or F, &c) in the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + \frac{x}{2}, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, the signs to be prefixed to the said two corresponding terms being always the same.

In the third place it is evident that in the first, or upper, horizontal row of terms in the above-mentioned compound series

$$\left\{ \begin{array}{l} 1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c \\ B^2x^2, 2 BCx^3, 2 BDx^4, 2 BEx^5, \&c \\ C^2x^4, 2 CDx^5, \&c \end{array} \right\}$$

(which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$) the coefficients of $x, x^2, x^3, x^4, x^5, \&c$, are $2 B, 2 C, 2 D, 2 E, 2 F, \&c$, or exactly the doubles of the coefficients of the same powers of x in the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, so that a new capital letter C, or D, or E, or F, &c, enters into every new term of the said first, or upper horizontal row of terms.

For each of these terms in the said upper horizontal row is the sum of two equal terms, of which one is placed in the highest horizontal row of terms of the original set of separate products arising from the multiplication of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, into itself, before the similar terms in the said separate products are added together at the bottom, and the other is placed at the bottom of the same vertical column of terms in which the former is the upper term. Thus, in the first vertical column of terms in the said original set of products arising from the multiplication of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, into itself, there are two equal terms, to wit, Bx and Bx , whose sum forms the term $2 Bx$ in the upper horizontal row of terms of the compound series at the bottom, to wit, the series

$$\begin{array}{l} 1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c \\ B^2x^2, 2 BCx^3, 2 BDx^4, 2 BEx^5, \&c \\ C^2x^4, 2 CDx^5, \&c, \end{array}$$

which is equal to the square of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$; and in the second vertical column of the same original set of products, there are the terms Cx^2, B^2x^2 , and Cx^2 , of which the highest and lowest are equal to each other, and their sum is consequently equal to $2 Cx^2$; and in the third vertical column there are the terms Dx^3, BCx^3, BCx^3 , and Dx^3 , of which the highest and the lowest are equal to each other, and their sum is consequently equal to $2 Dx^3$; and in the fourth vertical column there are the terms $Ex^4, BDx^4, C^2x^4, BDx^4$, and Ex^4 , of which the highest and the lowest are equal to each other, and their sum is consequently equal to $2 Ex^4$; and in the fifth vertical column there are the terms $Fx^5, BEx^5, CDx^5, CDx^5, BEx^5$, and Fx^5 , of which the highest and the lowest are equal to each other, and their sum is consequently equal to $2 Fx^5$. And it is evident that the same thing would take place in all the following vertical columns of terms, to whatever number of terms

terms the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, should be continued : So that the following terms of the upper horizontal row of terms after the term $2 Fx^5$ must be $2 Gx^6, 2 Hx^7, 2 Ix^8, 2 Kx^9, 2 Lx^{10}, 2 Mx^{11}, \&c$, *ad infinitum*.

And in the fourth place it must be observed, that the capital letter C, or D, or E, or F, &c, which enters into the highest term of any of the said vertical columns of terms (which highest terms are $2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, 2 Gx^6, \&c$) will not enter into any of the other terms of the same vertical column, but the said other, or lower, terms of the said vertical column will involve in them only such capital letters as had appeared in the upper terms of the preceeding vertical columns. Thus for example, in the vertical column $2 Cx^2, B^2x^2$ (which is the first vertical column of the compound series in which the similar terms have been added to each other) the letter C enters only in the upper term $2 Cx^2$, and the other term B^2x^2 involves only the preceeding letter B; and in the second vertical column of terms in the same compound series, to wit, the vertical column $2 Dx^3, 2 BCx^3$, the letter D enters only in the upper term $2 Dx^3$, and the other term $2 BCx^3$ involves only the preceeding letters B and C; and in the third vertical column $2 Ex^4, 2 BDx^4, C^2x^4$, the letter E enters only in the upper term $2 Ex^4$, and the two other terms $2 BDx^4$, and C^2x^4 , involve only the preceeding letters B, C, and D; and in the fourth vertical column $2 Fx^5, 2 BEx^5, 2 CDx^5$, the letter F enters only in the upper term $2 Fx^5$, and the other two terms $2 BEx^5$ and $2 CDx^5$ involve only the preceeding letters B, C, D, and E.

And it is easy to perceive, from the nature of the multiplication by which the said compound series is obtained, that the same thing would take place in all the following vertical columns of terms in the said compound series, to whatever number of terms the said series should be continued. We may therefore conclude that there will be one, and but one, new capital letter B, or C, or D, or E, or F, &c, contained in every new vertical column of terms in the above-mentioned compound series, which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and that this new letter will appear only in the upper term of such vertical column.

This is a most important remark, and ought to be well understood and remembered.

In the 5th place it is evident, that when the signs $+$ and $-$, which are to be prefixed to Bx and Cx^2 and Dx^3 , or any greater number of the terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, after the first term 1 , have been determined, the signs that are to be prefixed to all the several terms in the next following vertical columns, except the highest term, will be thereby determined likewise. Thus, if the signs $+$ and $-$, which are to be prefixed to the second, third, and fourth terms, Bx, Cx^2 , and Dx^3 , of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are determined, the signs of all the terms, except the highest, in the next following vertical column of terms, which involves in it the next co-efficient E, to wit, the column containing the terms $2 Ex^4, 2 BDx^4, C^2x^4$, will be determined likewise; because they will involve in them only the old letters B and C and D, which are the co-efficients of the powers of x in.

x in the terms Bx , Cx^2 , and Dx^3 , of which the signs are already supposed to be determined. For example, if the first four terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are $1 + Bx - Cx^2 + Dx^3$ (as in truth they are found to be) the said following vertical column of terms, which involves in it the letter E , to wit, the column containing the terms $2Ex^4, 2BDx^4$, and C^2x^4 , will be $2Ex^4 + 2BDx^4 + C^2x^4$. For $2BDx^4$ and C^2x^4 being produced by the multiplications of Bx into Dx^3 and of Cx^2 into Cx^2 , it is evident that, when it is ascertained that the sign $+$ is to be prefixed to Bx and to Dx^3 , and that the sign $-$ is to be prefixed to Cx^2 , it will necessarily follow that BDx^4 will be $(= + Bx \times + Dx^3) = + BDx^4$, and consequently that $2BDx^4$ will be $= + 2BDx^4$, and that $Cx^2 \times Cx^2$ will be $(= - Cx^2 \times - Cx^2) = + C^2x^4$, or that the sign $+$ must be prefixed both to the term $2BDx^4$ and to the term C^2x^4 . This is a necessary consequence of the common rules of multiplication in Algebra, according to which when the signs of the factors of any product are known or determined, the sign to be prefixed to the product may be thence determined likewise.

These five observations being well understood and assented to as sufficiently demonstrated, we may now deduce from them a proof of the proposition that was stated in the latter part of art. 7, to wit, that it is possible for a series of the form $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, to exist, in which the co-efficients $C, D, E, F, \&c$, of the third, fourth, fifth, sixth, and other following terms of it, to wit, $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, shall be of such magnitudes, and the said terms themselves shall be connected with the two first terms $1 + Bx$, or $1 + \frac{x}{2}$, by the signs $+$ and $-$, or by addition and subtraction, in such a manner, that every following term of the compound series

$$\left\{ \begin{array}{l} 1 + 2Bx, \quad 2Cx^2, \quad 2Dx^3, \quad 2Ex^4, \quad 2Fx^5, \quad \&c \\ B^2x^2, \quad 2BCx^3, \quad 2BDx^4, \quad 2BEx^5, \quad \&c \\ C^2x^4, \quad 2CDx^5, \quad \&c \end{array} \right\}$$

$$\left\{ \begin{array}{l} \text{or } 1 + x \quad 2Cx^2, \quad 2Dx^3, \quad 2Ex^4, \quad 2Fx^5, \quad \&c \\ B^2x^2, \quad 2BCx^3, \quad 2BDx^4, \quad 2BEx^5, \quad \&c \\ C^2x^4, \quad 2CDx^5, \quad \&c \end{array} \right\}$$

shall be equal to nothing, or that some of its members (for each of these terms is evidently a compound quantity, or quantity consisting of more than one member) shall be marked with the sign $+$, and others of them shall be marked with the sign $-$, and that the sum of the latter members in each term, which are marked with the sign $-$, shall be equal to the sum of the former members in the same term, which are marked with the sign $+$, so as to exactly counterbalance, or destroy, the said former members, and make the whole term be equal to nothing; in consequence of which the whole of the said compound series will be equal to its two first terms $1 + 2Bx$, or $1 + 2 \times \frac{1}{2}x$, or $1 + x$. This proposition may be proved in the manner following.

9. Since the second term Bx of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, is to have the sign $+$ prefixed to it, so that the said series will be $1 + Bx,$

$+ Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, it follows (by art. 8, obs. 5) that the quantity B^2x^2 , which forms the second member of the third term, $2 Cx^2, B^2x^2$, of the foregoing compound series

$$\left\{ \begin{array}{l} 1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c \\ B^2x^2, 2 BCx^3, 2 BDx^4, 2 BEx^5, \\ C^2x^4, 2 CDx^5, \end{array} \right\}$$

must also have the sign $+$ prefixed to it. For it is the product of the multiplication of $+ Bx$ into $+ Bx$, which is $+ B^2x^2$.

And, since the said second member B^2x^2 of the third term $2 Cx^2, B^2x^2$ of the said compound series is to have the sign $+$ prefixed to it, it is evident that, in order to make the whole of the said term $2 Cx^2, B^2x^2$, or $2 Cx^2 + B^2x^2$, be equal to 0, we must prefix the contrary sign $-$ to its first member $2 Cx^2$, and at the same time must take C of such a magnitude that the said first member $2 Cx^2$ shall be equal to the second member B^2x^2 , that is, we must take C equal to $\frac{B^2}{2}$, or (because B has been already found to be equal to $\frac{1}{2}$) we must take C equal to $\frac{1}{2} \times \frac{1}{4}$, or $\frac{1}{8}$. And then we shall have the whole third term $2 Cx^2, B^2x^2 = - 2 Cx^2 + B^2x^2 = - 2 \times \frac{1}{8} x^2 + \frac{1}{2} \times \frac{1}{2} x^2 = - \frac{1}{4} x^2 + \frac{1}{4} x^2 = 0$.

And, because the sign $-$ is to be prefixed to $2 Cx^2$ in the aforesaid compound series, which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or third, term Cx^2 of the said simple series; and consequently the three first terms of the said simple series will be $1 + Bx - Cx^2$, or $1 + \frac{1}{2} x - \frac{1}{8} xx$, or $1 + \frac{x}{2} - \frac{xx}{8}$.

And therefore, if the three first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are $1 + \frac{x}{2} - \frac{xx}{8}$, the three first terms of the foregoing compound series

$$\left\{ \begin{array}{l} 1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c \\ B^2x^2, 2 BCx^3, 2 BDx^4, 2 BEx^5, \&c \\ C^2x^4, 2 CDx^5, \&c \end{array} \right\}$$

(which is equal to the square of the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$) will be

$$\left\{ \begin{array}{l} 1 + 2 Bx - Cx^2 \\ + B^2x^2 \end{array} \right\},$$

or $1 + 2 \times \frac{1}{2} x - 0$, or $1 + x - 0$; which are equal to $1 + x$.

10. The fourth term of the foregoing compound series is $2 Dx^3$, $2 BCx^3$, or (because the three first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, have been shewn to be $1 + Bx - Cx^2$) $2 Dx^3 - 2 BCx^3$.

Now, since the second member, $2 BCx^3$, of this fourth term, $2 Dx^3 - 2 BCx^3$, has the sign $-$ prefixed to it, it is evident that, in order to make the whole term be equal to 0, we must prefix the sign $+$ to its first member $2 Dx^3$, and we must at the same time take D of such a magnitude as shall make $2 Dx^3$ be exactly equal to $2 BCx^3$, that is, we must take $D = BC$, or (because it has been shewn that B is $= \frac{1}{2}$, and that C is $= \frac{1}{8}$) we must take $D = \frac{1}{2} \times \frac{1}{8}$, or $\frac{1}{16}$.

And, because the sign $+$ is to be prefixed to $2 Dx^3$ in the aforesaid compound series, which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, or $1 + Bx - Cx^2, Dx^3, Ex^4, Fx^5, &c$, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or fourth, term, Dx^3 , of the said simple series, and consequently the four first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$, or $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$.

And therefore, if the four first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, are $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$, the four first terms of the foregoing compound series (which is equal to the square of the said simple series) will be

$$\left\{ \begin{array}{l} 1 + 2Bx - Cx^2 + 2Dx^3 \\ + B^2x^2 - 2BCx^3 \end{array} \right\},$$

or $1 + 2 \times \frac{1}{2}x - 0 + 0$, or $1 + x - 0 + 0$; which are equal to $1 + x$.

11. The fifth term of the foregoing compound series

$$\left\{ \begin{array}{l} 1, 2Bx, 2Cx^2, 2Dx^3, 2Ex^4, 2Fx^5, &c \\ B^2x^2, 2BCx^3, 2BDx^4, 2BEx^5, &c \\ C^2x^4, 2CDx^5, &c \end{array} \right\}$$

or (prefixing the proper signs to the several terms that involve only the letters B, C , and D , which have been already investigated) of the compound series

$$\left\{ \begin{array}{l} 1 + 2Bx - 2Cx^2 + 2Dx^3, \quad 2Ex^4, \quad 2Fx^5, \quad &c \\ + B^2x^2 - 2BCx^3 + 2BDx^4, \quad 2BEx^5, \quad &c \\ + C^2x^4 - 2CDx^5, \quad &c \end{array} \right\}$$

is $2Ex^4 + 2BDx^4 + C^2x^4$.

Now, since the sign $+$ is prefixed to the second and third members, $2BDx^4$ and C^2x^4 , of this fourth term, it is evident that, in order to make the whole

whole term be equal to 0, we must prefix the contrary sign — to its first member $2 Ex^4$, and must likewise take E of such a magnitude that $2 Ex^4$ shall be equal to the sum of the other two members $2 BDx^4$ and C^2x^4 ; that is, we must take $E = BD + \frac{C^2}{2} = \frac{1}{2} \times \frac{1}{16} + \frac{1}{2} \times \frac{1}{8} \times \frac{1}{8} = \frac{1}{32} + \frac{1}{128} = \frac{4}{128} + \frac{1}{128} = \frac{5}{128}$.

And, because the sign — is to be prefixed to the quantity $2 Ex^4$ in the foregoing compound series which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + Bx - Cx^2 + Dx^3 - Ex^4, Fx^5, \&c$, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or fifth, term, Ex^4 , of the said simple series; and consequently the five first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3 - Ex^4$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$, or $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$.

And therefore, if the five first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$, the five first terms of the foregoing compound series (which is equal to the square of the said simple series) will be

$$1 + 2Bx - 2Cx^2 + 2Dx^3 - 2Ex^4 + B^2x^2 - 2BCx^3 + 2BDx^4 + C^2x^4,$$

or $1 + 2 \times \frac{1}{2}x - 0 + 0 - 0$, or $1 + x - 0 + 0 - 0$; which are equal to $1 + x$.

12. The sixth term of the foregoing compound series

$$\left\{ \begin{array}{l} 1, 2Bx, 2Cx^2, 2Dx^3, 2Ex^4, 2Fx^5, \&c \\ B^2x^2, 2BCx^3, 2BDx^4, 2BEx^5, \&c \\ C^2x^4, 2CDx^5, \&c \end{array} \right\}$$

or (prefixing the proper signs to the several terms that involve only the letters B, C, D, and E, which have been already investigated) of the compound series

$$\left\{ \begin{array}{l} 1 + 2Bx - 2Cx^2 + 2Dx^3 - 2Ex^4 + 2Fx^5 \&c \\ + B^2x^2 - 2BCx^3 + 2BDx^4 - 2BEx^5 \&c \\ + C^2x^4 - 2CDx^5, \&c \end{array} \right\},$$

is $2Fx^5 - 2BEx^5 - 2CDx^5$.

Now, since the sign — is prefixed to the second and third members, $2BEx^5$ and $2CDx^5$, of this sixth term, it is evident, that, in order to make the whole term be equal to nothing, we must prefix the contrary sign + to its first member $2Fx^5$, and must likewise take F of such a magnitude that the said first member

member $2Fx^5$ shall be equal to the sum of the other two members, $2BEx^5$ and $2CDx^5$; that is, we must take F equal to $BE + CD$, or (because it has been shewn that B is $= \frac{1}{2}$, and that C is $= \frac{1}{8}$, and that D is $= \frac{1}{16}$, and that E is $= \frac{5}{128}$) we must take F equal to $\frac{1}{2} \times \frac{5}{128} + \frac{1}{8} \times \frac{1}{16}$, or to $\frac{5}{256} + \frac{1}{128}$, or to $\frac{5}{256} + \frac{2}{256}$, or to $\frac{7}{256}$.

And, because the sign $+$ is to be prefixed to the quantity $2Fx^5$ in the foregoing compound series which is equal to the square of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5, \&c$, it must likewise (by art. 8, obs. 1) be prefixed to the corresponding, or sixth, term, Fx^5 , of the said simple series; and consequently the six first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5$, or $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5$, or $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}$.

And therefore, if the six first terms of the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ are $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}$, the six first terms of the foregoing compound series (which is equal to the square of the said simple series) will be

$$\left. \begin{array}{l} 1 + 2Bx - 2Cx^2 + 2Dx^3 - 2Ex^4 + 2Fx^5 \\ + B^2x^2 - 2BCx^3 + 2BDx^4 - 2BEx^5 \\ + C^2x^4 - 2CDx^5 \end{array} \right\},$$

or $1 + 2 \times \frac{1}{2}x - 0 + 0 - 0 + 0$, or $1 + x - 0 + 0 - 0 + 0$; which are equal to $1 + x$.

13. And in like manner it will evidently be possible to take the following numeral co-efficients $G, H, I, K, L, M, \&c$ of $x^6, x^7, x^8, x^9, x^{10}, x^{11}$, and of the other following powers of x , in the seventh, eighth, ninth, tenth, eleventh, and twelfth, and other following terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$, of such magnitudes, and so to connect the said seventh, eighth, ninth, tenth, eleventh, twelfth, and other following terms of the said series with its first term 1 by the signs $+$ and $-$, or by addition and subtraction, as to make the whole of the seventh, and of the eighth, and of the ninth, and of the tenth, and of the eleventh, and of the twelfth, and of every following term of the compound series which is the square of the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$, be equal to 0 , and consequently to make the whole of the said compound series be equal to its two first terms $1 + x$. And therefore we may conclude that there is a certain infinite series of the aforesaid form, to wit, $1, Bx, Cx^2, Dx^3,$

$Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$, which is equal to $\sqrt{1+x}$, or to $\sqrt[2]{1+x}$, or the square-root of the binomial quantity $1+x$. Q. E. D.

14. Having

14. Having thus proved the truth of this proposition in the case of the square-root of the binomial quantity $1 + x$, or when the number n in the general expression $\sqrt[n]{1 + x}$, or $\sqrt[n]{1 + x}$, is equal to 2, we will now proceed to shew that it is also true in the case of the cube-root of $1 + x$, or when n is equal to 3; after which we will endeavour to shew that it will also be true in the case of any other root of $1 + x$, or when n is equal to any other whole number whatsoever.

Of the cube root of the binomial quantity $1 + x$.

15. Now it is evident (as has been already observed in art. 4) that $\sqrt[3]{1 + x}$, or $\sqrt[3]{1 + x}$, or the cube-root of the binomial quantity $1 + x$, will be equal to an infinite series of this form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, if it is possible to assign such magnitudes to the numeral co-efficients $B, C, D, E, F, \&c$, of the powers of x in the second and other following terms of this series, and so to connect the said second and other following terms of the said series with the first term 1 , by the signs $+$ and $-$, or by addition and subtraction, that the second term of the compound series which is the cube of the said series, or is the product that arises by multiplying it twice into itself, shall be equal to x and shall have the sign $+$ prefixed to it, or shall be added to 1 (which must evidently be the first term of the said cube or compound series), and that the third term of the said compound series shall be equal to nothing, and that the fourth term of it shall also be equal to nothing, and that every following term of it shall in like manner be equal to nothing, or that each of the said terms, after the said second term, shall consist of two, or more, members, and that some of the said members of each term shall be marked with the sign $+$, and the other members of the same term shall be marked with the sign $-$, and that the sum of the latter members of each term, which are marked with the sign $-$, shall be equal to the sum of the former members of the same term, which are marked with the sign $+$, so as to counterbalance, or destroy, them, and to make the whole of the said term be equal to nothing. We must therefore shew that it is possible to assign such magnitudes to the numeral co-efficients $B, C, D, E, F, \&c$, and so to connect the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, with its first term 1 by the signs $+$ and $-$, as to produce the effects just now described.

16. It has been shewn above in art. 6, that, if the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, be multiplied once into itself, the product thence arising, or the square of the said simple series, will be the following compound series, to wit,

$$\left. \begin{array}{l} 1, 2 Bx, 2 Cx^2, 2 Dx^3, 2 Ex^4, 2 Fx^5, \&c \\ B^2x^2, 2 BCx^3, 2 BDx^4, 2 BEx^5, \&c \\ C^2x^4, 2 CDx^5, \&c \end{array} \right\}$$

Therefore,

Therefore, in order to obtain the cube of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, we must multiply this compound series (which is equal to its square) into the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, itself. This may be done in the manner following.

$$\begin{array}{cccccc} 1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & C^2x^4, & 2CDx^5, & \&c \end{array}$$

$$1, \quad Bx, \quad Cx^2, \quad Dx^3, \quad Ex^4, \quad Fx^5, \&c$$

$$\begin{array}{cccccc} 1, & 2Bx, & 2Cx^2, & 2Dx^3, & 2Ex^4, & 2Fx^5, \&c \\ & B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & & C^2x^4, & 2CDx^5, & \&c \\ Bx, & 2B^2x^2, & 2BCx^3, & 2BDx^4, & 2BEx^5, & \&c \\ & B^3x^3, & 2B^2Cx^4, & 2B^2Dx^5, & \&c \\ & & BC^2x^5, & \&c \\ Cx^2, & 2BCx^3, & 2C^2x^4, & 2CDx^5, & \&c \\ & B^2Cx^4, & 2BC^2x^5, & \&c \\ Dx^3, & 2BDx^4, & 2CDx^5, & \&c \\ & B^2Dx^5, & \&c \\ Ex^4, & 2BEx^5, & \&c \\ & Fx^5, & \&c \end{array}$$

$$\begin{array}{cccccc} 1, & 3Bx, & 3Cx^2, & 3Dx^3, & 3Ex^4, & 3Fx^5, \&c \\ & 3B^2x^2, & 6BCx^3, & 6BDx^4, & 6BEx^5, & \&c \\ & & B^3x^3, & 3C^2x^4, & 6CDx^5, & \&c \\ & & & 3B^2Cx^4, & 3B^2Dx^5, & \&c \\ & & & & 3BC^2x^5, & \&c \end{array}$$

Therefore this last compound series

$$\left. \begin{array}{cccccc} 1, & 3Bx, & 3Cx^2, & 3Dx^3, & 3Ex^4, & 3Fx^5, \&c \\ & 3B^2x^2, & 6BCx^3, & 6BDx^4, & 6BEx^5, & \&c \\ & & B^3x^3, & 3C^2x^4, & 6CDx^5, & \&c \\ & & & 3B^2Cx^4, & 3B^2Dx^5, & \&c \\ & & & & 3BC^2x^5, & \&c \end{array} \right\}$$

will

will be the cube of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, *ad infinitum*. Let this compound series (for the sake of brevity) be denoted by the Greek capital letter Γ .

We are therefore now to prove that it is possible to assign such magnitudes to the numeral co-efficients $B, C, D, E, F, \&c$, of the powers of x in the second and other following terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and so to connect the said second and other following terms of the said series with its first term 1 by the signs $+$ and $-$, that the second term $3 Bx$ of the said compound series Γ shall be equal to x and shall have the sign $+$ prefixed to it, or shall be added to the first term 1 of the said series, and that the whole compound third term of the said series, to wit, $3 Cx^2, 3 B^2x^2$, shall be equal to 0 , and that the whole compound fourth term of the said series, to wit, $3 Dx^3, 6 BCx^3, B^3x^3$, shall also be equal to 0 , and that the whole compound fifth term, to wit, $3 Ex^4, 6 BDx^4, 3 C^2x^4, 3 B^2Cx^4$, shall also be equal to 0 , and that the whole compound sixth term, to wit, $3 Fx^5, 6 BEx^5, 6 CDx^5, 3 B^2Dx^5, 3 BC^2x^5$, shall also be equal to 0 , and, in like manner, that every following compound term shall also be equal to 0 , and consequently that the whole of the said compound series Γ shall be equal to its two first terms, or to the binomial quantity $1 + x$.

17. Now, to the end that the second term $3 Bx$ of the compound series Γ may have the sign $+$ prefixed to it, we need only prefix the same sign $+$ to the second term Bx of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$; it being evident from the rules of multiplication that, if the sign $+$ be prefixed to the term Bx in the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, it must likewise be prefixed to the term $3 Bx$ in the series Γ , which arises by multiplying the said series twice into itself.

And to the end that $3 Bx$, or the second term of the compound series Γ , may be equal to x , we need only suppose that $3 B$ is equal to 1 , or that B is equal to $\frac{1}{3}$. For then we shall have $3 Bx = 3 \times \frac{1}{3} \times x = \frac{3}{3} \times x = x$.

Therefore, if the two first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, be $1 + \frac{1}{3}x$, or $1 + \frac{x}{3}$, the two first terms of the compound series Γ will be $1 + x$.

18. In the next place we must observe that in the compound series Γ (which is equal to the cube of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$) as well as in the former compound series (which was equal to the square of the said series) one new capital letter C , or D , or E , or $F, \&c$, and but one, will enter into every new compound term, or vertical column of terms; and this new letter will appear in the highest term of every such vertical column, and only in the said highest term, and the other terms of such vertical column, or those which are placed immediately under such highest term, will involve in them only such of the capital letters $B, C, D, E, F, \&c$, as had occurred in the preceeding compound

pound terms, or vertical columns of terms. And, further, when the magnitudes of any number of the said capital letters, B, C, D, E, F, &c (or co-efficients of the powers of x in the terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$) have been determined, and the signs $+$ and $-$, which are to be prefixed to those of the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$ of the said simple series, in which the said letters are involved, have been likewise determined, the magnitudes of all the quantities in the compound series Γ which involve the said letters of which the magnitudes have been determined, and the signs $+$ and $-$, that are to be prefixed to such quantities, will all be determined likewise.

These observations are obvious consequences from the rules of algebraïck multiplication and the manner in which the compound series Γ has been derived from the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, to wit, by multiplying it twice into itself. And they are true likewise, with respect to all higher powers of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, whatsoever, as well as with respect to its cube.

19. Since, by art. 17, it appears that Bx , or the second term of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, is to have the sign $+$ prefixed to it, it follows that the same sign $+$ must also be prefixed to the quantity $3B^2x^2$, which is the second member of the third term, $3Cx^2, 3B^2x^2$, of the compound series Γ , so that the said third term will be $3Cx^2 + 3B^2x^2$. Therefore, to the end that the whole of the said third term $3Cx^2 + 3B^2x^2$ may be equal to nothing, we must prefix the contrary sign $-$ to the first member of it, to wit, $3Cx^2$, and must likewise take C of such a magnitude that $3Cx^2$ shall be equal to $3B^2x^2$, or that $3C$ shall be equal to $3B^2$, that is, we must take C equal to B^2 , or to $\frac{1}{3} \times \frac{1}{3}$ or $\frac{1}{9}$. And then we shall have the said third term $3Cx^2 + 3B^2x^2 = -3Cx^2 + 3B^2x^2 = -3 \times \frac{1}{9}x^2 + 3 \times \frac{1}{9}x^2 = 0$.

And, because $3Cx^2$ in the compound series Γ has the sign $-$ prefixed to it, the third term Cx^2 in the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, which corresponds to it, must also have the sign $-$ prefixed to it, and consequently the three first terms of the said simple series will be $1 + Bx - Cx^2$, or $1 + \frac{1}{3}x - \frac{1}{9}xx$, or $1 + \frac{x}{3} - \frac{xx}{9}$.

And, therefore, if the three first terms of the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, are $1 + \frac{x}{3} - \frac{xx}{9}$, the three first terms of the compound series Γ (which is equal to the cube of the said simple series) will be

$$1 + 3Bx - 3Cx^2 + 3B^2x^2,$$

or $1 + 3 \times \frac{1}{3}x - 0$, or $1 + x - 0$; which are equal to $1 + x$.

20. The

20. The fourth term of the compound series Γ is $3 D x^3$, $6 BC x^3$, $B^3 x^3$, or (because the three first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, have been found to be $1 + Bx - Cx^2$) $3 D x^3 - 6 BC x^3 + B^3 x^3$, or (because B has been found to be equal to $\frac{1}{3}$, and C has been found to be equal to $\frac{1}{9}$) $3 D x^3 - 6 \times \frac{1}{3} \times \frac{1}{9} x^3 + \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} x^3$, or $3 D x^3 - \frac{6}{27} x^3 + \frac{1}{27} x^3$, or $3 D x^3 - \frac{5}{27} x^3$. Now, since the sign $-$ is prefixed to the second member of this term, to wit, $\frac{5}{27} x^3$, it is evident that, in order to make the whole of the said term be equal to 0, we must prefix the sign $+$ to the first member of it, $3 D x^3$, and we must also take D of such a magnitude as will make the said first member of it $3 D x^3$ be equal to the second member $\frac{5}{27} x^3$, that is, we must take D equal to $\frac{1}{3} \times \frac{5}{27}$, or $\frac{5}{81}$. And, because the sign $+$ is to be prefixed to the quantity $3 D x^3$ in the compound series Γ , we must prefix the same sign $+$ to the correspondent term $D x^3$ of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and consequently the four first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{1}{3}x - \frac{1}{9}xx + \frac{5}{81}x^3$, or $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81}$.

And therefore, if the four first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81}$, the four first terms of the compound series Γ (which is equal to the cube of the said simple series) will be $1 + 3 Bx - 0 + 0$, or $1 + 3 \times \frac{1}{3}x - 0 + 0$, or $1 + x - 0 + 0$; which are equal to $1 + x$.

21. The fifth term of the said compound series Γ is $3 Ex^4$, $6 BD x^4$, $3 C^2 x^4$, $3 B^2 C x^4$, or (prefixing the proper signs $+$ and $-$ to the three last members of this term, to wit, $6 BD x^4$, $3 C^2 x^4$, and $3 B^2 C x^4$, which involve only the co-efficients B, C , and D , which have already been investigated) $3 Ex^4 + 6 BD x^4 + 3 C^2 x^4 - 3 B^2 C x^4$, or (because B has been found to be $= \frac{1}{3}$, and C has been found to be $= \frac{1}{9}$, and D has been found to be $= \frac{5}{81}$) $3 Ex^4 + 6 \times \frac{1}{3} \times \frac{5}{81} x^4 + 3 \times \frac{1}{9} \times \frac{1}{9} x^4 - 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{9} x^4$, or $3 Ex^4 + \frac{10}{81} x^4 + \frac{3}{81} x^4 - \frac{3}{81} x^4$, or $3 Ex^4 + \frac{10}{81} x^4$. Now, in order to make this whole term, $3 Ex^4 + \frac{10}{81} x^4$ (of which the second member $\frac{10}{81} x^4$ is marked with the sign $+$) be equal to 0, we must prefix the contrary sign $-$ to its first member $3 Ex^4$, and we must likewise take E of such a magnitude that $3 Ex^4$ shall be equal to $\frac{10}{81} x^4$, that is, we must take $E = \frac{1}{3} \times \frac{10}{81} = \frac{10}{243}$. And, because

the sign $-$ is to be prefixed to $3Ex^4$ in the compound series Γ , it must also be prefixed to the corresponding term, Ex^4 , in the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$; and consequently the five first terms of the said simple series will be $1 + Bx - Cx^2 + Dx^3 - Ex^4$, or $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4$, or $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$.

And therefore, if the five first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, are $1 + \frac{x}{3} - \frac{xx}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$, the first five terms of the compound series Γ (which is equal to the cube of the said series) will be $1 + 3Bx - 0 + 0 - 0$, or $1 + 3 \times \frac{1}{3}x - 0 + 0 - 0$, or $1 + x - 0 + 0 - 0$; which are equal to $1 + x$.

22. And the sixth term of the said compound series Γ is $3Fx^5, 6BEx^5, 6CDx^5, 3B^2Dx^5, 3BC^2x^5$, or (prefixing the proper signs $+$ and $-$ to the second, and third, and fourth, and fifth members of this term, to wit, $6BEx^5, 6CDx^5, 3B^2Dx^5$, and $3BC^2x^5$, which involve only the co-efficients B, C, D , and E , which have already been investigated) $3Fx^5 - 6BEx^5 - 6CDx^5 + 3B^2Dx^5 + 3BC^2x^5$, or (because B has been found to be $= \frac{1}{3}$, and C to be $= \frac{1}{9}$, and D to be $= \frac{5}{81}$, and E to be $= \frac{10}{243}$) $3Fx^5 - 6 \times \frac{1}{3} \times \frac{10}{243}x^5 - 6 \times \frac{1}{9} \times \frac{5}{81}x^5 + 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{5}{81}x^5 + 3 \times \frac{1}{3} \times \frac{1}{9} \times \frac{1}{9}x^5$, or $3Fx^5 - \frac{20}{243}x^5 - \frac{10}{243}x^5 + \frac{5}{243}x^5 + \frac{3}{243}x^5$, or $3Fx^5 - \frac{30}{243}x^5 + \frac{8}{243}x^5$, or $3Fx^5 - \frac{22}{243}x^5$. Now, in order to make this whole term $3Fx^5 - \frac{22}{243}x^5$

(of which the second member $\frac{22}{243}x^5$ is marked with the sign $-$) be equal to 0, we must prefix the contrary sign $+$ to its first member $3Fx^5$, and we must likewise take F of such a magnitude that $3Fx^5$ shall be equal to $\frac{22}{243}x^5$, that is, we

must take $F = \frac{1}{3} \times \frac{22}{243}$, or $\frac{22}{729}$. And, because the sign $+$ is to be prefixed to $3Fx^5$ in the compound series Γ , it must also be prefixed to the corresponding term Fx^5 of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$; and consequently the first six terms of the said series will be $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5$, or $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \frac{22}{729}x^5$, or $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729}$.

And therefore, if the six first terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$, are $1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729}$, the first six terms of the compound series Γ (which is equal to the cube of the said series) will

will be $1 + 3Bx - 0 + 0 - 0 + 0$, or $1 + 3 \times \frac{1}{3}x - 0 + 0 - 0 + 0$, or $1 + x - 0 + 0 - 0 + 0$; which are equal to $1 + x$.

23. And in the same manner it will evidently be possible to take the following numeral co-efficients, G, H, I, K, L, M, &c, of $x^6, x^7, x^8, x^9, x^{10}, x^{11},$ &c, in the 7th, 8th, 9th, 10th, 11th, 12th, and other following terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11},$ &c, of such magnitudes, and so to connect the said 7th, 8th, 9th, 10th, 11th, 12th, and other following terms of the said series with its first term 1 by the signs + and -, or by addition and subtraction, as to make the whole of the seventh term, and the whole of the eighth term, and the whole of the ninth term, and of the tenth term, and of the eleventh term, and of the twelfth term, and of every following term, of the compound series Γ (which is equal to the cube of the said simple series) be equal to nothing, and consequently to make the whole of the said compound series be equal to its two first terms $1 + x$. And therefore we may conclude that there is a certain infinite series of the aforesaid form, to wit, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11},$ &c, which is equal to $\sqrt[3]{1+x}$, or to $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1 + x$.

Q. E. D.

24. We have now shewn that an infinite series of the foregoing form, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5,$ &c, may exist, that shall be equal to $\sqrt[2]{1+x}$, or $\sqrt[2]{1+x}$, or the square-root of the binomial quantity $1 + x$; and that another infinite series of the same form, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5,$ &c, but with different values of the co-efficients B, C, D, E, F, &c, may exist, that shall be equal to $\sqrt[3]{1+x}$, or $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1 + x$. It remains that we prove that, if n be any other whole number whatsoever greater than 3, it will always be possible for an infinite series of the same form $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5,$ &c, to exist, that shall be equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1 + x$.

Of the nth root of the binomial quantity $1 + x$, when n is any whole number whatsoever.

25. Now the possibility of the existence of such a series in all values of the index n may be deduced from an attentive consideration of the two foregoing compound serieses (which are equal to the square and the cube of the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5,$ &c) and of the several properties

of their terms described above in art. 8, and resulting from the nature of multiplication. For, it will appear, upon such a consideration of those serieses, that, if we were to raise the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, to the fourth power, and to the fifth power, and to the sixth power, and to the seventh power, and to any greater number of its following powers whatsoever, by continual multiplications of the next preceeding powers of it into the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, itself, the observations contained in art. 8 would always be true of all these powers, or products, as well as of the two former and less complicated compound serieses which are equal to its square and cube.

26. And hence it follows that, if the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, were to be raised to the n th power (n being any whole number whatsoever) the first, or upper, horizontal row of terms in the compound series that would be equal to such n th power of it, would be $1, nBx, nCx^2, nDx^3, nEx^4, nFx^5, &c$ *ad infinitum*; just as, when n was $= 2$, we have seen that it was $1, 2Bx, 2Cx^2, 2Dx^3, 2Ex^4, 2Fx^5, &c$; and, when n was $= 3$, we have seen that it was $1, 3Bx, 3Cx^2, 3Dx^3, 3Ex^4, 3Fx^5, &c$.

And it will likewise be evident that the second term, or term involving x , in the said compound series will be only a single quantity, to wit, nBx , but that every following term of the said compound series involving any of the following powers of x , to wit, $xx, x^3, x^4, x^5, x^6, x^7, &c$, will be a compound term, or will consist of two, or more, single quantities.

And, 3dly, it will be evident, that in the third term, or term involving xx (which will be the first compound term, or vertical column of terms) the capital letter C, which enters in the highest term of the said vertical column, to wit, in the term nCx^2 , will not be contained in the other, or lower, terms of the same vertical column; but the said lower terms will only involve the preceeding capital letter B. And, in like manner, the capital letter D, which enters in the highest term of the next vertical column of terms, involving x^3 , to wit, in the term nDx^3 , will not be contained in the other, or lower, terms of the same vertical column; but the said lower terms will only involve the preceeding capital letters B and C. And, in like manner, the capital letter E, or F, or G, or H, &c, that will enter in the highest term of any subsequent vertical column of terms, will not be contained in the other, or lower, terms of the same vertical column; but the said lower terms will involve only the capital letters preceeding the said capital letter that enters in the highest term of the said vertical column.

And therefore, 4thly, if the magnitudes of the capital letters contained in any given number of terms in the first, or upper, horizontal row of terms of the said compound series, to wit, $1, nBx, nCx^2, nDx^3, nEx^4, nFx^5, &c$, have been determined, and the signs $+$ and $-$, which are to be prefixed to the said terms

in

in the said horizontal row, have also been determined, and consequently the signs which are to be prefixed to the corresponding terms of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$ (which are always the same with the former signs, by art. 8, obs. 1.) have also been determined, the magnitudes of all the terms of the next following vertical column of terms, except the highest term, will be thereby determined; and likewise the signs $+$ and $-$ which are to be prefixed to all the said terms of the said next following vertical column of terms, except the highest term, will also be thereby determined.

And therefore, 5thly, since both the magnitudes of all the terms in the said next vertical column of terms, except the highest term, and the signs $+$ and $-$, that are to be prefixed to the said terms, are determined, in the case here supposed, or when the preceeding terms of the first, or upper horizontal row of terms $1, nBx, nCx^2, nDx^3, nEx^4, nFx^5, &c$, and the signs $+$ and $-$, that are to be prefixed to them, have been determined, it follows that in the same case *the resulting value* of all the terms of the said next vertical column of terms, except the highest term, or the value resulting from the computation of their separate values, and the addition of those separate values to each other, or the subtraction of some of them from the others, or from the first term 1 , of the said compound series, according as the signs $+$ or $-$ are prefixed to them, will likewise be determined.

27. If these conclusions are allowed to be just, they will enable us to prove that the said compound series (which is equal to the n th power of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$) may be made to be equal to the binomial quantity $1 + x$, and consequently that the said simple series may be made to be equal to the n th root of the said binomial quantity, by taking the co-efficients $B, C, D, E, F, &c$, of certain proper magnitudes, and by connecting the second, and third, and other following, terms of the said simple series, to wit, the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, with its first term 1 by the signs $+$ and $-$, or by addition and subtraction, in a certain proper manner;—I say, the foregoing conclusions, if admitted to be just, will enable us to prove this proposition in the manner following.

In the first place, let the sign $+$ be prefixed to the second term Bx of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$. And it will follow that the second term nBx of the compound series which is equal to the n th power of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, or $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, must likewise have the same sign $+$ prefixed to it; and consequently the two first terms of the said compound series will be $1 + nBx$.

Secondly, let us suppose B to be equal to $\frac{1}{n}$. Then will nBx be $= n \times \frac{1}{n}x = x$, and consequently the two first terms $1 + nBx$ of the compound series which

which is equal to the n th power of the simple series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, will be $1 + n \times \frac{1}{n} x$, or $1 + x$.

Thirdly, since the sign $+$ is prefixed to the second term Bx of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and B has been taken $=$ to $\frac{1}{n}$, let the result of the value of the lower term of the first vertical column of terms in the said compound series (in which lower term only the capital letter B will enter) be computed, and be marked with its proper sign $+$ or $-$; and then let the contrary sign be prefixed to $n Cx^2$, or the upper term of the said vertical column; and let C be taken of such a magnitude as to make $n Cx^2$ equal to the said result. And, lastly, let the same sign $+$ or $-$, which is prefixed to $n Cx^2$, be also prefixed to the corresponding term, Cx^2 , of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$. And the consequence will be that the whole of the said vertical column, or third term of the said compound series, will be equal to 0 , and consequently that the three first terms of the said compound series will be $1 + x, 0$; which are equal to $1 + x$.

Fourthly, since the values of B and C are now determined, and likewise the signs $+$ and $-$, which are to be prefixed to the terms Bx and Cx^2 in the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, let the values of the several lower terms of the vertical column of terms which involves x^3 , and of which the highest term is $n Dx^3$, be computed; which will evidently be possible, because all the said lower terms will involve only the two capital letters B and C , which have been already determined: and let the values of the said several lower terms be added to each other, or some of them be subtracted from the others, or from the first term 1 of the said compound series, according as the sign $+$ or the sign $-$ is prefixed to them; and let the result of such additions and subtractions be marked with its proper sign $+$ or $-$. And then let the contrary sign be prefixed to the highest term, $n Dx^3$, of the said vertical column, which involves the new capital letter D ; and let the said capital letter D be taken of such a magnitude that the said highest term $n Dx^3$ shall be equal to the said result of the values of all the lower terms placed under it in the same vertical column. And, lastly, let the same sign which is prefixed to $n Dx^3$, be also prefixed to the corresponding term Dx^3 of the simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$. And it is evident that the whole of the said vertical column of terms of which $n Dx^3$ is the highest term, or the whole of the fourth term of the said compound series will be equal to 0 , and consequently that the four first terms of the said compound series will be $1 + x, 0, 0$; which are equal to $1 + x$.

And, in like manner, by computing the several lower terms of the next vertical column of terms, of which $n Ex^4$ is the highest term, and finding the result of them, and prefixing the proper sign $+$ or $-$ to such result, and then prefixing the contrary sign to the said highest term $n Ex^4$, and supposing E to be of such a magnitude as to make $n Ex^4$ be equal to the said result, and, lastly, prefixing to the corresponding term, Ex^4 , of the said simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, the same sign as was prefixed to the said highest term,
 $n Ex^4$,

$n E x^4$, of the said vertical column involving x^4 , we shall make the whole of the said vertical column, or the whole fifth term of the said compound series, to be equal to 0, and consequently the five first terms of it to be $1 + x, 0, 0, 0, 0$; which are equal to $1 + x$.

And in the same manner it will be possible to assign certain proper magnitudes, or values, to the following capital letters F, G, H, I, K, L, M, &c, which are involved in the following terms $F x^5, G x^6, H x^7, I x^8, K x^9, L x^{10}, M x^{11}$, &c, of the said simple series $1, B x, C x^2, D x^3, E x^4, F x^5, G x^6, H x^7, I x^8, K x^9, L x^{10}, M x^{11}$, &c (to whatever number of terms the said series may be continued) and to connect the said terms with the first term 1 of the said series by the signs $+$ and $-$, or by addition and subtraction, in such a manner, that the whole of the sixth term, and of every following term, of the compound series that is equal to the n th power of the said simple series, or that arises by the multiplication of the said series $n - 1$ times into itself, shall also be equal to 0, and consequently that the whole of the said compound series shall be $1 + x, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$, &c *ad infinitum*; which are equal to $1 + x$. And therefore an infinite series of the foregoing form, $1, B x, C x^2, D x^3, E x^4, F x^5, G x^6,$

$H x^7, I x^8, K x^9, L x^{10}, M x^{11}$, &c, may exist, which shall be equal to $\sqrt[n]{1 + x}$, or to $\sqrt[n]{1 + x}$, or to the n th root of the binomial quantity $1 + x$.

Q. E. D.

An Investigation of the Series

$$1 + \frac{1}{2} Ax - \frac{\frac{n-1}{2n}}{2n} Bx^2 + \frac{\frac{2n-1}{3n}}{3n} Cx^3 - \frac{\frac{3n-1}{4n}}{4n} Dx^4 + \frac{\frac{4n-1}{5n}}{5n} Ex^5 - \&c,$$

which is set down above in Art. 2, as being equal to

$$\sqrt[n]{1+x} \text{ or } \sqrt[n]{1+x}.$$

28. Let $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c,$ be the series that is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$; in which series the capital letters $B, C, D, E, F, \&c,$ stand for certain numbers, or numeral co-efficients of the several powers of x , which are hitherto unknown, and which it is the object of this investigation to discover. And further, since we have seen that the second term, Bx , of this series, is always to be added to the first term 1 , let us prefix the sign $+$ to the said second term; and then we shall have $\sqrt[n]{1+x} =$ the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$

29. Further, let all the terms of the said series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c,$ except the first, be represented by the letter y ; so that $\sqrt[n]{1+x}$ shall be equal $1 + y$.

Then will the n th power of $\sqrt[n]{1+x}$ be equal to the n th power of $1 + y$; that is, $1 + x$ will be $= (1 + y)^n =$ (by the binomial theorem in the case of integral powers, which has been already demonstrated above in the last tract but one, in pages 153, 154, &c, 169,) to the series $1 + \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c;$ and consequently, (subtracting 1 from both sides,) x will be equal to the series $\frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c.$

For the sake of brevity, let the small letters $c, d, e,$ and $f, \&c,$ be substituted instead of $\frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4},$ and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}, \&c,$ respectively, in the last equation. And we shall then have $x = ny + cy^2 + dy^3 + ey^4 + fy^5 + \&c.$

30. In the next place let the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c,$ which is equal to y , be raised to the second, third, fourth, fifth, and other following powers,

powers, in order to have the values of $y^2, y^3, y^4, y^5, \&c$, expressed in the powers of x . This may be done in the manner following.

$$\begin{array}{r}
 y = Bx, \quad Cx^2, \quad Dx^3, \quad Ex^4, \quad Fx^5, \quad \&c \\
 y = Bx, \quad Cx^2, \quad Dx^3, \quad Ex^4, \quad Fx^5, \quad \&c \\
 \hline
 B^2x^2, \quad BCx^3, \quad BDx^4, \quad BEx^5, \quad \&c \\
 \quad \quad BCx^3, \quad C^2x^4, \quad CDx^5, \quad \&c \\
 \quad \quad \quad \quad BDx^4, \quad CDx^5, \quad \&c \\
 \quad \quad \quad \quad \quad \quad BEx^5, \quad \&c \\
 \hline
 y^2 = \left\{ \begin{array}{l} B^2x^2, 2BCx^3, 2BDx^4, 2BEx^5, \&c \\ \quad \quad \quad C^2x^4, 2CDx^5, \&c \end{array} \right. \\
 \\
 y = Bx, Cx^2, Dx^3, \&c \\
 \hline
 B^3x^3, 2B^2Cx^4, 2B^2Dx^5, \&c \\
 \quad \quad B^2Cx^4, \quad BC^2x^5, \&c \\
 \quad \quad \quad \quad 2BC^2x^5, \&c \\
 \quad \quad \quad \quad \quad \quad B^2Dx^5, \&c \\
 \hline
 y^3 = \left\{ \begin{array}{l} B^3x^3, 3B^2Cx^4, 3B^2Dx^5, \&c \\ \quad \quad \quad 3BC^2x^5, \&c \end{array} \right. \\
 \\
 y = Bx, Cx^2, \&c \\
 \hline
 B^4x^4, 3B^3Cx^5, \&c \\
 \quad \quad B^3Cx^5, \&c \\
 \hline
 y^4 = B^4x^4, 4B^3Cx^5, \&c \\
 y = Bx, \&c \\
 \hline
 y^5 = B^5x^5, \&c
 \end{array}$$

Now let these values of $y, y^2, y^3, y^4, y^5, \&c$, be substituted instead of $y, y^2, y^3, y^4, y^5, \&c$, respectively, in the equation $x = ny + cy^2 + dy^3 + ey^4 + fy^5 + \&c$. And we shall have

$$x = \left\{ \begin{array}{l} nBx, \quad nCx^2, \quad nDx^3, \quad nEx^4, \quad nFx^5, \quad \&c \\ + cB^2x^2, 2cBCx^3, 2cBDx^4, 2cBEx^5, \&c \\ \quad \quad \quad 2cC^2x^4, 2cCDx^5, \&c \\ + dB^3x^3, 3dB^2Cx^4, 3dB^2Dx^5, \&c \\ \quad \quad \quad 3dBC^2x^5, \&c \\ + eB^4x^4, 4eB^3Cx^5, \&c \\ \quad \quad \quad + fB^5x^5, \&c. \end{array} \right.$$

By means of this equation we may determine the values of as many of the co-efficients $B, C, D, E, F, \&c$, as we think proper, by proceeding in the manner following.

31. To find the value of B , we must proceed as follows. Divide all the terms of this equation by x ; and we shall have

$$1 = \left\{ \begin{array}{l} nB, \quad nCx, \quad nDx^2, \quad nEx^3, \quad nFx^4, \quad \&c \\ + cB^2x, \quad 2cBCx^2, \quad 2cBDx^3, \quad 2cBEx^4, \quad \&c \\ \quad \quad \quad 2cC^2x^3, \quad 2cCDx^4, \quad \&c \\ + dB^3x^2, \quad 3dB^2Cx^3, \quad 3dB^2Dx^4, \quad \&c \\ \quad \quad \quad 3dBC^2x^4, \quad \&c \\ + eB^4x^3, \quad 4eB^3Cx^4, \quad \&c \\ \quad \quad \quad + fB^5x^4, \quad \&c \end{array} \right.$$

This equation is always true, of how small a magnitude soever we suppose x to be taken. And therefore it will be true when $x = 0$. But, when $x = 0$, all the terms that involve x in them will be equal to 0 likewise, and consequently the equation will be $1 = nB$. Therefore B will be $= \frac{1}{n}$; and consequently the two first terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{1}{n}}$ will be $1 + \frac{1}{n}x$, or $1 + \frac{x}{n}$. Q. E. I.

This value of the co-efficient B agrees with that which we before found for it in Art. 23, in which it appeared that the three first terms of the series that is equal to $\overline{1+x}^{\frac{1}{n}}$ would be $1 + \frac{x}{n} - \left[\frac{n-1}{2nn} \right] xx$, or $1 + \frac{x}{n} - \left[\frac{n-1}{2nn} \right] xx$.

32. To find the value of C , the co-efficient of the third term, Cx^2 , of the said series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{1}{n}}$, and likewise to determine, which of the two signs $+$ and $-$ is to be prefixed to the said term, or whether the said term is to be added to the two foregoing terms $1 + Bx$, or $1 + \frac{x}{n}$, or to be subtracted from them, we must proceed as follows.

Since 1, which forms the left-hand side of the equation obtained in Art. 31, is equal to nB , which is the first term of the upper line of terms on the right-hand side of that equation, it follows that all the other terms on the right-hand side of that equation, taken together, must be equal to 0; that is, the compound series

$$\begin{array}{l} nCx, \quad nDx^2, \quad nEx^3, \quad nFx^4, \quad \&c \\ + cB^2x, \quad 2cBCx^2, \quad 2cBDx^3, \quad 2cBEx^4, \quad \&c \\ \quad \quad \quad 2cC^2x^3, \quad 2cCDx^4, \quad \&c \\ + dB^3x^2, \quad 3dB^2Cx^3, \quad 3dB^2Dx^4, \quad \&c \\ \quad \quad \quad 3dBC^2x^4, \quad \&c \\ + eB^4x^3, \quad 4eB^3Cx^4, \quad \&c \\ \quad \quad \quad + fB^5x^4, \quad \&c \end{array}$$

will be $= 0$; that is, some of the terms of this series must be subtracted from the others, and the sum of those that are so subtracted must be equal to the sum of the other terms from which they are subtracted.

Now let all the terms of this compound series be divided by x . And it is evident that the terms that were subtracted from the others, and which were equal to them before such division, will still be equal to them after it. Therefore the new series that will result from such division, will still be $= 0$; that is, the compound series

$$\begin{array}{ccccccc} nC, & nDx, & nEx^2, & nFx^3, & \&c \\ + cB^2, & 2cBCx, & 2cBDx^2, & 2cBEx^3, & \&c \\ & & 2cC^2x^2, & 2cCDx^3, & \&c \\ & + dB^3x, & 3dB^2Cx^2, & 3dB^2Dx^3, & \&c \\ & & & 3dBC^2x^3, & \&c \\ & & + eB^4x^2, & 4eB^3Cx^3, & \&c \\ & & & + fB^5x^3, & \&c \end{array}$$

will be $= 0$. And this equation will be true in all the possible magnitudes of x ; and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms of the said series that involve x in them will be equal to 0 likewise, and the whole series will be reduced to the two terms $nC + cB^2$. Therefore these two terms $nC + cB^2$ will be $= 0$; and consequently nC must be subtracted from cB^2 , and marked with the sign $-$. For, if it were to be added to cB^2 , it would not be possible that $nC + cB^2$, (which on that supposition would mean the sum of the two quantities nC and cB^2) could be equal to 0. Therefore nC must have the sign $-$ prefixed to it; and consequently the third term, Cx^2 , of

the assumed series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is $= \sqrt[n]{1 + x}$, (from which third term the said quantity nC has been derived by multiplication and division in the course of this investigation,) must likewise have the sign $-$ prefixed to it, and must be subtracted from the two former terms $1 + Bx$, or $1 + \frac{x}{n}$; so that the three first terms of the said assumed series, which is equal to $\sqrt[n]{1 + x}$, will be $1 + Bx - Cx^2$, or $1 + \frac{x}{n} - Cx^2$.

And, for determining the magnitude of the co-efficient C , we shall have the equation $-nC + cB^2 = 0$; whence cB^2 will be $= nC$, and C will be $= \frac{1}{n} \times cB^2 = \frac{1}{n} \times \frac{n}{1} \times \frac{n-1}{2} \times B^2 = \frac{n-1}{2} \times B^2 = \frac{n-1}{2} \times \frac{1}{nn} = \frac{n-1}{2n} \times \frac{1}{n}$. Therefore the three first terms of the assumed series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1 + x}$, will be $1 + \frac{x}{n} - \frac{1}{n} \times \frac{n-1}{2n} xx$, or $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bxx$.

Q. E. D.

33. To find the value of D , the co-efficient of Dx^3 , the fourth term of the assumed series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1 + x}$, and

and to determine whether the sign $+$ or the sign $-$ is to be prefixed to the said fourth term, we must proceed as follows.

In the course of the last article we obtained the following equation; to wit,

$$\left. \begin{array}{l} nC, \quad nDx, \quad nEx^2, \quad nFx^3, \quad \&c \\ + cB^2, \quad 2cBCx, \quad 2cBDx^2, \quad 2cBE x^3, \quad \&c \\ \quad \quad \quad 2cC^2x^2, \quad 2cCDx^3, \quad \&c \\ + dB^3x, \quad 3dB^2Cx^2, \quad 3dB^2Dx^3, \quad \&c \\ \quad \quad \quad 3dBC^2x^3, \quad \&c \\ + eB^4x^2, \quad 4eB^3Cx^3, \quad \&c \\ \quad \quad \quad + fB^5x^3, \quad \&c \end{array} \right\} = 0.$$

And it was also found in the last article, that the term nC in this equation was to have the sign $-$ prefixed to it. Therefore the terms $2cBCx$ and $3dB^2Cx^2$, and $4eB^3Cx^3$, in the same equation, (which contain the simple power of C , or are produced by multiplications into C ,) must likewise have the sign $-$ prefixed to them; but the terms $2cC^2x^2$ and $3dBC^2x^3$, (which contain the square of C ,) must have the sign $+$ prefixed to them, because $-C \times -C = +C^2$. Therefore, if these several terms have their proper signs prefixed to them, the said equation will be as follows; to wit,

$$\left. \begin{array}{l} -nC, \quad nDx, \quad nEx^2, \quad nFx^3, \quad \&c \\ + cB^2 - 2cBCx, \quad 2cBDx^2, \quad 2cBE x^3, \quad \&c \\ \quad \quad \quad + 2cC^2x^2, \quad 2cCDx^3, \quad \&c \\ + dB^3x - 3dB^2Cx^2, \quad 3dB^2Dx^3, \quad \&c \\ \quad \quad \quad + 3dBC^2x^3, \quad \&c \\ + eB^4x^2 - 4eB^3Cx^3, \quad \&c \\ \quad \quad \quad + fB^5x^3, \quad \&c \end{array} \right\} = 0.$$

But it has been also shown in the last article, that $-nC + cB^2$ is $= 0$. Therefore, if we drop these two terms, the remaining terms on the left-hand side of the equation will still be equal to 0; that is,

$$\left. \begin{array}{l} nDx, \quad nEx^2, \quad nFx^3, \quad \&c \\ - 2cBCx, \quad 2cBDx^2, \quad 2cBE x^3, \quad \&c \\ \quad \quad \quad + 2cC^2x^2, \quad 2cCDx^3, \quad \&c \\ + dB^3x - 3dB^2Cx^2, \quad 3dB^2Dx^3, \quad \&c \\ \quad \quad \quad + 3dBC^2x^3, \quad \&c \\ + eB^4x^2 - 4eB^3Cx^3, \quad \&c \\ \quad \quad \quad + fB^5x^3, \quad \&c \end{array} \right\} \text{will be } = 0.$$

And, if we divide all the terms on the left-hand side of this equation by x , the equation will still be true, or the terms on the left-hand side of it will still be equal to 0, or those of the said terms which are marked, or to be marked, with the sign $-$, and are to be subtracted from the other terms, will still be equal to the terms from which they are subtracted; and therefore the equation will be as follows, to wit,

$$\left. \begin{array}{lll} nD, & nEx, & nFx^2, \&c \\ -2cBC, & 2cBDx, & 2cBE x^2, \&c \\ & +2cC^2x, & 2cCD x^2, \&c \\ +dB^3 - 3dB^2Cx, & 3dB^2Dx^2, \&c \\ & +3dBC^2x^2, \&c \\ & +eB^4x - 4eB^3Cx^2, \&c \\ & +fB^5x^2, \&c \end{array} \right\} = 0.$$

And this equation will be true in all the possible magnitudes of x : and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will be $nD - 2cBC + dB^3 = 0$. Therefore, by adding $2cBC$ to both sides, we shall have $nD + dB^3 = 2cBC$, that is, (because d is $= \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = c \times \frac{n-2}{3}$) $nD + \frac{n-2}{3} \times cB^3 = 2cBC$, or (because B is $= \frac{1}{n}$) $nD + \frac{n-2}{3} \times c \times \frac{1}{n^3} = 2c \times \frac{1}{n} \times \frac{1}{n} \times \frac{n-1}{2n}$, or (because C is $= \frac{1}{n} \times \frac{n-1}{2n}$) $nD + \frac{n-2}{3} \times c \times \frac{1}{n^3} = 2c \times \frac{1}{n} \times \frac{1}{n} \times \frac{n-1}{2n}$, or $nD + \frac{n-2}{3n^3} \times c = \frac{n-1}{3n^3} \times c$, and (adding $\frac{3}{3n^3} \times c$ to both sides,) $nD + \frac{n+1}{3n^3} \times c = \frac{3n}{3n^3} \times c$. Now, because n is a whole number, and consequently greater than 1, $n + n$ must be greater than $n + 1$, that is, $2n$ must be greater than $n + 1$, and, *à fortiori*, $3n$ (which is greater than $2n$) will be greater than $n + 1$. Therefore $\frac{3n}{3n^3} \times c$ will be greater than $\frac{n+1}{3n^3} \times c$; and consequently, to the end that $nD + \frac{n+1}{3n^3} \times c$ may be equal to $\frac{3n}{3n^3} \times c$, it is necessary that nD should be added to $\frac{n+1}{3n^3} \times c$, and not subtracted from it. We must therefore prefix the sign $+$ to nD , and consequently to the fourth term, Dx^3 , of the assumed series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, (which is equal to $\overline{1+x}^{\frac{1}{n}}$), from which fourth term the term nD was derived by the operations of multiplication and division in the course of this investigation. Therefore the four first terms of the said series, which is equal to $\overline{1+x}^{\frac{1}{n}}$, will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{1}{n}x - \frac{1}{n} + \frac{n-1}{2n}xx + Dx^3$; which was one of the points we were to determine.

And to determine the magnitude of the co-efficient D , we shall have the said equation $+nD + \frac{n+1}{3n^3} \times c = \frac{3n}{3n^3} \times c$; whence nD will be $= \frac{3n}{3n^3} \times c - \frac{n+1}{3n^3} \times c = \frac{2n-1}{3n^3} \times c$, or (because c is $= \frac{n}{1} \times \frac{n-1}{2}$) nD will be $= \frac{2n-1}{3n^3} \times n$

$\times \frac{n}{1} \times \frac{n-1}{2} = \frac{2n-1}{3n^2} \times \frac{n-1}{2} = \frac{2n-1}{3n} \times \frac{n-1}{2n}$; and consequently D will be $= \frac{2n-1}{3n} \times \frac{n-1}{2n} \times \frac{1}{n} = \frac{2n-1}{3n} \times C$. Therefore Dx^3 , or the fourth term of

the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ which is equal to $1 + x \sqrt[n]{\frac{1}{n}}$, will be $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3$, or $\frac{2n-1}{3n} Cx^3$; and consequently the four first terms of the said series will be $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3$, or $1 + \frac{1}{n} Ax - \left[\frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 \right]$. Q. E. I.

34. By reasoning in the same manner as in the foregoing articles on the equation obtained in the last article, to wit, the equation

$$\left. \begin{array}{l} nD, \quad nEx, \quad nFx^2, \quad \&c \\ -2cBC, \quad 2cBDx, \quad 2cBE x^2, \quad \&c \\ \quad \quad + 2cC^2 x, \quad 2cCD x^2, \quad \&c \\ +dB^3 - 3dB^2Cx, \quad 3dB^2Dx^2, \quad \&c \\ \quad \quad + 3dBC^2 x^2, \quad \&c \\ \quad \quad + eB^4 x - 4eB^3Cx^2, \quad \&c \\ \quad \quad + fB^5 x^2, \quad \&c \end{array} \right\} = 0,$$

or (if we prefix the sign $+$ to the terms nD , $2cBDx$, and $3dB^2Dx^2$, and the sign $-$ to the term $2cCDx^2$), the equation

$$\left. \begin{array}{l} +nD, \quad nEx, \quad nFx^2, \quad \&c \\ -2cBC + 2cBDx, \quad 2cBE x^2, \quad \&c \\ \quad \quad + 2cC^2 x - 2cCD x^2, \quad \&c \\ +dB^3 - 3dB^2Cx + 3dB^2Dx^2, \quad \&c \\ \quad \quad + 3dBC^2 x^2, \quad \&c \\ \quad \quad + eB^4 x - 4eB^3Cx^2, \quad \&c \\ \quad \quad + fB^5 x^2, \quad \&c \end{array} \right\} = 0,$$

we may obtain the two following equations for the determination of the values of E and F, and of the signs that are to be prefixed to them; to wit, the equation

$$nE + 2cBD + 2cC^2 - 3dB^2C + eB^4 = 0,$$

$$\text{or } nE + 2cBD + 2cC^2 + eB^4 = 3dB^2C,$$

for the determination of E; and the equation

$$nF - 2cBE - 2cCD + 3dB^2D + 3dBC^2$$

$$- 4eB^3C + fB^5 = 0, \text{ or}$$

$$nF + 3dB^2D + 3dBC^2 + fB^5 =$$

$2cBE + 2cCD + 4eB^3C$, for the determination of F. But the labour of resolving these equations by substituting in them, instead of the small letters

e, d, c, f , their respective values, to wit, $\frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}, \frac{n}{1} \times \frac{n-1}{2}$

$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, and, instead of the capital letters B, C, D and E, their respective values, to wit, $\frac{1}{n}$, $\frac{1}{n} \times \frac{n-1}{2n}$, $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}$ (for this last quantity will be found to be the value of E,) will be found to be very great; more especially in the latter equation, by which the value of F is to be determined. And in the investigation of the following co-efficients of the powers of x , to wit, the co-efficients G, H, I, K, L, &c, the intricacy of the calculations becomes so excessive as to make the discovery of these co-efficients in this method become absolutely impracticable.

35. And further, if this method of investigating the values of the co-efficients of the terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and the signs that are to be prefixed to them, was not, after the first four or five terms, so exceedingly troublesome as soon to become impracticable, it would still be liable to another objection. For, to whatever number of terms we had carried the investigation,—as, for example, if we had discovered twenty terms of the said series,—it would still be impossible to see, from this method of obtaining these terms, that the next, or twenty-first, term (which we had not actually investigated by resolving the simple equation that belongs to it,) would observe the same law of generation, or derivation from the preceeding terms, which had been found to take place amongst the twenty terms which had been investigated: so that we should not be able to discover with certainty any more terms of the said

series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, (which is equal to $1 + x \sqrt[n]{n}$), than we should actually have investigated. And therefore I shall dwell no longer on the foregoing method of investigating the values of these co-efficients, but shall proceed to explain another method of investigating them, which will be both much easier to practise in the few co-efficients we may think it necessary to investigate, and will, from the simplicity and regularity of the several simple equations involving the co-efficients C, D, E, F, &c, by the resolution of which the values of those co-efficients are to be determined, enable us to perceive that the law of the generation, or derivation, of the said co-efficients one from another, which takes place in the first four or five terms which we shall have actually investigated, must likewise take place in all the remaining terms of the series, to whatever number they may be supposed to be continued.

Another Investigation of the Series

$$1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3$$

$$- \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \frac{1}{n} \times$$

$$\frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 - \&c,$$

$$\text{or, } 1 + \frac{1}{n}Ax - \sqrt{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 -$$

$$\sqrt{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 - \&c,$$

which is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$; by which the law of the continuation of the co-efficients A, B, C, D, E, F, G, H, I, K, L, &c, will be apparent.

36. In the course of the foregoing investigation of the series that is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, of which we have found the four first terms to be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3$, or $1 + \frac{1}{n}Ax - \sqrt{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3$, the principal difficulties that we had to encounter related to the discovery of the third and fourth terms $\frac{n-1}{2n}Bxx$, and $\frac{2n-1}{3n}Cx^3$; of which the former is to be marked with the sign $-$, or to be subtracted from the two first terms $1 + \frac{1}{n}Ax$, or $1 + \frac{1}{n}x$; and the latter, to wit, $\frac{2n-1}{3n}Cx^3$, is to be marked with the sign $+$, or to be added to the said two first terms. But it was clear beyond a doubt, both from the said foregoing investigation and from the three preliminary observations in art. 21, 22, 23, — — 26, that the two first terms of the said series would be 1 and $\frac{1}{n}x$, or $\frac{x}{n}$, and that the second of these terms is to be added to the first; and also that the form of the said series (which is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$) would be 1, Bx, Cx², Dx³, Ex⁴, Fx⁵, &c, or $1 + \frac{1}{n}x$, Cx², Dx³, Ex⁴, Fx⁵, &c, in which the several powers of x follow each other in their natural order, without any interruption. We may therefore assume it as a truth sufficiently ascertained, and a legitimate ground-

work of the investigation we are now going to explain, "that $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, will be equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ " This being premised, the said investigation will be as follows.

37. It having been proved in art. 21, 22, 23, — — 26, that, if x is of any magnitude less than 1, the quantity $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, will be equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$; it follows that, if y be any quantity greater than x , but less than 1, the quantity $\sqrt[n]{1+y}$, or $\sqrt[n]{1+y}$, will, in like manner, be equal to the series $1 + \frac{1}{n}y, Cy^2, Dy^3, Ey^4, Fy^5, \&c.$ Now let d be equal to the difference by which y exceeds x , so that $x + d$ shall be $= y$; and let $x + d$ be substituted instead of y in the last equation $\sqrt[n]{1+y} = 1 + \frac{1}{n}y, Cy^2, Dy^3, Ey^4, Fy^5, \&c.$ And we shall then have $\sqrt[n]{1+x+d} =$ the series $1 + \frac{1}{n}x + \frac{1}{n}d, C \times \overline{x+d}^2, D \times \overline{x+d}^3, E \times \overline{x+d}^4, F \times \overline{x+d}^5, \&c =$ the series $1 + \frac{1}{n}x + \frac{1}{n}d, C \times \overline{xx + 2xd + dd}, D \times \overline{xx^2 + 3x^2d + 3xd^2 + d^3}, E \times \overline{xx^3 + 4x^3d + 6x^2d^2 + 4xd^3 + d^4}, F \times \overline{xx^4 + 5x^4d + 10x^3d^2 + 10x^2d^3 + 5xd^4 + d^5}, \&c =$ the compound series

$$\begin{array}{cccccc} 1 + \frac{1}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\ + \frac{1}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\ & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\ & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\ & & & Ed^4, & 5Fxd^4, & \&c \\ & & & & Fd^5, & \&c. \end{array}$$

38. Let f be $= 1 + x$; and we shall have $f + d = 1 + x + d$, and $\sqrt[n]{f+d} = \sqrt[n]{1+x+d}$. But, because $f + d$ is $= f \times \sqrt[n]{1 + \frac{d}{f}}$, it follows that $\sqrt[n]{f+d}$ will be $= f^{\frac{1}{n}} \times \sqrt[n]{1 + \frac{d}{f}}$. But, because $\sqrt[n]{1+x}$ is equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$; it follows that $\sqrt[n]{1 + \frac{d}{f}}$ will, in like manner, be equal to the series $1 + \frac{1}{n} \times \frac{d}{f}, C \times \overline{\frac{d}{f}}^2, D \times \overline{\frac{d}{f}}^3, E \times \overline{\frac{d}{f}}^4, F \times \overline{\frac{d}{f}}^5, \&c = 1 + \frac{1}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c.$

&c. Therefore $f^{\frac{1}{n}} \times \sqrt[n]{1 + \frac{d}{f}}$ will be $= f^{\frac{1}{n}} \times$ the series $1 + \frac{1}{n} \times \frac{d}{f},$
 $C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5},$ &c $=$ the series $f^{\frac{1}{n}} + \frac{1}{n} \times f^{\frac{1}{n}} \times$
 $\frac{d}{f}, C \times f^{\frac{1}{n}} \times \frac{d^2}{f^2}, D \times f^{\frac{1}{n}} \times \frac{d^3}{f^3}, E \times f^{\frac{1}{n}} \times \frac{d^4}{f^4}, F \times f^{\frac{1}{n}} \times \frac{d^5}{f^5},$ &c.

Therefore $\sqrt[n]{f + d}$ (which is $= f^{\frac{1}{n}} \times \sqrt[n]{1 + \frac{d}{f}}$) will also be equal to the
 series $f^{\frac{1}{n}} + \frac{1}{n} \times f^{\frac{1}{n}} \times \frac{d}{f}, C \times f^{\frac{1}{n}} \times \frac{d^2}{f^2}, D \times f^{\frac{1}{n}} \times \frac{d^3}{f^3}, E \times f^{\frac{1}{n}} \times$
 $\frac{d^4}{f^4}, F \times f^{\frac{1}{n}} \times \frac{d^5}{f^5},$ &c. Therefore, if we substitute $1 + x$ in this last equa-
 tion instead of f , to which it is equal, we shall have $\sqrt[n]{1 + x + d} =$ the series
 $\sqrt[n]{1 + x} + \frac{1}{n} \times \sqrt[n]{1 + x} \times \frac{d}{1 + x}, C \times \sqrt[n]{1 + x} \times \frac{d^2}{(1 + x)^2}, D \times \sqrt[n]{1 + x} \times$
 $\frac{d^3}{(1 + x)^3}, E \times \sqrt[n]{1 + x} \times \frac{d^4}{(1 + x)^4}, F \times \sqrt[n]{1 + x} \times \frac{d^5}{(1 + x)^5},$ &c.

39. But it has been shewn, in art. 37, that $\sqrt[n]{1 + x + d}$ is equal to the
 compound series

$$\begin{array}{cccccc} 1 + \frac{1}{n} x, & C x^2, & D x^3, & E x^4, & F x^5, & \&c \\ + \frac{1}{n} d, & 2 C x d, & 3 D x^2 d, & 4 E x^3 d, & 5 F x^4 d, & \&c \\ & C d^2, & 3 D x d^2, & 6 E x^2 d^2, & 10 F x^3 d^2, & \&c \\ & & D d^3, & 4 E x d^3, & 10 F x^2 d^3, & \&c \\ & & & E d^4, & 5 F x d^4, & \&c \\ & & & & F d^5, & \&c. \end{array}$$

Therefore the series $\sqrt[n]{1 + x} + \frac{1}{n} \times \sqrt[n]{1 + x} \times \frac{d}{1 + x}, C \times \sqrt[n]{1 + x} \times$
 $\frac{d^2}{(1 + x)^2}, D \times \sqrt[n]{1 + x} \times \frac{d^3}{(1 + x)^3}, E \times \sqrt[n]{1 + x} \times \frac{d^4}{(1 + x)^4}, F \times \sqrt[n]{1 + x} \times$
 $\frac{d^5}{(1 + x)^5},$ &c will be equal to the said compound series

$$\begin{array}{cccccc} 1 + \frac{1}{n} x, & C x^2, & D x^3, & E x^4, & F x^5, & \&c \\ + \frac{1}{n} d, & 2 C x d, & 3 D x^2 d, & 4 E x^3 d, & 5 F x^4 d, & \&c \\ & C d^2, & 3 D x d^2, & 6 E x^2 d^2, & 10 F x^3 d^2, & \&c \\ & & D d^3, & 4 E x d^3, & 10 F x^2 d^3, & \&c \\ & & & E d^4, & 5 F x d^4, & \&c \\ & & & & F d^5, & \&c. \end{array}$$

Therefore, if we subtract $\overline{1+x}^{\frac{1}{n}}$ from the left-hand side of this equation, and the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1+x}^{\frac{1}{n}}$), from the right-hand side of it, we shall have the series $\frac{1}{n} \times \overline{1+x}^{\frac{1}{n}} \times \frac{d}{1+x}, C \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^2}{1+x^2}, D \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^3}{1+x^3}, E \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^4}{1+x^4}, F \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^5}{1+x^5}, \&c =$ the compound series

$$\begin{array}{cccccc} \frac{1}{n}d, & 2Cx d, & 3Dx^2 d, & 4Ex^3 d, & 5Fx^4 d, & \&c \\ & Cd^2, & 3Dx d^2, & 6Ex^2 d^2, & 10Fx^3 d^2, & \&c \\ & & Dd^3, & 4Ex d^3, & 10Fx^2 d^3, & \&c \\ & & & Ed^4, & 5Fx d^4, & \&c \\ & & & & Fd^5, & \&c; \end{array}$$

and, if we divide all the terms by d (which is involved in every term on both sides of this equation), we shall have the series

$$\begin{array}{cccccc} \frac{1}{n} \times \overline{1+x}^{\frac{1}{n}} \times \frac{1}{1+x}, & C \times \overline{1+x}^{\frac{1}{n}} \times \frac{d}{1+x^2}, & D \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^2}{1+x^3}, & E \\ \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^3}{1+x^4}, & F \times \overline{1+x}^{\frac{1}{n}} \times \frac{d^4}{1+x^5}, & \&c = \text{the compound series} \\ \frac{1}{n}, & 2Cx, & 3Dx^2, & 4Ex^3, & 5Fx^4, & \&c \\ & Cd, & 3Dx d, & 6Ex^2 d, & 10Fx^3 d, & \&c \\ & & Dd^2, & 4Ex d^2, & 10Fx^2 d^2, & \&c \\ & & & Ed^3, & 5Fx d^3, & \&c \\ & & & & Fd^4, & \&c \end{array}$$

40. This equation is always true, how small soever we may suppose d to be. And therefore it will also be true when d is $= 0$. But, when d is $= 0$, all the terms in the equation that involve d will be equal to 0 likewise; and conse-

quently the equation will then be as follows, to wit, $\frac{1}{n} \times \overline{1+x}^{\frac{1}{n}} \times \frac{1}{1+x} =$ the series $\frac{1}{n}, 2Cx, 3Dx^2, 4Ex^3, 5Fx^4, \&c$. Therefore, if we multiply all

the terms of this equation by n , we shall have $\overline{1+x}^{\frac{1}{n}} \times \frac{1}{1+x} =$ the series $1, 2nCx, 3nDx^2, 4nEx^3, 5nFx^4, \&c$; and, if we multiply both sides of this last equation by $1+x$, we shall have $\overline{1+x}^{\frac{1}{n}} =$ the compound series

$$\begin{array}{cccccc} 1, & 2nCx, & 3nDx^2, & 4nEx^3, & 5nFx^4, & \&c \\ + & x, & 2nCx^2, & 3nDx^3, & 4nEx^4, & \&c \end{array}$$

41. But

41. But $\overline{1 + x}^{\frac{1}{n}}$ is equal to the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ Therefore the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ will be equal to the compound series

$$1, 2nCx, 3nDx^2, 4nEx^3, 5nFx^4, \&c \\ + x, 2nC x^2, 3nD x^3, 4nE x^4, \&c;$$

and consequently (subtracting 1 from both sides of the equation) we shall have the series $\frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c =$ the compound series

$$2nCx, 3nDx^2, 4nEx^3, 5nFx^4, \&c \\ + x, 2nC x^2, 3nD x^3, 4nE x^4, \&c; \text{ and,}$$

lastly, dividing all the terms by x , we shall have the simple series $\frac{1}{n}, Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$2nC, 3nDx, 4nEx^2, 5nFx^3, \&c \\ + 1, 2nC x, 3nD x^2, 4nE x^3, \&c;$$

by the help of which equation we may determine both the values of the several coefficients $C, D, E, F, \&c.$ of the terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ of the assumed series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c.$ or $1 + \frac{1}{n}x, Cx^2, Dx^3,$

$Ex^4, Fx^5, \&c.$ which is equal to $\overline{1 + x}^{\frac{1}{n}}$, and the signs $+$ and $-$ that are to be prefixed to the said terms respectively, by reasonings similar to those used in the former investigation, in deriving the values of the co-efficients $B, C,$ and D from the much more complicated equation obtained above in art. 30. This may be done in the manner following:

42. In the first place we may determine both the sign to be prefixed to the term Cx^2 in the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1 + x}^{\frac{1}{n}}$), and the magnitude of the co-efficient C , by proceeding as follows.

The final equation obtained in the last article, to wit, the equation between the simple series $\frac{1}{n}, Cx, Dx^2, Ex^3, Fx^4, \&c.$ and the compound series

$$2nC, 3nDx, 4nEx^2, 5nFx^3, \&c \\ + 1, 2nC x, 3nD x^2, 4nE x^3, \&c, \text{ is}$$

always true, how small soever we may suppose x to be. And therefore it will be true when x is $= 0$. But when x is $= 0$, all the terms in the foregoing equation that involve x in them will be equal to 0 likewise; and the said equation will consequently then be $\frac{1}{n} = 2nC, + 1$, or $\frac{1}{n} = 1, 2nC$; that is, $\frac{1}{n}$ will be equal to 1, together with $2nC$ either added to it, or subtracted from it, as may be necessary to produce the said equality. But, because n is a whole number, $\frac{1}{n}$ must be less than 1, and therefore cannot be equal to 1 together with

with $2nC$ added to it; therefore it must be equal to 1 with $2nC$ subtracted from it. Therefore the term $2nC$, in the equation $\frac{1}{n} = 1, 2nC$, must have the sign — prefixed to it: and consequently the terms $2nCx$ and Cx , in the final equation obtained in art. 41, and Cx^2 , the third term of the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (from which third term the term $2nC$ has been derived, by various multiplications and divisions, in the course of the preceding investigation), must likewise have the sign — prefixed to them. So that the said third term Cx^2 of the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5,$

$\&c$, which is equal to $\overline{1 + x}^{\frac{1}{n}}$, must be subtracted from the two first terms of the said series; and the three first terms of the said series will therefore be $1 + \frac{1}{n}x - Cxx$. Q. E. I.

And, to determine the magnitude of C , we shall have the equation $\frac{1}{n} = 1 - 2nC$; whence (adding $2nC$ to both sides) we shall have $\frac{1}{n} + 2nC = 1$; and (subtracting $\frac{1}{n}$ from both sides) $2nC = 1 - \frac{1}{n} = \frac{n-1}{n}$; and (dividing both sides by $2n$) $C = \frac{n-1}{n} \times \frac{1}{2n} = \frac{n-1}{2n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2n}$.

Therefore the three first terms of the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1 + x}^{\frac{1}{n}}$) will be $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx$, or $1 + \frac{1}{n}Ax - \left[\frac{n-1}{2n}\right]Bxx$. Q. E. I.

43. Secondly, to find the value of the co-efficient D , and the sign that is to be prefixed to Dx^3 , the fourth term of the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4,$

$Fx^5, \&c$, which is equal to $\overline{1 + x}^{\frac{1}{n}}$, we must proceed as follows.

Since it has been shewn in the last article that the sign — is to be prefixed to the terms $2nC$ and $2nCx$ and Cx , in the final equation obtained in art. 41, to wit, the equation $\frac{1}{n}, Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$\begin{array}{l} 2nC, \quad 3nDx, \quad 4nEx^2, \quad 5nFx^3, \quad \&c \\ + 1, \quad 2nCx, \quad 3nDx^2, \quad 4nEx^3, \quad \&c, \end{array}$$

accordingly prefixed to the said terms; and then the said equation will be as follows, to wit, $\frac{1}{n} - Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$\begin{array}{l} - 2nC, \quad 3nDx, \quad 4nEx^2, \quad 5nFx^3, \quad \&c \\ + 1 - 2nCx, \quad 3nDx^2, \quad 4nEx^3, \quad \&c. \end{array}$$

Now let $2Cx$ be added to both sides of this equation; and we shall then have

$$\frac{1}{n} +$$

$\frac{1}{n} + Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$\begin{aligned} & - 2nC, 3nDx, 4nEx^2, 5nFx^3, \&c \\ & + 1 - 2nC, 3nDx^2, 4nEx^3, \&c. \\ & + 2Cx \end{aligned}$$

But it was shewn in the last article that $\frac{1}{n}$ is $= 1 - 2nC$. Therefore, if we subtract $\frac{1}{n}$ and $1 - 2nC$ from the opposite sides of this equation, the remainders will be equal; that is, the series $+ Cx, Dx^2, Ex^3, Fx^4, \&c$, will be $=$ the compound series

$$\begin{aligned} & 3nDx, 4nEx^2, 5nFx^3, \&c \\ & - 2nC, 3nDx^2, 4nEx^3, \&c \\ & + 2Cx \end{aligned}$$

Therefore (dividing all the terms by x) we shall have $C, Dx, Ex^2, Fx^3, \&c =$ the compound series

$$\begin{aligned} & 3nD, 4nEx, 5nFx^2, \&c \\ & - 2nC, 3nDx, 4nEx^2, \&c \\ & + 2C \end{aligned}$$

This equation will be true, how small soever we may suppose x to be. And therefore it will be true also when x is $= 0$. But, when x is $= 0$, all the terms which involve x will be equal to 0 likewise; and consequently the equation will be $C = 3nD - 2nC + 2C$. Therefore, if we add $2nC$ to both sides, we shall have $C + 2nC = 3nD + 2C$; and (subtracting C from both sides) $2nC = 3nD + C$, or $2nC = C, 3nD$; that is, $2nC$ will be equal to C , together with $3nD$ either added to it, or subtracted from it, as may be necessary to produce the said equality. But, since n is a whole number, and consequently greater than 1, $2nC$ must be greater than C , and therefore cannot be equal to C with $3nD$ subtracted from it, but must be equal to C with $3nD$ added to it. Therefore the sign, $+$ must be prefixed to $3nD$, and consequently to the terms $3nDx$ and $3nDx^2$ in the last equation, and to the term Dx^2 in the final equation obtained in art. 41; and likewise to the fourth term, Dx^3 , of the series $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal

to $\sqrt[n]{1+x}$; from which fourth term Dx^3 all the other terms that involve D were derived, by the operations of multiplication and division, in the course of the foregoing investigation. Therefore the four first terms of the said series 1

$$\begin{aligned} & \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c, \text{ which is equal to } \sqrt[n]{1+x}, \text{ will be } 1 + \\ & \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + Dx^3, \text{ or } 1 + \frac{1}{n}x - Cxx + Dx^3, \text{ or } 1 + \frac{1}{n}Ax \\ & - \frac{n-1}{2n}Bxx + Dx^3. \end{aligned}$$

Q. E. I.

And, to determine the magnitude of D , we shall have the equation $2nC = C + 3nD$; whence $3nD$ will be $= 2nC - C = \overline{2n-1} \times C$, and D will

wise be prefixed to $E x^4$, the fifth term of the assumed series $1 + \frac{1}{n} x, C x^2,$

$D x^3, E x^4, F x^5, \&c$ (which is equal to $\overline{1 + x}^{\frac{1}{n}}$); from which fifth term the said term $4 n E$ is derived, by the operations of multiplication and division, in the course of the foregoing investigations. Therefore the said fifth term, $E x^4$, of the said series, must be subtracted from the first term 1 ; and consequently the five first terms of the said series will be $1 + \frac{1}{n} x - C x^2 + D x^3 - E x^4$, or $1 + \frac{1}{n} A x - \left[\frac{n-1}{2n} B x^2 + \frac{2n-1}{3n} C x^3 - E x^4 \right]$. Q. E. I.

And, to determine the magnitude of the co-efficient E , we shall have the equation $+ D = 3 n D - 4 n E$; whence $D + 4 n E$ will be $= 3 n D$, and $4 n E$ will be $= 3 n D - D = \overline{3 n - 1} \times D$, and E will be $= \frac{3 n - 1}{4 n} \times D = \frac{3 n - 1}{4 n} \times \frac{2 n - 1}{3 n} \times \frac{n-1}{2 n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2 n} \times \frac{2 n - 1}{3 n} \times \frac{3 n - 1}{4 n}$. Therefore the four first terms of the series $1 + \frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal

to $\overline{1 + x}^{\frac{1}{n}}$, will be $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2 n} x^2 + \frac{1}{n} \times \frac{n-1}{2 n} \times \frac{2 n - 1}{3 n} x^3 - \frac{1}{n} \times \frac{n-1}{2 n} \times \frac{2 n - 1}{3 n} \times \frac{3 n - 1}{4 n} x^4$, or $1 + \frac{1}{n} A x - \left[\frac{n-1}{2 n} B x^2 + \frac{2 n - 1}{3 n} C x^3 - \frac{3 n - 1}{4 n} D x^4 \right]$. Q. E. I.

45. To determine the sign that is to be prefixed to $F x^5$, the sixth term of the series $1 + \frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\overline{1 + x}^{\frac{1}{n}}$, and to find the magnitude of the co-efficient F , we must proceed as follows.

In the last article we obtained the equation $+ D, E x, F x^2, G x^3, H x^4, \&c =$ the compound series

$$4 n E, 5 n F x, 6 n G x^2, 7 n H x^3, \&c \\ + 3 n D, 4 n E x, 5 n F x^2, 6 n G x^3, \&c$$

or (if we prefix the sign $-$ to the terms $4 n E, 4 n E x$, and $E x$, agreeably to what was found in the last article to be necessary)

$$+ D - E x, F x^2, G x^3, H x^4, \&c = \text{the compound series} \\ - 4 n E, 5 n F x, 6 n G x^2, 7 n H x^3, \&c \\ + 3 n D - 4 n E x, 5 n F x^2, 6 n G x^3, \&c.$$

Therefore, if we add $2 E x$ to both sides of this equation, we shall have

$$+ D + E x, F x^2, G x^3, H x^4, \&c = \text{the compound series} \\ - 4 n E, 5 n F x, 6 n G x^2, 7 n H x^3, \&c \\ + 3 n D - 4 n E x, 5 n F x^2, 6 n G x^3, \&c \\ + 2 E x.$$

And it has also been shewn in the last article, that D is $= 3 n D - 4 n E$. Therefore, if we subtract D and $3 n D - 4 n E$ from the opposite sides of the

last equation, the remainders will be equal; that is, the simple series $E x$, $F x^2$, $G x^3$, $H x^4$, &c will be equal to the compound series

$$\begin{aligned} & 5 n F x, \quad 6 n G x^2, \quad 7 n H x^3, \quad \&c \\ & - 4 n E x, \quad 5 n F x^2, \quad 6 n G x^3, \quad \&c \\ & + 2 E x \end{aligned} \quad ; \text{ and consequently}$$

(dividing all the terms by x) the simple series E , $F x$, $G x^2$, $H x^3$, &c will be equal to the compound series

$$\begin{aligned} & 5 n F, \quad 6 n G x, \quad 7 n H x^2, \quad \&c \\ & - 4 n E, \quad 5 n F x, \quad 6 n G x^2, \quad \&c \\ & + 2 E. \end{aligned}$$

And this equation is true, of how small a magnitude soever we suppose x to be taken: and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will then be $E = 5 n F - 4 n E + 2 E$. Therefore (adding $4 n E$ to both sides) we shall have $E + 4 n E = 5 n F + 2 E$, and (subtracting E from both sides) we shall have $4 n E = 5 n F + E$, or $4 n E = E + 5 n F$; that is $4 n E$ will be equal to E , together with $5 n F$ either added to it or subtracted from it, as may be necessary to produce such equality. But, because $4 n$ is greater than 1, $4 n E$ must be greater than E . Therefore, in order to make E be equal to $4 n E$, it will be necessary to add $5 n F$ to it; and consequently the sign $+$ must be prefixed to $5 n F$, and the last equation $4 n E = E + 5 n F$ will be $4 n E = E + 5 n F$. Therefore the sign $+$ must also be prefixed to the sixth term, $F x^5$, of the assumed series $1 + \frac{1}{n} x$, $C x^2$, $D x^3$, $E x^4$,

$F x^5$, $G x^6$, $H x^7$, &c, which is equal to $1 + x \sqrt[n]{\frac{1}{n}}$; from which sixth term the term $5 n F$ has been derived by the operations of multiplication and division in the course of this investigation. Therefore the said sixth term, $F x^5$, of the said series is to be added to the first term 1, and consequently the six first terms of the said series will be $1 + \frac{1}{n} x - C x^2 + D x^3 - E x^4 + F x^5$, or $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} x x + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5$, or $1 + \frac{1}{n} A x - \sqrt[n-1]{\frac{n-1}{2n}} B x^2 + \frac{2n-1}{3n} C x^3 - \frac{3n-1}{4n} D x^4 + F x^5$. Q. E. I.

And to determine the magnitude of F , we shall have the aforesaid equation $4 n E = E + 5 n F$; whence $5 n F$ will be $= 4 n E - E = \overline{4 n - 1} \times E$, and F will be $= \frac{4 n - 1}{5 n} \times E = \frac{4 n - 1}{5 n} \times \frac{3 n - 1}{4 n} \times \frac{2 n - 1}{3 n} \times \frac{n - 1}{2 n} \times \frac{1}{n}$, or $\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}$. Therefore the first six terms of the series $1 + \frac{1}{n} x$, $C x x$, $D x^3$, $E x^4$, $F x^5$, $G x^6$, $H x^7$, &c, which is equal to

$$\overline{1 + x \sqrt[n]{\frac{1}{n}}}, \text{ will be } 1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} x x + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5$$

$$\times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5, \text{ or}$$

$$1 + \frac{1}{n} A x - \left[\frac{n-1}{2n} B x^2 + \frac{2n-1}{3n} C x^3 - \left[\frac{3n-1}{4n} D x^4 + \frac{4n-1}{5n} E x^5 \right. \right.$$

Q. E. I.

46. The reader will observe, that in this way of investigating the values of C, D, E, F, &c, there is no more difficulty in determining the values of E and F, than in determining the values of C and D; because the simple equations, by the resolution of which the several new co-efficients are to be determined, consist always of only three terms, to wit, of one term derived from a term of the simple series $\frac{1}{n}$, C x, D x², E x³, F x⁴, &c, (which forms the left-hand side of the final equation obtained in art. 41,) by dividing it by the power of x with which it involved, and of two terms derived in like manner from two terms of the compound series

$$2n C, 3n D x, 4n E x^2, 5n F x^3, \&c$$

$$+ 1, 2n C x, 3n D x^2, 4n E x^3, \&c \text{ (which}$$

forms the right-hand side of the same final equation), by dividing them by the same power of x: whereas, in the former way of investigating the values of those co-efficients, set forth in art. 31, 32, and 33, the several simple equations, by the resolution of which the said co-efficients are to be determined, to wit, the equations $1 = n B$, and $c B^2 - n C = 0$, or $c B^2 = n C$, and $n D - 2 c B C + d B^3 = 0$, or $n D + d B^3 = 2 c B C$, and $n E + 2 c B D + 2 c C^2 - 3 d B^2 C + e B^4 = 0$, or $n E + 2 c B D + 2 c C^2 + e B^4 = 3 d B^2 C$, and $n F - 2 c B E - 2 c C D + 3 d B^2 D + 3 d B C^2 - 4 e B^3 C + f B^5 = 0$, or $n F + 3 d B^2 D + 3 d B C^2 + f B^5 = 2 c B E + 2 c C D + 4 e B^3 C$, increase continually in the number of their terms by the addition of two terms in every new equation; and likewise consist of terms that are more and more complicated continually, so as soon to make the labour of resolving them become intolerable. Therefore the present method of investigating these co-efficients is, in a practical view, very much to be preferred to the former method; though the reasonings used in that former method seem to be rather more direct and simple than those which we were obliged to have recourse to in art. 37, 38, 39, 40, and 41, in order to obtain the final equation set forth in the last of those articles. Both the methods therefore have their separate merits, and are worthy of our attention; and the former method serves as a proper introduction to the latter.

47. From the manner in which the grand final equation, $\frac{1}{n}$, C x, D x², E x³, F x⁴, &c = the compound series,

$$2n C, 3n D x, 4n E x^2, 5n F x^3, \&c$$

$$+ 1, 2n C x, 3n D x^2, 4n E x^3, \&c, \text{ in}$$

art. 41, was obtained (which was by the multiplication of the series $1, 2n C x, 3n D x^2, 4n E x^3, 5n F x^4, \&c$ into $1 + x$ in art. 40), it is easy to see that the several equations for determining the values of the following co-efficients G,

2 H 2

H, I,

H, I, K, L, &c (which we have not yet investigated), and the signs + and —, that are to be prefixed to them, would be as follows.

The first equation would be $F = 6nG$, $5nF$, or $F = 5nF$, $6nG$, or (since we have already seen in art. 45, that $5nF$ and F must have the sign + prefixed to them) $+F = +5nF$, $6nG$, and consequently (in order to make the said equation possible) $+F = 5nF - 6nG$; whence it will follow that Gx^6 , or the seventh term of the assumed series $1 + \frac{1}{n}x$, Cx^2 , Dx^3 ,

Ex^4 , Fx^5 , &c (which is equal to $\overline{1 + x}^{\frac{1}{n}}$), must be marked with the sign —, or be subtracted from the first term 1, and that G will be $= \frac{5n-1}{6n} \times F$.

And the second equation would be $G = 7nH$, $6nG$, or (because it has just now been shewn that the sign — is to be prefixed to $6nG$ and G) $-G = 7nH - 6nG$, and consequently (in order to make the said equation possible) $-G = +7nH - 6nG$; whence it will follow that Hx^7 , or the eighth term of the assumed series $1 + \frac{1}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c (which

is equal to $\overline{1 + x}^{\frac{1}{n}}$), will be marked with the sign +, or added to the first term 1, and that H will be $= \frac{6n-1}{7n} \times G$.

And the third equation would be $H = 8nI$, $7nH$, or $H = 7nH$, $8nI$, or (because it has been just now shewn that the terms $7nH$ and H must have the sign + prefixed to them), $+H = +7nH$, $8nI$, and consequently (in order to make the said equation possible,) $+H = 7nH - 8nI$; whence it will follow that Ix^8 , or the ninth term of the series $1 + \frac{1}{n}x$, Cx^2 , Dx^3 , Ex^4 ,

Fx^5 , &c (which is equal to $\overline{1 + x}^{\frac{1}{n}}$), will be marked with the sign —, or subtracted from the first term 1, and that I will be $= \frac{7n-1}{8n} \times H$.

And the fourth equation would be $I = 9nK$, $8nI$, or (because it has been just now shewn that the terms $8nI$ and I must have the sign — prefixed to them) $-I = 9nK - 8nI$, and consequently (in order to make the said equation possible,) $-I = +9nK - 8nI$; whence it will follow that Kx^9 , or the tenth term of the series $1 + \frac{1}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c (which is

equal to $\overline{1 + x}^{\frac{1}{n}}$), will be marked with the sign +, or must be added to the first term 1, and that K will be $= \frac{8n-1}{9n} \times I$.

And the 5th equation would be $K = 10nL$, $9nK$, or because it has been just now shewn that the terms $9nK$ and K must have the sign + prefixed to them) $+K = 10nL + 9nK$, or $+K = +9nK$, $10nL$, and consequently (in order to make the said equation possible) $+K = +9nK - 10nL$; whence it will follow that Lx^{10} , or the eleventh term of the series $1 +$

$\frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, &c$ (which is equal to $\sqrt[n]{1+x}$), will be marked with the sign $-$, or subtracted from the first term 1, and that L will be $= \frac{9^n - 1}{10n} \times K$.

And the like short and easy equations,

$$\begin{aligned} -L &= -10nL + 11nM, \\ +M &= +11nM - 12nN, \\ -N &= -12nN + 13nO, \\ +O &= +13nO - 14nP, \\ -P &= -14nP + 15nQ, \end{aligned}$$

&c, would be found for the determination of the values and the signs of the following co-efficients M, N, O, P, Q, &c, *ad infinitum*, or as far as we pleased to continue the investigation. We may therefore now conclude with certainty,

that the quantity $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$, is equal to the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} x^5 - &c$, or $1 + \frac{1}{n}Ax - \left[\frac{n-1}{2n}\right]Bx^2 + \frac{2n-1}{3n}Cx^3 - \left[\frac{3n-1}{4n}\right]Dx^4 + \frac{4n-1}{5n}Ex^5 - \left[\frac{5n-1}{6n}\right]Fx^6 + \frac{6n-1}{7n}Gx^7 - \left[\frac{7n-1}{8n}\right>Hx^8 + \frac{8n-1}{9n}Ix^9 - \left[\frac{9n-1}{10n}\right>Kx^{10} + &c$ *ad infinitum*, in which the signs of the terms that come after the second term $\frac{1}{n}x$ are alternately $-$ and $+$, and the law of the generation, or continuation of the terms one from another, is very manifest, every new generating fraction of the co-efficients C, D, E, F, G, H, I, K, L, &c, being derived from the generating fraction which immediately preceeds it, by the addition of n to both its numerator and its denominator.

Q. E. I.

48. This series agrees exactly with that which was set forth in art. 2, as being equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, and which had been derived from the binomial theorem in the case of integral powers.

A Review of the foregoing Investigation of the Series which is equal to

$$\sqrt[n]{1+x}, \text{ or } \sqrt[n]{1+x}.$$

49. I have now completed the second investigation of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, &c$, or $1 + \frac{1}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, &c$, which is equal

equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$. The ease with which it enables us to find the values and the signs of the several co-efficients C, D, E, F, G, H, I, K, L, &c, gives it a great superiority over the former investigation explained above in art. 28, 29, 30, &c—34, and raises it to an equality with any method of investigating this series that I have yet seen. Yet, from the number of the steps which it contains before we obtain the final equation in art. 41, it may, perhaps, be thought to be rather too abstruse and difficult; though none of the steps in it, taken separately, seem to deserve that character. To explain it, therefore, as fully as I can to the reader, and make the connection of the reasonings used in it as apparent as possible, I will here state them over again with somewhat more brevity than at first, in order to bring all the parts of the investigation into view at once.

Now the principal artifice of this investigation consists in finding two different expressions for the value of $\sqrt[n]{1+x+d}$. The first of these expressions is obtained by considering the trinomial quantity $1+x+d$ as being a binomial quantity, of which the first member is 1, and the second is $x+d$. By supposing y to be equal to $x+d$, and consequently $1+y$ to be equal to $1+x+d$, and assuming the series $1 + \frac{1}{n}y, Cy^2, Dy^3, Ey^4, Fy^5, \&c$, for

the value of $\sqrt[n]{1+y}$ (as we before assumed the series $1 + \frac{1}{n}x, Cx^2, Dx^3,$

$Ex^4, Fx^5, \&c$ for the value of $\sqrt[n]{1+x}$), and then substituting in the said series $1 + \frac{1}{n}y, Cy^2, Dy^3, Ey^4, Fy^5, \&c$, instead of $y, y^2, y^3, y^4, y^5, \&c$, the like powers of $x+d$, to wit, $x+d, x+d, x+d, x+d, x+d, \&c$, or $x+d, x^2+2xd+d^2, x^3+3x^2d+3xd^2+d^3, x^4+4x^3d+6x^2d^2+4xd^3+d^4, x^5+5x^4d+10x^3d^2+10x^2d^3+5xd^4+d^5,$

&c, we obtained the equation, $\sqrt[n]{1+x+d} =$ the compound series

$$\begin{array}{cccccc} 1 + \frac{1}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\ + \frac{1}{n}d, & 2Cx d, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\ & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\ & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\ & & & Ed^4, & 5Fxd^4, & \&c \\ & & & & Fd^5, & \&c. \end{array}$$

The second expression of the value of $\sqrt[n]{1+x+d}$ was obtained by considering the trinomial quantity $1+x+d$ as being a binomial quantity of which $1+x$ is the first member, and d the second. By supposing f to be equal to $1+x$, and consequently $f+d$ to be equal to $1+x+d$, and therefore $f \times \sqrt[n]{1+\frac{d}{f}}$ (which is $= f+d$) to be also equal to $1+x+d$, and assuming the series

series $1 + \frac{1}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c$, for the value of $\sqrt[n]{1 + \frac{d}{f}}$ (as we before assumed the series $1 + \frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, for the value of $\sqrt[n]{1 + x}$, and the series $1 + \frac{1}{n} y, C y^2, D y^3, E y^4, F y^5, \&c$, for the value of $\sqrt[n]{1 + y}$), we found $\sqrt[n]{1 + x + \frac{d}{f}}$ to be equal to $f^{\frac{1}{n}} \times$ the series $1 + \frac{1}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c$, and consequently to $f^{\frac{1}{n}} \times$ the series $1 + \frac{1}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c$, and therefore to the series $f^{\frac{1}{n}} + \frac{1}{n} \times f^{\frac{1}{n}} \times \frac{d}{f}, C \times f^{\frac{1}{n}} \times \frac{d^2}{f^2}, D \times f^{\frac{1}{n}} \times \frac{d^3}{f^3}, E \times f^{\frac{1}{n}} \times \frac{d^4}{f^4}, F \times f^{\frac{1}{n}} \times \frac{d^5}{f^5}, \&c$, that is (if we substitute $1 + x$ in the terms of this series instead of f , which is equal to it), to the series $\sqrt[n]{1 + x} + \frac{1}{n} \times \sqrt[n]{1 + x} \times \frac{d}{1 + x}, C \times \sqrt[n]{1 + x} \times \frac{d^2}{(1 + x)^2}, D \times \sqrt[n]{1 + x} \times \frac{d^3}{(1 + x)^3}, E \times \sqrt[n]{1 + x} \times \frac{d^4}{(1 + x)^4}, F \times \sqrt[n]{1 + x} \times \frac{d^5}{(1 + x)^5}, \&c$.

We then equated this latter expression of the value of $\sqrt[n]{1 + x + \frac{d}{f}}$ to the former, to wit, the compound series

$$\begin{array}{ccccccc} 1 + \frac{1}{n} x, & C x^2, & D x^3, & E x^4, & F x^5, & \&c \\ + \frac{1}{n} d, & 2 C x d, & 3 D x^2 d, & 4 E x^3 d, & 5 F x^4 d, & \&c \\ & C d, & 3 D x d^2, & 6 E x^2 d^2, & 10 F x^3 d^2, & \&c \\ & & D x^3, & 4 E x d^3, & 10 F x^2 d^3, & \&c \\ & & & E d^4, & 5 F x d^4, & \&c \\ & & & & F d^5, & \&c; \text{ and} \end{array}$$

then subtracted $\sqrt[n]{1 + x}$ from the left-hand side of this last equation, and the series $1 + \frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$ (which is equal to $\sqrt[n]{1 + x}$), from the right-hand side of the same equation; whereby we obtained the equation $\frac{1}{n}$

$$\times \sqrt[n]{1 + x} \times \frac{d}{1 + x}, C \times \sqrt[n]{1 + x} \times \frac{d^2}{(1 + x)^2}, D \times \sqrt[n]{1 + x} \times \frac{d^3}{(1 + x)^3}, E \times \sqrt[n]{1 + x} \times \frac{d^4}{(1 + x)^4}, F \times \sqrt[n]{1 + x} \times \frac{d^5}{(1 + x)^5}, \&c = \text{the compound series}$$

$$\frac{1}{n} d,$$

$$\begin{array}{cccccc}
 \frac{1}{n} d, & 2 C x d, & 3 D x^2 d, & 4 E x^3 d, & 5 F x^4 d, & \&c \\
 & C d^2, & 3 D x d^2, & 6 E x^2 d^2, & 10 F x^3 d^2, & \&c \\
 & & D d^3, & 4 E x d^3, & 10 F x^2 d^3, & \&c \\
 & & & E d^4, & 5 F x d^4, & \&c \\
 & & & & F d^5, & \&c;
 \end{array}$$

and (by dividing all the terms by d) the equation

$$\frac{1}{n} \times \frac{1}{1+x} \times \frac{1}{1+x} \times \frac{1}{1+x}, C \times \frac{1}{1+x} \times \frac{1}{1+x} \times \frac{d}{1+x}, D \times \frac{1}{1+x} \times \frac{1}{1+x} \times \frac{d^2}{1+x}, E \times \frac{1}{1+x} \times \frac{1}{1+x} \times \frac{d^3}{1+x}, F \times \frac{1}{1+x} \times \frac{1}{1+x} \times \frac{d^4}{1+x}, \&c = \text{the compound series}$$

$$\begin{array}{cccccc}
 \frac{1}{n}, & 2 C x, & 3 D x^2, & 4 E x^3, & 5 F x^4, & \&c \\
 & C d, & 3 D x d, & 6 E x^2 d, & 10 F x^3 d, & \&c \\
 & & D d^2, & 4 E x d^2, & 10 F x^2 d^2, & \&c \\
 & & & E d^3, & 5 F x d^3, & \&c \\
 & & & & F d^4, & \&c; \text{ and (by}
 \end{array}$$

supposing d to become $= 0$, and consequently all the terms that involve d to

become equal to 0 likewise) the equation $\frac{1}{n} \times \frac{1}{1+x} \times \frac{1}{1+x} = \text{the series}$

$\frac{1}{n}, 2 C x, 3 D x^2, 4 E x^3, 5 F x^4, \&c$; and (by multiplying both sides into n)

the equation $\frac{1}{1+x} \times \frac{1}{1+x} = 1, 2n C x, 3n D x^2, 4n E x^3, 5n F x^4, \&c$;

and, lastly (by multiplying both sides into $1+x$), the equation $\frac{1}{1+x} \times \frac{1}{1+x}$

$$\begin{array}{cccccc}
 = & 1, & 2n C x, & 3n D x^2, & 4n E x^3, & 5n F x^4, & \&c \\
 + & x, & 2n C x^2, & 3n D x^3, & 4n E x^4, & \&c.
 \end{array}$$

Having thus obtained a second expression of the value of $\frac{1}{1+x}$, involving the several powers of x , to wit, $x, x^2, x^3, x^4, x^5, \&c$, in their natural order, without interruption, and involving likewise the co-efficients $C, D, E, F, \&c$, of the powers of x in the series $1 + \frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, that was

at first assumed for the value of $\frac{1}{1+x}$, we equated these two expressions of the value of $\frac{1}{1+x}$ to each other, and thereby obtained the equation $1 + \frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c = \text{the compound series}$

$$\begin{array}{cccccc}
 1, & 2n C x, & 3n D x^2, & 4n E x^3, & 5n F x^4, & \&c \\
 + & x, & 2n C x^2, & 3n D x^3, & 4n E x^4, & \&c;
 \end{array}$$

and (by subtracting 1 from both sides) the equation $\frac{1}{n} x, C x^2, D x^3, E x^4, F x^5, \&c = \text{the compound series}$

$$2n C x,$$

$$2nCx, 3nDx^2, 4nEx^3, 5nFx^4, \&c \\ + x, 2nCx^2, 3nDx^3, 4nEx^4, \&c;$$

and, lastly (by dividing all the terms by x), the equation $\frac{1}{n}, Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$2nC, 3nDx, 4nEx^2, 5nFx^3, \&c \\ + 1, 2nC, 3nDx^2, 4nEx^3, \&c;$$

which is the grand final equation from which we are to deduce the several particular equations (consisting of only three terms each), by means of which the signs of the terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, of the series $1 + \frac{1}{n}x, Cx^2, Dx^3,$

$Ex^4, Fx^5, \&c$ (which is equal to $\sqrt[n]{1+x}$), and the magnitudes of the coefficients $C, D, E, F, \&c$, are to be determined.

50. These were the several steps of the foregoing investigation, which, the reader will probably observe, bear some resemblance to Mr Landen's investiga-

tion of the more general series that is equal to $\sqrt[n]{1+x}$, which has been printed above in this collection, in pages 170, 171, — 175, and of which an explanation has been given in pages 176, 177, 178, — 193. And I readily acknowledge that it was after a very close and attentive perusal and consideration of that investigation of Mr Landen, and in consequence of the ideas which that perusal suggested to me, that I discovered the investigation that has been here explained. It seems, however, to be in many respects different from that of Mr Landen, though it had been originally suggested by it; and, in particular, I hope it will be found much less abstruse and difficult.

An Example to the foregoing Series.

51. Let it be required to find, by means of the foregoing series, the cube-root of the binomial quantity $1+x$, or the value of $\sqrt[3]{1+x}$.

Here n is $= 3$. We shall therefore have

$$B = \frac{1}{n} \quad A = \frac{1}{3} A, \\ \text{and } C = \frac{n-1}{2n} B = \frac{3-1}{2 \times 3} B = \frac{2}{6} B, \\ \text{and } D = \frac{2n-1}{3n} C = \frac{6-1}{9} C = \frac{5}{9} C, \\ \text{and } E = \frac{3n-1}{4n} D = \frac{9-1}{12} D = \frac{8}{12} D, \\ \text{and } F = \frac{4n-1}{5n} E = \frac{12-1}{15} E = \frac{11}{15} E, \\ \text{and } G = \frac{5n-1}{6n} F = \frac{15-1}{18} F = \frac{14}{18} F, \\ \text{and } H = \frac{6n-1}{7n} G = \frac{18-1}{21} G = \frac{17}{21} G, \\ \text{and } I = \frac{7n-1}{8n} H = \frac{21-1}{24} H = \frac{20}{24} H,$$

$$\text{and } K = \frac{8n-1}{9n} I = \frac{24-1}{27} I = \frac{23}{27} I,$$

$$\text{and } L = \frac{9n-1}{10n} K = \frac{27-1}{30} K = \frac{26}{30} K,$$

$$\text{and } M = \frac{10n-1}{11n} L = \frac{30-1}{33} L = \frac{29}{33} L,$$

$$\text{and } N = \frac{11n-1}{12n} M = \frac{33-1}{36} M = \frac{32}{36} M,$$

$$\text{and } O = \frac{12n-1}{13n} N = \frac{36-1}{39} N = \frac{35}{39} N,$$

$$\text{and } P = \frac{13n-1}{14n} O = \frac{39-1}{42} O = \frac{38}{42} O,$$

$$\text{and } Q = \frac{14n-1}{15n} P = \frac{42-1}{45} P = \frac{41}{45} P,$$

$$\text{and } R = \frac{15n-1}{16n} Q = \frac{45-1}{48} Q = \frac{44}{48} Q,$$

$$\text{and } S = \frac{16n-1}{17n} R = \frac{48-1}{51} R = \frac{47}{51} R,$$

$$\text{and } T = \frac{17n-1}{18n} S = \frac{51-1}{54} S = \frac{50}{54} S.$$

Therefore the series $I + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \frac{6n-1}{7n} Gx^7 - \frac{7n-1}{8n} Hx^8 + \frac{8n-1}{9n} Ix^9 - \frac{9n-1}{10n} Kx^{10} + \frac{10n-1}{11n} Lx^{11} - \frac{11n-1}{12n} Mx^{12} + \frac{12n-1}{13n} Nx^{13} - \frac{13n-1}{14n} Ox^{14} + \frac{14n-1}{15n} Px^{15} - \frac{15n-1}{16n} Qx^{16} + \frac{16n-1}{17n} Rx^{17} - \frac{17n-1}{18n} Sx^{18} + \&c$ will, in this case, be $= I + \frac{1}{3} Ax - \frac{2}{6} Bx^2 + \frac{5}{9} Cx^3 - \frac{8}{12} Dx^4 + \frac{11}{15} Ex^5 - \frac{14}{18} Fx^6 + \frac{17}{21} Gx^7 - \frac{20}{24} Hx^8 + \frac{23}{27} Ix^9 - \frac{26}{30} Kx^{10} + \frac{29}{33} Lx^{11} - \frac{32}{36} Mx^{12} + \frac{35}{39} Nx^{13} - \frac{38}{42} Ox^{14} + \frac{41}{45} Px^{15} - \frac{44}{48} Qx^{16} + \frac{47}{51} Rx^{17} - \frac{50}{54} Sx^{18} + \&c = I + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19683} - \frac{935x^8}{59049} + \frac{21,505x^9}{1,594,323} - \frac{55,913x^{10}}{4,782,969} + \frac{147,407x^{11}}{14,348,907} - \frac{1,179,256x^{12}}{129,140,163} + \frac{3,174,920x^{13}}{387,420,489} - \frac{8,617,640x^{14}}{1,162,261,467} + \frac{70,664,648x^{15}}{10,460,353,203} - \frac{194,327,782x^{16}}{31,381,059,609} + \frac{537,259,162x^{17}}{94,143,178,827} - \frac{13,431,479,050x^{18}}{2,541,865,828,329} + \&c$; which therefore, is $= \sqrt[3]{1+x}$, or the cube-root of the binomial quantity $I + x$.

Q. E. I.

52. Note. This series for expressing the value of $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $I + x$, and a similar series for expressing the value of $\sqrt[3]{1-x}$, or the cube-root of the residual quantity $I - x$, will enable us to

* The co-efficients of $x^{13}, x^{14}, x^{15}, x^{16}, x^{17}, x^{18}$ here given are equal to the co-efficients of the same powers of x given above in page 192, but are reduced to somewhat smaller numbers by dividing both the numerators and denominators of the said former co-efficients by 13 and 17.

extend

extend the well-known rules for resolving cubick equations of these two forms, to wit, $x^3 + qx = r$ and $x^3 - qx = r$, called *Cardan's Rules*, to that case of the second of these equations which they naturally are not fitted to solve, and which therefore has obtained the name of *the irreducible case*; which is the case in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$. This case must be subdivided into two branches, according as $\frac{rr}{4}$ is greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, (though less than $\frac{q^3}{27}$) and as it is less than $\frac{q^3}{54}$. When $\frac{rr}{4}$ is greater than $\frac{q^3}{54}$, though less than $\frac{q^3}{27}$, the equation $x^3 - qx = r$ may be resolved by extending to it, by a certain peculiar train of reasoning, the second of Cardan's rules, or that by which the same equation is to be resolved when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; which extension may be performed by the help of the two infinite serieses that express the cube-roots of the binomial quantity $1 + x$ and the residual quantity $1 - x$, in a manner that is explained at considerable length in a Paper of mine published in the Philosophical Transactions for the year 1778, without any mention of impossible roots, or impossible quantities of any kind, or even of negative quantities. But when $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, the method explained in that Paper will not enable us to find the value of x in that equation, because the series obtained for the said value will not be a converging series. In this branch therefore of the irreducible case of the equation $x^3 - qx = r$, we are obliged to have recourse to another method of proceeding, and to determine the value of x by deriving it from that of the lesser of the two roots of the opposite equation $qx - x^3 = r$, which lesser root may be obtained by a similar extension of Cardan's first rule, or that by which he finds the root of the equation $x^3 + qx = r$, or $qx + x^3 = r$, to the equation $qx - x^3 = r$, by means of the said two serieses for expressing the cube-roots of $1 + x$ and $1 - x$; which extension may be made in a clear and intelligible manner, without any mention of impossible roots, or other impossible quantities, or even of negative quantities, any more than in the extension of Cardan's other rule. But this extension of Cardan's first rule to the resolution of the equation $qx - x^3 = r$ has not yet been published.

Of the Binomial Theorem in the case of $\sqrt[n]{1+x}$,
or of the n th root of the m th power of the bi-
nomial quantity $1+x$, or the m th power of
its n th root, when m is any whole number
whatsoever, and n any other whole number
greater than m .

53. Having now shewn in a pretty full, and, I hope, satisfactory manner, that the quantity $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$,

$1 + x$, is equal to the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 - \&c \text{ ad infinitum}$, or $1 + \frac{1}{n}Ax - \sqrt{\frac{n-1}{2n}}Bx^2 + \frac{2n-1}{3n}Cx^3 - \sqrt{\frac{3n-1}{4n}}Dx^4 + \frac{4n-1}{5n}Ex^5 - \sqrt{\frac{5n-1}{6n}}Fx^6 + \frac{6n-1}{7n}Gx^7 - \sqrt{\frac{7n-1}{8n}}Hx^8 + \frac{8n-1}{9n}Ix^9 - \sqrt{\frac{9n-1}{10n}}Kx^{10} + \&c, \text{ ad infinitum}$, agreeably to what was asserted in

art. 2; I shall now proceed to consider the quantity $\sqrt[n]{1+x}$, or the n th root of the n th power of the binomial quantity $1+x$, or (which comes to the same thing) the n th power of its n th root, in the first case of it, or when n , the index of the root, is greater than m the index of the power, and shall endeavour to shew that

in this case, the said quantity $\sqrt[n]{1+x}$ will be equal to the series $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5 - \&c \text{ ad infinitum}$, or (putting $A = 1$, and $B = \frac{m}{n}$, or the co-efficient of x , and $C = \frac{m}{n} \times \frac{n-m}{2n}$, or the co-efficient of x^2 , and $D, E, F, G, H, \&c$, for the co-efficients of x^3, x^4, x^5, x^6, x^7 , and the following powers of x , in the fourth, fifth, sixth, seventh, eighth, and other following terms of the series,) to the series $1 + \frac{m}{n}Ax - \sqrt{\frac{n-m}{2n}}Bx^2 + \frac{2n-m}{3n}Cx^3 - \sqrt{\frac{3n-m}{4n}}Dx^4 + \frac{4n-m}{5n}Ex^5 - \sqrt{\frac{5n-m}{6n}}Fx^6 + \frac{6n-m}{7n}Gx^7 - \sqrt{\frac{7n-m}{8n}}Hx^8 + \frac{8n-m}{9n}Ix^9 - \sqrt{\frac{9n-m}{10n}}Kx^{10} + \&c, \text{ ad infinitum}$, agreeably to what was asserted in art. 3. Now this may be done by methods of reasoning exactly similar to those which we have already made use of in finding the foregoing series, which is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$. These methods may be explained as follows.

Observations preparatory to the Investigation of the Series set forth in the foregoing article as

being equal to the quantity $\sqrt[n]{1+x}$, when n is greater than m .

54. In the first place, we must observe that the quantity $\sqrt[n]{1+x}$, or the n th power of the n th root of the binomial quantity $1+x$, must be equal to a series

series of quantities of the same form with the series that is equal to $\sqrt[n]{1+x}$, or the n th root itself of the same binomial quantity, that is, to a series of the form $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, of which the first term is 1 , and the second and third, and other following terms, involve in them the several powers of x , to wit, $x, x^2, x^3, x^4, x^5, \&c$, in their natural order without any interruption.

For, if a series of the said form $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, be multiplied into itself any number of times, or raised to any power denoted by the whole number m , it is evident that the product, or m th power of the said series, must always be a series of the same form, or a series of which 1 will be the first term, and of which the second and third, and other following terms, will involve the several powers of x in their natural order without any interruption. And

therefore, since $\sqrt[n]{1+x}$ has been shewn to be equal to a series of the said form $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, it follows that $\sqrt[n]{1+x}^m$, or the m th power of $\sqrt[n]{1+x}$, must also be equal to a series of the same form, $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$.
Q. E. D.

55. In the second place, we must observe that the second term, Bx , of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x}^m$, must be added to the first term 1 , and consequently marked with the sign $+$.

For, since $1+x$ is greater than 1 , the n th root of $1+x$, or the first of $n-1$ geometrical mean proportionals between 1 and $1+x$, must also be greater than 1 ; and therefore, *à fortiori*, the m th power of the said n th root (which is greater than the said n th root itself) must also be greater than 1 . Consequently the

series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x}^m$, must also be greater than 1 . And this must be true, of however small a magnitude we suppose x to be taken, so long as it has any magnitude at all. But, in order that the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, may be always greater than 1 , however small the magnitude of x be taken, it is necessary that the second term Bx should be added to the first term 1 . For, if Bx were subtracted from 1 , it would be possible, by diminishing x , to make all the terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, that come after the second term Bx , become so much less than Bx , that the sum of them all put together should be less than Bx , whatever might be the magnitudes of the numeral co-efficients $C, D, E, F, \&c$; in which case it is evident that the whole series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, would (even though all the terms after Bx should be added to the first term 1 , and the series should consequently be $1 - Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \&c$) be less than the first term 1 ; which, we have shewn, it never can be. Therefore Bx cannot be subtracted from 1 , but must be added to it; and the series that is

equal to $\sqrt[n]{1+x}^m$, must consequently be of this form, $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$.
Q. E. D.

56. The

56. The truth of the foregoing observation, to wit, that the second term, Bx , of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x}$, is to be added to the first term 1 , and not subtracted from it, will likewise appear by raising a few of the powers of the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c$

(which is equal to $\sqrt[n]{1+x}$), by actual multiplication. For we shall find, in all the powers that we so raise, that the second term is always added to the first; and we shall at the same time be easily able to perceive, from the manner of the operation, that the same thing must take place in all higher powers whatsoever of the same series, and consequently in the m th power of it, or in the series that

is equal to $\sqrt[n]{1+x}$. Now the second, third, fourth, and fifth powers of the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c$, carried only to three terms, may be raised in the manner following.

$$\sqrt[n]{1+x}^{\frac{1}{n}} = 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c$$

$$\sqrt[n]{1+x}^{\frac{1}{n}} = 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c$$

$$1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c$$

$$+ \frac{1}{n}x + \frac{1}{n} \times \frac{1}{n}xx - \&c$$

$$- \frac{1}{n} \times \frac{n-1}{2n}xx - \&c$$

$$\sqrt[n]{1+x}^{\frac{2}{n}} = 1 + \frac{2}{n}x - \frac{1}{n} \times \frac{2n-4}{2n}xx \&c$$

$$\sqrt[n]{1+x}^{\frac{1}{n}} = 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \&c$$

$$1 + \frac{2}{n}x - \frac{1}{n} \times \frac{2n-4}{2n}x^2 \&c$$

$$+ \frac{1}{n}x + \frac{1}{n} \times \frac{2}{n}x^2 - \&c$$

$$- \frac{1}{n} \times \frac{n-1}{2n}x^2 - \&c$$

$$\sqrt[n]{1+x}^{\frac{3}{n}} = 1 + \frac{3}{n}x - \frac{1}{n} \times \frac{3n-9}{2n}x^2 \&c$$

$$\sqrt[n]{1+x}^{\frac{1}{n}} = 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c$$

$$1 + \frac{3}{n}x - \frac{1}{n} \times \frac{3n-9}{2n}x^2 \&c$$

$$+ \frac{1}{n}x + \frac{1}{n} \times \frac{3}{n}x^2 - \&c$$

$$- \frac{1}{n} \times \frac{n-1}{2n}x^2 - \&c$$

$$\overline{1+x}^{\frac{4}{n}} = 1 + \frac{4}{n}x - \frac{1}{n} \times \frac{4n-16}{2n}x^2 \&c$$

$$\overline{1+x}^{\frac{1}{n}} = 1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c$$

$$1 + \frac{4}{n}x - \frac{1}{n} \times \frac{4n-16}{2n}x^2 \&c$$

$$+ \frac{1}{n}x + \frac{1}{n} \times \frac{4}{n}x^2 \&c$$

$$- \frac{1}{n} \times \frac{n-1}{2n}x^2 - \&c$$

$$\overline{1+x}^{\frac{5}{n}} = 1 + \frac{5}{n}x - \frac{1}{n} \times \frac{5n-25}{2n}x^2 \&c.$$

In all these powers of the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c$ (which is equal to $\overline{1+x}^{\frac{1}{n}}$), the second term is added to the first term 1; and from the manner of raising these powers (which is done by multiplying the last series, or the series equal to the last power of $\overline{1+x}^{\frac{1}{n}}$, by the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \&c$, and consequently by adding the product of the last series multiplied by $\frac{1}{n}x$ to the product of the same series multiplied by 1, or, properly speaking, not multiplied at all), it is evident that the same thing will always take place, or the second term will be always added to the first term, in all higher powers of the same series whatsoever, and consequently in the m th power of it, or in the series which is equal to $\overline{1+x}^{\frac{m}{n}}$. Q. E. D.

57. And, in the 3d place, it is evident likewise from the foregoing operations (whereby we have obtained the three first terms of the serieses that are equal to

$\overline{1+x}^{\frac{2}{n}}$, $\overline{1+x}^{\frac{3}{n}}$, $\overline{1+x}^{\frac{4}{n}}$, and $\overline{1+x}^{\frac{5}{n}}$, respectively), that the co-efficient of the second term of each of these powers of the series that is equal to

$\overline{1+x}^{\frac{1}{n}}$, is always a fraction, of which the index of the power to which the va-

lue of $\overline{1+x}^{\frac{1}{n}}$ is raised is the numerator, and n is the denominator. Thus, the

co-efficient of the second term of the series that is equal to $\overline{1+x}^{\frac{2}{n}}$, to wit, the series $1 + \frac{2}{n}x - \frac{1}{n} \times \frac{2n-4}{2n}x^2 \&c$, is the fraction $\frac{2}{n}$; and the co-effi-

cient of the second term of the series that is equal to $\overline{1+x}^{\frac{3}{n}}$, to wit, the series

$1 + \frac{3}{n}x - \frac{1}{n} \times \frac{3n-9}{2n}x^2 \&c$, is the fraction $\frac{3}{n}$; and the co-efficient of the

second

second term of the series that is equal to $\sqrt[n]{1+x^4}$, to wit, the series $1 + \frac{4}{n}x - \frac{1}{n} \times \frac{4n-16}{2n} x^2 \&c$, is the fraction $\frac{4}{n}$; and the co-efficient of the second term of the series that is equal to $\sqrt[n]{1+x^5}$, to wit, the series $1 + \frac{5}{n}x - \frac{1}{n} \times \frac{5n-25}{2n} x^2 \&c$, is the fraction $\frac{5}{n}$. And it is evident that the same thing will take place in all higher powers of $\sqrt[n]{1+x^I}$, or of the series $1 + \frac{I}{n}x - \frac{1}{n} \times \frac{n-I}{2n} x^2 + \&c$, and consequently that the co-efficient of the second term of the series that is equal to $\sqrt[n]{1+x^m}$ will be $\frac{m}{n}$, and that the said second term itself will be $\frac{m}{n}x$. Therefore the two first terms of the series that is equal to $\sqrt[n]{1+x^m}$ will be $1 + \frac{m}{n}x$. And these three observations will always be true, whether n be greater than m , or m be greater than n .

These things being premised, the values of the following co-efficients, C, D, E, F, &c, of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x^m}$ (m being any whole number whatsoever, and n any other whole number greater than m), may be investigated in the manner following.

An Investigation of the third, fourth, fifth, sixth, and other following terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1+x^m}$, or the m th power of the n th root of the binomial quantity $1+x$, when m is any whole number whatsoever, and n is any other whole number greater than m .

58. Let y be put = the series $\frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or $1+y$ be equal the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to

$\sqrt[n]{1+x^m}$. Then will $\sqrt[n]{1+y^n}$ be equal to the n th power of $\sqrt[n]{1+x^m}$, or to $\sqrt[n]{1+x^m}$. But, by the binomial theorem in the case of integral powers (which has been demonstrated above in the preceeding tract in pages 153, 154, &c — 169,) $\sqrt[n]{1+x^m}$ is = the series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2} x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$

$\times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 + \&c$, and $(1+y)^n$ is = the series $1 + \frac{n}{1} y + \frac{n}{1} \times \frac{n-1}{2} y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} y^5 + \&c$. Therefore the series $1 + \frac{m}{1} x + \frac{m}{1} \times \frac{m-1}{2} x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 + \&c$, will be equal to the series $1 + \frac{n}{1} y + \frac{n}{1} \times \frac{n-1}{2} y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} y^5 + \&c$; and consequently (subtracting 1 from both sides of the equation), the series $\frac{m}{1} x + \frac{m}{1} \times \frac{m-1}{2} x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} x^5 + \&c$, will be equal to the series $\frac{n}{1} y + \frac{n}{1} \times \frac{n-1}{2} y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} y^5 + \&c$.

59. To shorten the expressions of these co-efficients, let us put $Q = \frac{m}{1} \times \frac{m-1}{2}$, $R = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $S = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, and $T = \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}$, and $q = \frac{n}{1} \times \frac{n-1}{2}$, $r = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $s = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $t = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$. And we shall then have $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c = ny + qy^2 + ry^3 + sy^4 + ty^5 + \&c$.

60. We must now raise the several powers of the series $\frac{m}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c, or Bx , Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c (which is equal to y), in order to have the values of y^2 , y^3 , y^4 , y^5 , &c, expressed in powers of x . This may be done in the manner following:

$$\begin{array}{r}
 y = Bx, \quad Cx^2, \quad Dx^3, \quad Ex^4, \quad Fx^5, \quad \&c \\
 y = Bx, \quad Cx^2, \quad Dx^3, \quad Ex^4, \quad Fx^5, \quad \&c \\
 \hline
 B^2x^2, \quad BCx^3, \quad BDx^4, \quad BEx^5, \quad \&c \\
 \quad BCx^3, \quad C^2x^4, \quad CDx^5, \quad \&c \\
 \quad \quad BDx^4, \quad CDx^5, \quad \&c \\
 \quad \quad \quad BEx^5, \quad \&c
 \end{array}$$

$$y^2 = B^2x^2, \quad 2BCx^3, \quad 2BDx^4, \quad 2BEx^5, \quad \&c \\
 \quad \quad C^2x^4, \quad 2CDx^5, \quad \&c$$

$$\begin{aligned}
 y &= \frac{Bx, \quad Cx^2, \quad Dx^3, \quad \&c}{B^3x^3, \quad 2B^2Cx^4, \quad 2B^2Dx^5, \quad \&c} \\
 &\quad \quad \quad BC^2x^5, \quad \&c \\
 &\quad \quad \quad B^2Cx^4, \quad 2BC^2x^5, \quad \&c \\
 &\quad \quad \quad B^2Dx^5, \quad \&c \\
 y^3 &= \frac{B^3x^3, \quad 3B^2Cx^4, \quad 3B^2Dx^5, \quad \&c}{3BC^2x^5, \quad \&c} \\
 y &= \frac{Bx, \quad Cx^2, \quad \&c}{B^4x^4, \quad 3B^3Cx^5, \quad \&c} \\
 &\quad \quad \quad B^3Cx^5, \quad \&c \\
 y^4 &= \frac{B^4x^4, \quad 4B^3Cx^5, \quad \&c}{y = Bx, \quad \&c} \\
 y^5 &= B^5x^5, \quad \&c.
 \end{aligned}$$

61. By substituting these powers of the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, instead of $y, y^2, y^3, y^4, y^5, \&c$, in the equation $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c = ny + qy^2 + ry^3 + sy^4 + ty^5 + \&c$, we shall have the simple series $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c =$ the compound series

$$\begin{aligned}
 &nBx, \quad nCx^2, \quad nDx^3, \quad nEx^4, \quad nFx^5, \quad \&c \\
 &\quad + qB^2x^2, \quad 2qBCx^3, \quad 2qBDx^4, \quad 2qBEx^5, \quad \&c \\
 &\quad \quad \quad qC^2x^4, \quad 2qCDx^5, \quad \&c \\
 &\quad \quad \quad + rB^3x^3, \quad 3rB^2Cx^4, \quad 3rB^2Dx^5, \quad \&c \\
 &\quad \quad \quad \quad \quad 3rBC^2x^5, \quad \&c \\
 &\quad \quad \quad + sB^4x^4, \quad 4sB^3Cx^5, \quad \&c \\
 &\quad \quad \quad \quad \quad + tB^5x^5, \quad \&c.
 \end{aligned}$$

Therefore, if we divide all the terms of this equation by x , we shall have the simple series $m + Qx + Rx^2 + Sx^3 + Tx^4 + \&c =$ the compound series

$$\begin{aligned}
 &nB, \quad nCx, \quad nDx^2, \quad nEx^3, \quad nFx^4, \quad \&c \\
 &\quad + qB^2x, \quad 2qBCx^2, \quad 2qBDx^3, \quad 2qBEx^4, \quad \&c \\
 &\quad \quad \quad qC^2x^3, \quad 2qCDx^4, \quad \&c \\
 &\quad \quad \quad + rB^3x^2, \quad 3rB^2Cx^3, \quad 3rB^2Dx^4, \quad \&c \\
 &\quad \quad \quad \quad \quad 3rBC^2x^4, \quad \&c \\
 &\quad \quad \quad + sB^4x^3, \quad 4sB^3Cx^4, \quad \&c \\
 &\quad \quad \quad \quad \quad + tB^5x^4, \quad \&c.
 \end{aligned}$$

And this equation will be true, of however small a magnitude we suppose x to be taken: and consequently it will be true also when x is equal to 0. But, when x is = 0, all the terms in this equation that involve x in them will be equal to 0 likewise, and consequently the equation will be as follows, to wit, $m = nB$. Therefore B will be $= \frac{m}{n}$, agreeably to what has been already shewn in art. 57, and consequently the two first terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $1 + x \sqrt[n]{m}$ will be $1 + \frac{m}{n}x$.

Q. E. I.

62. Since

62. Since m is equal to nB , it follows that, if we subtract m and nB from the opposite sides of the last equation, the remainders will be equal; that is, the simple series $Qx + Rx^2 + Sx^3 + Tx^4, + \&c$, will be equal to the compound series

$$\begin{array}{ccccccc} nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\ + qB^2x, & 2qBCx^2, & 2qBDx^3, & 2qBEx^4, & \&c \\ & & qC^2x^3, & 2qCDx^4, & \&c \\ & + rB^3x^2, & 3rB^2Cx^3, & 3rB^2Dx^4, & \&c \\ & & & 3rBC^2x^4, & \&c \\ & & + sB^4x^3, & 4sB^3Cx^4, & \&c \\ & & & + tB^5x^4, & \&c. \end{array}$$

Therefore, if we divide all the terms by x , the quotients on both sides will be equal; that is, the simple series $Q + Rx + Sx^2 + Tx^3 + \&c$, will be equal to the compound series

$$\begin{array}{ccccccc} nC, & nDx, & nEx^2, & nFx^3, & \&c \\ + qB^2, & 2qBCx, & 2qBDx^2, & 2qBEx^3, & \&c \\ & & qC^2x^2, & 2qCDx^3, & \&c \\ & + rB^3x, & 3rB^2Cx^2, & 3rB^2Dx^3, & \&c \\ & & & 3rBC^2x^3, & \&c \\ & & + sB^4x^2, & 4sB^3Cx^3, & \&c \\ & & & + tB^5x^3, & \&c. \end{array}$$

And this equation will always be true, to how small a quantity soever we suppose x to be diminished; and consequently it will be true also when x is $= 0$. But, when x is $= 0$, all the terms on both sides of the equation, that involve x , will also be equal to 0, and consequently the equation will be as follows, to wit,

$$Q = nC + qB^2, \text{ that is, (because } Q \text{ is } = \frac{m}{1} \times \frac{m-1}{2}, \text{ and } q \text{ is } = \frac{n}{1} \times \frac{n-1}{2})$$

$$\frac{m}{1} \times \frac{m-1}{2} \text{ will be } = nC + \frac{n}{1} \times \frac{n-1}{2} \times B^2, \text{ or (because } B \text{ is } = \frac{m}{n}) \frac{m}{1} \times$$

$$\frac{m-1}{2} \text{ will be } = nC + \frac{n}{1} \times \frac{n-1}{2} \times \frac{m}{n} \times \frac{m}{n}, \text{ or } \frac{m}{1} \times \frac{m-1}{2} \text{ will be } = nC$$

$$+ \frac{n-1}{2n} \times m^2, \text{ or } mn \times \frac{m-1}{2n} \text{ will be } = nC + \frac{n-1}{2n} \times m^2, \text{ or } \frac{m^2n-mn}{2n} \text{ will}$$

$$\text{be } = nC + \frac{m^2n-m^2}{2n}; \text{ whence (adding } \frac{mn}{2n} \text{ to both sides) we shall have } \frac{m^2n}{2n} =$$

$$nC + \frac{m^2n-m^2+mn}{2n}, \text{ and (adding } \frac{m^2}{2n} \text{ to both sides) } \frac{m^2n+m^2}{2n} = nC + \frac{m^2n+mn}{2n},$$

$$\text{and (subtracting } \frac{m^2n}{2n} \text{ from both sides), } \frac{m^2}{2n} = nC + \frac{mn}{2n}; \text{ that is, } \frac{m^2}{2n} \text{ will be}$$

$$\text{equal to } \frac{mn}{2n}, \text{ together with } nC, \text{ either added to it or subtracted from it, as may}$$

$$\text{be necessary to produce such equality. But, because } n \text{ is supposed to be greater}$$

$$\text{than } m, \frac{mn}{2n} \text{ will be greater than } \frac{mm}{2n}; \text{ and consequently } nC \text{ must be subtracted}$$

$$\text{from } \frac{mn}{2n} \text{ in order to make it equal to } \frac{mm}{2n}, \text{ or } \frac{m^2}{2n}. \text{ We shall therefore have } \frac{m^2}{2n}$$

$$= -nC + \frac{mn}{2n}, \text{ and consequently (adding } nC \text{ to both sides) } \frac{m^2}{2n} + nC =$$

$$\frac{mn}{2n}, \text{ and (subtracting } \frac{m^2}{2n} \text{ from both sides) } nC = \frac{mn}{2n} - \frac{m^2}{2n} = \frac{mn-m^2}{2n} =$$

$\frac{m \times n - m}{2n} = m \times \frac{n - m}{2n}$, and consequently $C = \frac{m}{n} \times \frac{n - m}{2n}$. Therefore the third term, Cx^2 , of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $1 + x \sqrt[n]{m}$, must have the sign $-$ prefixed to it, and will be $= \frac{m}{n} \times \frac{n - m}{2n} \times x^2$, and the three first terms of the said series will be $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n - m}{2n} x^2$. Q. E. I.

63. It has been shewn, in the last article, that Q is $= nC + qB^2$, or (because it has also been shewn that nC is to be subtracted from qB^2 , or to have the sign $-$ prefixed to it) $-nC + qB^2$. Therefore, if we subtract Q and $-nC + qB^2$ from the opposite sides of the equation which was obtained in that article by the division of x , the remainders will be equal to each other; that is, the simple series $Rx + Sx^2 + Tx^3 + \&c$ will be equal to the compound series

$$\begin{array}{lll} nDx, & nEx^2, & nFx^3, \&c \\ 2qBCx, & 2qBDx^2, & 2qBEx^3, \&c \\ & qC^2x^2, & 2qCDx^3, \&c \\ + rB^3x, & 3rB^2Cx^2, & 3rB^2Dx^3, \&c \\ & 3rBC^2x^3, \&c \\ & + sB^4x^2, & 4sB^3Cx^3, \&c \\ & & + tB^5x^3, \&c: \text{ or, if} \end{array}$$

we prefix the sign $-$ to the terms $2qBCx, 3rB^2Cx^2$, and $4sB^3Cx^3$ (in which the simple power of C occurs with the powers of B), and the sign $+$ to the terms $qC^2x^2, 3rBC^2x^3$ (in which the square of C occurs by itself and with B), the simple series $Rx + Sx^2 + Tx^3 + \&c$ will be equal to the compound series

$$\begin{array}{lll} nDx, & nEx^2, & nFx^3, \&c \\ - 2qBCx, & 2qBDx^2, & 2qBEx^3, \&c \\ & + qC^2x^2, & 2qCDx^3, \&c \\ + rB^3x - 3rB^2Cx^2, & 3rB^2Dx^3, \&c \\ & + 3rBC^2x^3, \&c \\ + sB^4x^2 - 4sB^3Cx^3, & \&c \\ & + tB^5x^3, \&c. \end{array}$$

Therefore, if we divide all the terms of this equation by x , we shall have the simple series $R + Sx + Tx^2 + \&c =$ the compound series

$$\begin{array}{lll} nD, & nEx, & nFx^2, \&c \\ - 2qBC, & 2qBDx, & 2qBEx^2, \&c \\ & + qC^2x, & 2qCDx^2, \&c \\ + rB^3 - 3rB^2Cx, & 3rB^2Dx^2, \&c \\ & + 3rBC^2x^2, \&c \\ + sB^4x - 4sB^3Cx^2, & \&c \\ & + tB^5x^2, \&c. \end{array}$$

And this equation will be always true, to how small a quantity soever we suppose x to be diminished: and therefore it will be true also when x is $= 0$. But, when

when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewise, and consequently the equation will be as follows, to wit, $R = n D - 2 q B C + r B^3$; by examining and resolving which equation we may both determine whether the sign $+$ or the sign $-$ is to be prefixed to the quantity $n D$, and consequently to $D x^3$, or the fourth term of the series $1 + B x, C x^2, D x^3,$

$E x^4, F x^5, \&c$, which is equal to $1 + x \Big| ^m$ (from which fourth term the said quantity $n D$ has been derived by the operations of multiplication and division in the course of the foregoing processes), and likewise what is the magnitude of D , and consequently that of $D x^3$. Now this resolution will be found to be a matter of some intricacy. It may, however, be performed in the manner following:

64. By substituting $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$ in this equation instead of R , and $\frac{n}{1} \times \frac{n-1}{2}$ instead of q , and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ instead of r , and $\frac{m}{n}$ instead of B , and $\frac{m}{n} \times \frac{n-m}{2n}$ instead of C , it will become as follows, to wit, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} = n D - n \times \frac{n-1}{2} \times \frac{m}{n} \times \frac{m}{n} \times \frac{n-m}{2n} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{m^3}{n^3}$, or $\frac{m^3-3m^2+2m}{6} = n D - m \times \frac{n-1}{2} \times \frac{m}{n} \times \frac{n-m}{2n} + \frac{n^3-3n^2+2n}{6} \times \frac{m^3}{n^3}$, or $\frac{m^3n^2-3m^2n^2+2mn^2}{6nn} = n D - \frac{3m^2}{3n} \times \frac{n-1}{2} \times \frac{n-m}{2n} + m^3 \times \frac{n^3-3n^2+2n}{6nn}$
 $= n D - 3m^2 \times \frac{nn-nm-n+m}{6nn} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = n D - \frac{3m^2n^2+3m^3n-3m^2n-3m^3}{6nn}$
 $+ \frac{m^3n^2-3m^3n+2m^3}{6nn} = n D - \frac{3m^2n^2+3m^3n-m^3+m^3n^2}{6nn}$; and consequently (adding $\frac{3m^2n^2}{6nn}$ to both sides) we shall have $\frac{m^3n^2+2mn^2}{6nn} = n D + \frac{3m^2n-m^3+m^3n^2}{6nn}$, and (adding $\frac{m^3}{6nn}$ to both sides) $\frac{m^3n^2+2mn^2+m^3}{6nn} = n D + \frac{3m^2n+m^3n^2}{6nn}$, and (subtracting $\frac{m^3n^2}{6nn}$ from both sides) $\frac{2mn^2+m^3}{6nn} = n D + \frac{3m^2n}{6nn}$.

Now $\frac{2mn^2+m^3}{6nn}$ is greater than $\frac{3m^2n}{6nn}$, as may be thus demonstrated. Since n is greater than m , mn^2 will be greater than m^2n , and consequently $2m^2n + mn^2$ will be greater than $2m^2n + m^2n$, or than $3m^2n$. Further, since mn^2 , m^2n , and m^3 are in continued proportion, the common ratio being that of n to m , it follows from Euclid's Elements, Book 5, Prop. 25, that the sum of the two extreme terms will be greater than twice the middle term; that is, $mn^2 + m^3$ will be greater than $2m^2n$. Therefore $mn^2 + mn^2 + m^3$ will be greater than $mn^2 + 2m^2n$, or $2mn^2 + m^3$ will be greater than $mn^2 + 2m^2n$. But it has been shewn that $mn^2 + 2m^2n$ is greater than $m^2n + 2m^2n$, or $3m^2n$. Therefore $2mn^2 + m^3$ (which is greater than $mn^2 + 2m^2n$) will, *à fortiori*, be greater than $3m^2n$. Therefore $\frac{2mn^2+m^3}{6nn}$ will be greater than $\frac{3m^2n}{6nn}$. Q. E. D.

Since

Since therefore $\frac{2mn^2+m^3}{6nn}$ is greater than $\frac{3m^2n}{6nn}$, but equal to $nD + \frac{3m^2n}{6nn}$, it follows that nD must be added to $\frac{3m^2n}{6nn}$ in order to produce the said equality. Therefore the sign $+$ must be prefixed to nD , and consequently to Dx^3 , or the fourth term of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{m}{n}}$, or the said fourth term must be added to the first term 1 . Therefore the four first terms of the said series will be $1 + Bx - Cx^2 + Dx^3$, or $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + Dx^3$; which was the first point that was to be determined.

And, secondly, the magnitude of D may be determined by means of the equation last-obtained, to wit, $\frac{2mn^2+m^3}{6nn} = nD + \frac{3m^2n}{6nn}$, or $\frac{2mn^2+m^3}{6nn} = +nD + \frac{3m^2n}{6nn}$. For, by subtracting $\frac{3m^2n}{6nn}$ from both sides, we shall have $nD = \frac{2mn^2-3m^2n+m^3}{6nn} = \frac{n-m}{6nn} \times \frac{2nm-m^2}{1} = m \times \frac{n-m}{2n} \times \frac{2n-m}{3n}$, and consequently (dividing both sides by n) $D = \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}$, or $D = C \times \frac{2n-m}{3n}$, or $\frac{2n-m}{3n} \times C$. Therefore the four first terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{m}{n}}$, are $1 + \frac{m}{n}x - \frac{m}{n} \times \frac{n-m}{2n}x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}x^3$, or $1 + \frac{m}{n}Ax - \left[\frac{n-m}{2n}B\right]x^2 + \frac{2n-m}{3n}Cx^3$.

Q. E. I.

65. It is possible, in the like manner, to determine the signs $+$ or $-$ that are to be prefixed to the following terms, $Ex^4, Fx^5, Gx^6, \&c$, of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, \&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$), and to find the values of the co-efficients $E, F, G, \&c$, by the resolution of the like simple equations which may be derived from the fundamental equation obtained in art. 61. The equation for determining the sign and value of the fifth term, Ex^4 , of the said series, is as follows, to wit, $S = nE + 2qBD + qC^2 - 3rB^2C + sB^4$; and the equation for determining the sign and value of the sixth term, Fx^5 , of the said series, is as follows, to wit, $T = nF - 2qBE - 2qCD + 3rB^2D + 3rBC^2 - 4sB^3C + tB^5$. But the computations that would be necessary to the resolution of these equations would (as the reader will easily conceive from the difficulty we have found in resolving the last equation for determining the sign and magnitude of Dx^3) be excessively complicated and troublesome; and in the equations for determining the following terms $Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, \&c$, of the said series, the intricacy of the calculations would be so great as to make the resolution of those equations absolutely impracticable. And therefore I shall proceed no further in this method of investigating

investigating the terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which equal to $\sqrt[n]{1+x}^m$.

66. And further we may observe, concerning this method of investigating the terms of the said series, that, if it were not so excessively laborious in the practice as we have seen it to be after the two or three first terms of it, but we were able to compute twenty or thirty of its terms with only a moderate degree of trouble, it would still be impossible to ascertain, by means of the terms so computed, the law of their generation, or continuation, one from another, so as to be able to conclude with certainty that the said law must take place with respect to the terms that had not actually been computed, as well as with respect to those that had: which is the object we ought principally to aim at when we are investigating the terms of this series. I shall therefore on this account, as well as on account of the increasing intricacy of the calculations that occur in it, desist from all further prosecution of this method of investigating the terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is $= \sqrt[n]{1+x}^m$; and shall proceed to lay before the reader another method of investigating it, which will both be much easier to practise than the foregoing method, and will also enable us to discover the law by which the co-efficients B, C, D, E, F, G, H, I, K, L, &c, of the powers of x are generated, or formed, one from another; and to conclude that the said law must take place in the terms which have not been actually computed or investigated, as well as in those that have. This method is exactly similar to that by which we investigated the terms of the series that is equal to $\sqrt[n]{1+x}^1$, or $\sqrt[n]{1+x}$, in art. 36, 37, 38, 39, 40, and 41.

Another Investigation of the third, fourth, fifth, sixth, and other following terms of the Series

$$1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c,$$

which is equal to $\sqrt[n]{1+x}^m$, or the m th power of the n th root of the binomial quantity $1+x$, when m is any whole number whatsoever, and n is any other whole number greater than m .

67. It has been shewn above, in the three observations contained in art. 54, 55, 56, and 57, that, if x be of any magnitude less than 1, the quantity $\sqrt[n]{1+x}^m$, or the m th power of $\sqrt[n]{1+x}$ or of the n th root of $1+x$, will be equal to the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, in which C, D, E, F, &c, represent certain numeral co-efficients of $x^2, x^3, x^4, x^5, \&c$, that are always the same, whatever

whatever may be the magnitude of x . It follows, therefore, that if we suppose x to be increased from any one particular magnitude denoted by x , to any other magnitude (less than 1) that is denoted by y , the quantity $\sqrt[n]{1+y}$, or the m th power of $\sqrt[n]{1+y}$ or of the n th root of $1+y$, will be equal to the series $1 + \frac{m}{n}y, Cy^2, Dy^3, Ey^4, Fy^5, \&c.$ Now let d be equal to the difference by which y exceeds x ; so that $x+d$ shall be equal to y : and let $x+d$ be substituted instead of y in the last equation $\sqrt[n]{1+y} = 1 + \frac{m}{n}y, Cy^2, Dy^3, Ey^4, Fy^5, \&c.$ And we shall then have $\sqrt[n]{1+x+d} =$ the series $1 + \frac{m}{n} \times \overline{x+d}, C \times \overline{x+d}^2, D \times \overline{x+d}^3, E \times \overline{x+d}^4, F \times \overline{x+d}^5, \&c =$ the series $1 + \frac{m}{n} \times \overline{x+d}, C \times \overline{x^2+2xd+d^2}, D \times \overline{x^3+3x^2d+3xd^2+d^3}, E \times \overline{x^4+4x^3d+6x^2d^2+4xd^3+d^4}, F \times \overline{x^5+5x^4d+10x^3d^2+10x^2d^3+5xd^4+d^5}, + \&c =$ the compound series

$$\begin{aligned} &1 + \frac{m}{n}x, \quad Cx^2, \quad Dx^3, \quad Ex^4, \quad Fx^5, \quad \&c \\ &+ \frac{m}{n}d, \quad 2Cx^2d, \quad 3Dx^2d, \quad 4Ex^3d, \quad 5Fx^4d, \quad \&c \\ &\quad Cx^2d^2, \quad 3Dx^2d^2, \quad 6Ex^3d^2, \quad 10Fx^4d^2, \quad \&c \\ &\quad Dd^3, \quad 4Ex^3d^3, \quad 10Fx^4d^3, \quad \&c \\ &\quad E d^4, \quad 5F x d^4, \quad \&c \\ &\quad F d^5, \quad \&c. \end{aligned}$$

68. Let f be $= 1+x$. And we shall have $f+d = 1+x+d$, and $\sqrt[n]{f+d} = \sqrt[n]{1+x+d}$. But $f+d$ is $= f \times \left[1 + \frac{d}{f}\right]$. Therefore $\sqrt[n]{f+d}$ will be $= f^{\frac{m}{n}} \times \sqrt[n]{1 + \frac{d}{f}}$. But, because $\sqrt[n]{1+x}$ is equal to the series

$1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, it follows that $\sqrt[n]{1 + \frac{d}{f}}$ will, in like manner, be equal to the series $1 + \frac{m}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c =$ the series $1 + \frac{m}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c.$ Therefore $f^{\frac{m}{n}} \times \sqrt[n]{1 + \frac{d}{f}}$ will be equal to $f^{\frac{m}{n}} \times$ the series $1 + \frac{m}{n} \times \frac{d}{f}, C \times \frac{d^2}{f^2}, D \times \frac{d^3}{f^3}, E \times \frac{d^4}{f^4}, F \times \frac{d^5}{f^5}, \&c =$ the series $f^{\frac{m}{n}} + \frac{m}{n} \times f^{\frac{m}{n}} \times \frac{d}{f}, C \times f^{\frac{m}{n}} \times \frac{d^2}{f^2}, D \times f^{\frac{m}{n}} \times \frac{d^3}{f^3}, E \times f^{\frac{m}{n}} \times \frac{d^4}{f^4}, F \times f^{\frac{m}{n}} \times \frac{d^5}{f^5}, \&c.$

$\times f^{\frac{m}{n}} \times \frac{d^5}{f^5}$, &c. Therefore $\overline{f+d}^{\frac{m}{n}}$ (which is $= f^{\frac{m}{n}} \times \overline{1+\frac{d}{f}}^{\frac{m}{n}}$) will also be equal to the series $f^{\frac{m}{n}} + \frac{m}{n} \times f^{\frac{m}{n}} \times \frac{d}{f}$, $C \times f^{\frac{m}{n}} \times \frac{d^2}{f^2}$, $D \times f^{\frac{m}{n}} \times \frac{d^3}{f^3}$, $E \times f^{\frac{m}{n}} \times \frac{d^4}{f^4}$, $F \times f^{\frac{m}{n}} \times \frac{d^5}{f^5}$, &c. Therefore, if we substitute $1+x$ in this last equation instead of f , to which it is equal, we shall have $\overline{1+x+d}^{\frac{m}{n}} =$ the series $\overline{1+x}^{\frac{m}{n}} + \frac{m}{n} \times \overline{1+x}^{\frac{m}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^2}{(1+x)^2}$, $D \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^3}{(1+x)^3}$, $E \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^4}{(1+x)^4}$, $F \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^5}{(1+x)^5}$, &c.

69. But it has been shewn in art. 67, that $\overline{1+x+d}^{\frac{m}{n}}$ is equal to the compound series

$$\begin{array}{cccccc} 1 + \frac{m}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\ + \frac{m}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\ & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\ & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\ & & & Ed^4, & 5Fxd^4, & \&c \\ & & & & Fd^5, & \&c. \end{array}$$

Therefore the series $\overline{1+x}^{\frac{m}{n}} + \frac{m}{n} \times \overline{1+x}^{\frac{m}{n}} \times \frac{d}{1+x}$, $C \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^2}{(1+x)^2}$, $D \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^3}{(1+x)^3}$, $E \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^4}{(1+x)^4}$, $F \times \overline{1+x}^{\frac{m}{n}} \times \frac{d^5}{(1+x)^5}$, &c, will be equal to the said compound series

$$\begin{array}{cccccc} 1 + \frac{m}{n}x, & Cx^2, & Dx^3, & Ex^4, & Fx^5, & \&c \\ + \frac{m}{n}d, & 2Cxd, & 3Dx^2d, & 4Ex^3d, & 5Fx^4d, & \&c \\ & Cd^2, & 3Dxd^2, & 6Ex^2d^2, & 10Fx^3d^2, & \&c \\ & & Dd^3, & 4Exd^3, & 10Fx^2d^3, & \&c \\ & & & Ed^4, & 5Fxd^4, & \&c \\ & & & & Fd^5, & \&c. \end{array}$$

70. Now let $\overline{1+x}^{\frac{m}{n}}$ be subtracted from the left-hand side of this last equation; and the series $1 + \frac{m}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c (which is equal to $\overline{1+x}^{\frac{m}{n}}$), be subtracted from the right-hand side of it. Then, it is evident, the remainders will be equal to each other; that is, the series $\frac{m}{n} \times \overline{1+x}^{\frac{m}{n}}$

$\times \frac{d}{1+x}, C \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d^2}{1+x^2}, D \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d^3}{1+x^3}, E \times \frac{1}{1+x}^{\frac{m}{n}}$
 $\times \frac{d^4}{1+x^4}, F \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d^5}{1+x^5}, \&c,$ will be equal to the compound series

$$\begin{array}{cccccc} \frac{m}{n} d, & 2 C x d, & 3 D x^2 d, & 4 E x^3 d, & 5 F x^4 d, & \&c \\ C d^2, & 3 D x d^2, & 6 E x^2 d^2, & 10 F x^3 d^2, & \&c \\ D d^3, & 4 E x d^3, & 10 F x^2 d^3, & \&c \\ E d^4, & 5 F x d^4, & \&c \\ F d^5, & \&c. \end{array}$$

And consequently, if we divide all the terms by d , we shall have the series

$\frac{m}{n} \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{1}{1+x}, C \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d}{1+x^2}, D \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d^2}{1+x^3}, E$
 $\times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d^3}{1+x^4}, F \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{d^4}{1+x^5}, \&c =$ the compound series

$$\begin{array}{cccccc} \frac{m}{n}, & 2 C x, & 3 D x^2, & 4 E x^3, & 5 F x^4, & \&c \\ C d, & 3 D x d, & 6 E x^2 d, & 10 F x^3 d, & \&c \\ D d^2, & 4 E x d^2, & 10 F x^2 d^2, & \&c \\ E d^3, & 5 F x d^3, & \&c \\ F d^4, & \&c. \end{array}$$

71. This equation is always true, how small soever we may suppose d to be: and therefore it will be true also when d is $= 0$. But, when d is $= 0$, all the terms on both sides the equation that involve d will be equal to 0 likewise; and

consequently the equation will then be as follows, to wit, $\frac{m}{n} \times \frac{1}{1+x}^{\frac{m}{n}} \times \frac{1}{1+x} =$ the series $\frac{m}{n}, 2 C x, 3 D x^2, 4 E x^3, 5 F x^4, \&c$. Therefore, if we

multiply all the terms by the fraction $\frac{n}{m}$, we shall have $\frac{1}{1+x}^{\frac{m}{n}} \times \frac{1}{1+x} =$ the series $1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \frac{5n}{m} F x^4, \&c$; and, if we multiply

both sides of this last equation by $1+x$, we shall have $\frac{1}{1+x}^{\frac{m}{n}} =$ the compound series

$$1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \frac{5n}{m} F x^4, \&c.$$

$$+ x, \frac{2n}{m} C x^2, \frac{3n}{m} D x^3, \frac{4n}{m} E x^4, \&c.$$

72. But $\frac{1}{1+x}^{\frac{m}{n}}$ is also equal to the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$. Therefore the said series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, will be equal to the compound series

$$1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \frac{5n}{m} F x^4, \&c. \\ + x, \frac{2n}{m} C x^2, \frac{3n}{m} D x^3, \frac{4n}{m} E x^4, \&c;$$

and consequently (subtracting 1 from both sides of the equation) we shall have the series $\frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c =$ the compound series

$$\frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \frac{5n}{m} F x^4, \&c, \\ + x, \frac{2n}{m} C x^2, \frac{3n}{m} D x^3, \frac{4n}{m} E x^4, \&c;$$

and, lastly (dividing all the terms by x), we shall have the simple series $\frac{m}{n}, C x, D x^2, E x^3, F x^4, \&c$, equal to the compound series

$$\frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \frac{5n}{m} F x^3, \&c \\ + 1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \&c;$$

by the help of which equation we may discover both “which of the signs + and — are to be prefixed to the several co-efficients $C, D, E, F, \&c$ (and consequently to the corresponding terms $C x^2, D x^3, E x^4, F x^5, \&c$, in the series

$1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $1 + x)^{\frac{m}{n}}$,” and also,

“what are the magnitudes of the said co-efficients $C, D, E, F, \&c$, respectively.” This may be done by proceeding in the manner following :

73. In the first place, since the simple series $\frac{m}{n}, C x, D x^2, E x^3, F x^4, \&c$, is equal to the compound series

$$\frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \frac{5n}{m} F x^3, \&c \\ + 1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \&c,$$

and this equation is always true, of how small a magnitude soever we suppose x to be taken, it follows that it will also be true when $x = 0$. But, when $x = 0$, all the terms in the equation that involve x will be equal to 0 likewise; that is, all the terms on the left-hand side of the equation, except the first term $\frac{m}{n}$, and all the terms on the right-hand side of the equation, except the two first terms of the two lines of terms, to wit, $\frac{2n}{m} C + 1$, will be equal to 0. Therefore $\frac{m}{n}$ will be $=$ to $\frac{2n}{m} C + 1$, or to $1, \frac{2n}{m} C$, or to 1 together with $\frac{2n}{m} C$, either added to it or subtracted from it, as may be necessary to produce such equality. Now, because n is supposed to be greater than m , $\frac{m}{n}$ will be less than 1, and consequently $\frac{2n}{m} C$ must be subtracted from 1, in order to make it equal

to $\frac{m}{n}$. Therefore the sign $-$ must be prefixed to $\frac{2n}{m} C$ in the equation $\frac{m}{n} = \frac{2n}{m} C + 1$, or $\frac{m}{n} = 1, \frac{2n}{m} C$; and consequently the sign $-$ must also be prefixed to the third term $C x^2$ of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$ (which is equal to $\sqrt[n]{1 + x^{\frac{m}{n}}}$), from which third term the said quantity $\frac{2n}{m} C$ has been derived by the operations of multiplication and division in the course of the foregoing processes. Therefore the three first terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1 + x^{\frac{m}{n}}}$, will be $1 + \frac{m}{n} x - C x^2$. Q. E. I.

And the magnitude of the co-efficient C may likewise be determined by means of the equation $\frac{m}{n} = 1, \frac{2n}{m} C$, or $\frac{m}{n} = 1 - \frac{2n}{m} C$. For, by adding $\frac{2n}{m} C$ to both sides, we shall have $\frac{m}{n} + \frac{2n}{m} C = 1$; and, by subtracting $\frac{m}{n}$ from both sides, we shall have $\frac{2n}{m} C = 1 - \frac{m}{n} = \frac{n-m}{n}$; and (by dividing both sides by $\frac{2n}{m}$, or multiplying them into $\frac{m}{2n}$) $C = \frac{m}{2n} \times \frac{n-m}{n} = \frac{m}{n} \times \frac{n-m}{2n}$. Therefore the three first terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is $= \sqrt[n]{1 + x^{\frac{m}{n}}}$, will be $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2$, or $1 + \frac{m}{n} A x - \sqrt[n]{\frac{n-m}{2n}} B x^2$. Q. E. I.

74. To determine the sign that is to be prefixed to the fourth term, $D x^3$, of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1 + x^{\frac{m}{n}}}$, and to find the magnitude of its co-efficient D , we must proceed as follows:

Since the sign $-$ is to be prefixed to the co-efficient C , it must likewise be prefixed to all the terms which involve it in the grand, fundamental equation obtained in art. 72, to wit, the equation between the simple series $\frac{m}{n}, C x, D x^2, E x^3, F x^4, \&c$, and the compound series

$$\begin{aligned} & \frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \frac{5n}{m} F x^3, \&c \\ & + 1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \&c; \end{aligned}$$

which equation will therefore be as follows, to wit, the simple series $\frac{m}{n} - C x, D x^2, E x^3, F x^4, \&c =$ the compound series

$$\begin{aligned} & - \frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \frac{5n}{m} F x^3, \&c \\ & + 1 - \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \&c. \end{aligned}$$

Let

Let $2Cx$ be added to the two sides of this equation; and we shall then have the simple series $\frac{m}{n} + Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$\begin{aligned} & - \frac{2n}{m} C, \frac{3n}{m} Dx, \frac{4n}{m} Ex^2, \frac{5n}{m} Fx^3, \&c \\ & + 1, - \frac{2n}{m} Cx, \frac{3n}{m} Dx^2, \frac{4n}{m} Ex^3, \&c \\ & + 2Cx. \end{aligned}$$

But it has been shewn in the last article, that $\frac{m}{n}$ is $= 1 - \frac{2n}{m} C$. Therefore if we subtract $\frac{m}{n}$ and $1 - \frac{2n}{m} C$ from the opposite sides of the last equation, the remainders will be equal to each other; that is, the simple series $Cx, Dx^2, Ex^3, Fx^4, \&c$, will be equal to the compound series

$$\begin{aligned} & \frac{3n}{m} Dx, \frac{4n}{m} Ex^2, \frac{5n}{m} Fx^3, \&c \\ & - \frac{2n}{m} Cx, \frac{3n}{m} Dx^2, \frac{4n}{m} Ex^3, \&c \\ & + 2Cx; \text{ and consequently (dividing all the terms by } x), \\ & \text{we shall have the simple series } C, Dx, Ex^2, Fx^3, \&c = \text{the compound series} \end{aligned}$$

$$\begin{aligned} & \frac{3n}{m} D, \frac{4n}{m} Ex, \frac{5n}{m} Fx^2, \&c \\ & - \frac{2n}{m} C, \frac{3n}{m} Dx, \frac{4n}{m} Ex^2, \&c \\ & + 2C. \end{aligned}$$

And this equation will be true, how small soever we may suppose x to be: and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will be $C = \frac{3n}{m} D - \frac{2n}{m} C + 2C$. Therefore $C + \frac{2n}{m} C$ will be $= \frac{3n}{m} D + 2C$, and $\frac{2n}{m} C$ will be $= \frac{3n}{m} D + C$, or $C, \frac{3n}{m} D$; that is, $\frac{2n}{m} C$ will be equal to C together with $\frac{3n}{m} D$ either added to it or subtracted from it, as may be necessary to produce the said equality. But, because n is greater than m , $\frac{2n}{m} C$ will be greater than $\frac{2m}{m} C$, or than $2C$, and, *à fortiori*, greater than C . Therefore, in order to make C and $\frac{3n}{m} D$ be equal to $\frac{2n}{m} C$, $\frac{3n}{m} D$ must be added to C , and consequently must have the sign $+$ prefixed to it. Therefore the sign $+$ must also be prefixed to the fourth term, Dx^3 , of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$), from which fourth term the quantity $\frac{3n}{m} D$ has been derived by the operations of multiplication and division in the course of the foregoing processes. Therefore the four first terms of the said series will be $1 + \frac{m}{n}x - Cx^2 + Dx^3$. Q. E. I.

And

And the magnitude of the co-efficient D may likewise be determined by means of the equation $\frac{2n}{m} C = C + \frac{3n}{m} D$. For, by subtracting C from both sides, we shall have $\frac{3n}{m} D = \frac{2n}{m} C - C = \left[\frac{2n}{m} - 1 \right] \times C = \frac{2n-m}{m} \times C$, and (by dividing both sides by $\frac{3n}{m}$, or multiplying them into $\frac{m}{3n}$) $D = \frac{2n-m}{3n} \times C$, or (because C is $= \frac{m}{n} \times \frac{n-m}{2n}$) $D = \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}$. Q. E. I.

Therefore the four first terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5$, &c, which is equal to $\overline{1 + x}^{\frac{m}{n}}$, are $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3$, or $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3$. Q. E. I.

75. To determine the sign that is to be prefixed to the fifth term, $E x^4$, of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5$, &c, which is equal to $\overline{1 + x}^{\frac{m}{n}}$, and to find the magnitude of the co-efficient E, we must proceed as follows:

It has been shewn in the last article, that the simple series C, D x, E x², F x³, &c is = the compound series

$$\begin{aligned} & \frac{3n}{m} D, \frac{4n}{m} E x, \frac{5n}{m} F x^2, \&c \\ & - \frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \&c \end{aligned}$$

+ 2 C, or (prefixing the sign + to the terms which involve the co-efficient D), that the simple series C + D x, E x², F x³, &c, is = the compound series

$$\begin{aligned} & + \frac{3n}{m} D, \frac{4n}{m} E x, \frac{5n}{m} F x^2, \&c \\ & - \frac{2n}{m} C + \frac{3n}{m} D x, \frac{4n}{m} E x^2, \&c \end{aligned}$$

+ 2 C. And it has been shewn also in the same article, that C is = $+\frac{3n}{m} D - \frac{2n}{m} C + 2 C$. Therefore, if we subtract C from the left-hand side of the last equation, and $+\frac{3n}{m} D - \frac{2n}{m} C + 2 C$ from the right-hand side of it, the remainders will be equal; that is, the simple series + D x, E x², F x³, &c, will be = the compound series

$$\begin{aligned} & \frac{4n}{m} E x, \frac{5n}{m} F x^2, \&c \\ & + \frac{3n}{m} D x, \frac{4n}{m} E x^2, \&c; \text{ and consequently (dividing} \end{aligned}$$

all the terms by x) the simple series + D, E x, F x², &c, will be = the compound series

$$\begin{aligned} & \frac{4n}{m} E, \frac{5n}{m} F x, \&c \\ & + \frac{3n}{m} D, \frac{4n}{m} E x, \&c. \end{aligned}$$

And

And this equation is always true, of however small a magnitude we suppose x to be taken: and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewise, and consequently the equation will be $+ D = \frac{4n}{m} E + \frac{3n}{m} D$, or $+ D = + \frac{3n}{m} D, \frac{4n}{m} E$; that is, D will be equal to $\frac{3n}{m} D$ together with $\frac{4n}{m} E$, either added to it or subtracted from it, as may be necessary to produce the said equality. But, because n is greater than m , $\frac{3n}{m} D$ will be greater than D , and consequently $\frac{4n}{m} E$ must be subtracted from it, in order to make it equal to D . We must therefore prefix the sign $-$ to the quantity $\frac{4n}{m} E$ in the equation $+ D = \frac{4n}{m} E + \frac{3n}{m} D$; which equation will therefore be $+ D = - \frac{4n}{m} E + \frac{3n}{m} D$, or $+ D = + \frac{3n}{m} D - \frac{4n}{m} E$. Therefore the sign $-$ must also be prefixed to the fifth term, $E x^4$, of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1+x}$; and consequently the five first terms of the said series will be $1 + \frac{m}{n} x - C x^2 + D x^3 - E x^4$. Q. E. I.

And the magnitude of the co-efficient E may likewise be determined by means of the equation $+ D = - \frac{4n}{m} E + \frac{3n}{m} D$, or $+ D = + \frac{3n}{m} D - \frac{4n}{m} E$. For, by adding $\frac{4n}{m} E$ to both sides, we shall have $D + \frac{4n}{m} E = \frac{3n}{m} D$, and, by subtracting D from both sides, $\frac{4n}{m} E = \frac{3n}{m} D - D = \left(\frac{3n}{m} - 1 \right) \times D = \frac{3n-m}{m} \times D$, and consequently $E = \frac{m}{4n} \times \frac{3n-m}{m} \times D = \frac{3n-m}{4n} \times D = \frac{3n-m}{4n} \times \frac{2n-m}{3n} \times \frac{n-m}{2n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}$. Q. E. I.

Therefore the five first terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1+x}$, will be $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4$, or $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4$.

76. To determine the sign that is to be prefixed to $F x^5$, the sixth term of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1+x}$, and to find the magnitude of the co-efficient F , we must proceed as follows:

In the last article it was shewn that the simple series $+ D, E x, F x^2, G x^3, \&c$, was equal to the compound series

$$\frac{4n}{m}$$

$$\frac{4^n}{m} E, \frac{5^n}{m} F x, \frac{6^n}{m} G x^2, \frac{7^n}{m} H x^3, \&c$$

$$+ \frac{3^n}{m} D, \frac{4^n}{m} E x, \frac{5^n}{m} F x^2, \frac{6^n}{m} G x^3, \&c,$$

or (prefixing the sign $-$ to the terms that involve the co-efficient E) that the simple series $+ D - E x, F x^2, G x^3, \&c$, was equal to the compound series

$$- \frac{4^n}{m} E, \frac{5^n}{m} F x, \frac{6^n}{m} G x^2, \frac{7^n}{m} H x^3, \&c$$

$$+ \frac{3^n}{m} D - \frac{4^n}{m} E x, \frac{5^n}{m} F x^2, \frac{6^n}{m} G x^3, \&c.$$

Add $2 E x$ to both sides of this equation. And we shall then have the simple series $D + E x, F x^2, G x^3, \&c =$ the compound series

$$- \frac{4^n}{m} E, \frac{5^n}{m} F x, \frac{6^n}{m} G x^2, \frac{7^n}{m} H x^3, \&c$$

$$+ \frac{3^n}{m} D - \frac{4^n}{m} E x, \frac{5^n}{m} F x^2, \frac{6^n}{m} G x^3, \&c$$

$$+ 2 E x.$$

But it has been shewn in the last article that $D = - \frac{4^n}{m} E + \frac{3^n}{m} D$, or $\frac{3^n}{m} D - \frac{4^n}{m} E$. Therefore, if we subtract D from the left-hand side of the last equation, and $\frac{3^n}{m} D - \frac{4^n}{m} E$ from its right-hand side, the remainders will be equal; that is, the simple series $+ E x, F x^2, G x^3, \&c$, will be $=$ the compound series

$$\frac{5^n}{m} F x, \frac{6^n}{m} G x^2, \frac{7^n}{m} H x^3, \&c$$

$$- \frac{4^n}{m} E x, \frac{5^n}{m} F x^2, \frac{6^n}{m} G x^3, \&c$$

$$+ 2 E x; \text{ and consequently (dividing all the terms by } x) \text{ the simple series } E, F x, G x^2, \&c \text{ will be } = \text{ the compound series}$$

$$\frac{5^n}{m} F, \frac{6^n}{m} G x, \frac{7^n}{m} H x^2, \&c$$

$$- \frac{4^n}{m} E, \frac{5^n}{m} F x, \frac{6^n}{m} G x^2, \&c$$

$$+ 2 E.$$

And this equation is always true, of however small a magnitude we suppose x to be taken; and therefore it will also be true when $x = 0$. But, when $x = 0$, all the terms in the equation that involve x will be equal to 0 likewise, and consequently the equation will be $E = \frac{5^n}{m} F - \frac{4^n}{m} E + 2 E$. Therefore (adding $\frac{4^n}{m} E$ to both sides) we shall have $E + \frac{4^n}{m} E = \frac{5^n}{m} F + 2 E$, and (subtracting E from both sides) we shall have $\frac{4^n}{m} E = \frac{5^n}{m} F + E$, or $\frac{4^n}{m} E = E, \frac{5^n}{m} F$; that is, $\frac{4^n}{m} E$ will be equal to E together with $\frac{5^n}{m} F$ either added to it or subtracted from it, as may be necessary to produce the said equality. But, because n is greater than m , $\frac{4^n}{m} E$ will be much greater than E ; and therefore

fore $\frac{5^n}{m} F$ must be added to E in order to make it equal to $\frac{4^n}{m} E$. We must therefore prefix the sign $+$ to the quantity $\frac{5^n}{m} F$ in the equation $\frac{4^n}{m} E = E, \frac{5^n}{m} F$; and consequently the said equation will be $\frac{4^n}{m} E = E + \frac{5^n}{m} F$. Therefore the sign $+$ must also be prefixed to the sixth term, $F x^5$, of the series that is equal to $\overline{1 + x}^{\frac{m}{n}}$, to wit, the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$; from which sixth term the quantity $\frac{5^n}{m} F$ has been derived above by the operations of multiplication and division in the course of the foregoing processes. Therefore the first six terms of the said series will be $1 + \frac{m}{n} x - C x^2 + D x^3 - E x^4 + F x^5$. Q. E. I.

And the magnitude of the co-efficient F may likewise be determined by means of the equation $\frac{4^n}{m} E = E + \frac{5^n}{m} F$. For, by subtracting E from both sides, we shall have $\frac{5^n}{m} F = \frac{4^n}{m} E - E = \left[\frac{4^n}{m} - 1 \right] \times E = \frac{4^n - m}{m} \times E$; and consequently F will be $= \frac{m}{5^n} \times \frac{4^n - m}{m} \times E = \frac{4^n - m}{5^n} \times E = \frac{4^n - m}{5^n} \times \frac{3^n - m}{4^n} \times \frac{2^n - m}{3^n} \times \frac{n - m}{2^n} \times \frac{m}{n}$, or $\frac{m}{n} \times \frac{n - m}{2^n} \times \frac{2^n - m}{3^n} \times \frac{3^n - m}{4^n} \times \frac{4^n - m}{5^n}$. Q. E. I.

Therefore the first six terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\overline{1 + x}^{\frac{m}{n}}$, are $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n - m}{2^n} x^2 + \frac{m}{n} \times \frac{n - m}{2^n} \times \frac{2^n - m}{3^n} x^3 - \frac{m}{n} \times \frac{n - m}{2^n} \times \frac{2^n - m}{3^n} \times \frac{3^n - m}{4^n} x^4 + \frac{m}{n} \times \frac{n - m}{2^n} \times \frac{2^n - m}{3^n} \times \frac{3^n - m}{4^n} \times \frac{4^n - m}{5^n} x^5$, or $1 + \frac{m}{n} A x - \frac{n - m}{2^n} B x^2 + \frac{2^n - m}{3^n} C x^3 - \frac{3^n - m}{4^n} D x^4 + \frac{4^n - m}{5^n} E x^5$.

77. Having thus gone through the investigations of the values of the four co-efficients C, D, E, F , at considerable length, I shall treat more concisely of the investigation of the following co-efficients $G, H, I, K, L, M, N, O, P, Q, R, S, T, \&c$, and shall only observe concerning them, that the signs $+$ and $-$, which are to be prefixed to them, may be determined, and their magnitudes discovered, by means of the following short and easy simple equations, which may be easily derived from the grand, fundamental, equation obtained above in art. 72; to wit, the equations

$$\begin{aligned} F &= \frac{6^n}{m} G, & \frac{5^n}{m} F, \\ G &= \frac{7^n}{m} H, & \frac{6^n}{m} G, \\ H &= \frac{8^n}{m} I, & \frac{7^n}{m} H, \end{aligned}$$

$$\begin{aligned}
 I &= \frac{9n}{m} K, \frac{8n}{m} I, \\
 K &= \frac{10n}{m} L, \frac{9n}{m} K, \\
 L &= \frac{11n}{m} M, \frac{10n}{m} L, \\
 M &= \frac{12n}{m} N, \frac{11n}{m} M, \\
 N &= \frac{13n}{m} O, \frac{12n}{m} N, \\
 O &= \frac{14n}{m} P, \frac{13n}{m} O, \\
 P &= \frac{15n}{m} Q, \frac{14n}{m} P, \\
 Q &= \frac{16n}{m} R, \frac{15n}{m} Q, \\
 R &= \frac{17n}{m} S, \frac{16n}{m} R, \\
 S &= \frac{18n}{m} T, \frac{17n}{m} S, \&c,
 \end{aligned}$$

or, (because F has the sign + prefixed to it, and $\frac{5n}{m}$ F is greater than F, and consequently $\frac{6n}{m}$ G must be subtracted from $\frac{5n}{m}$ F in order to make it equal to F),

$$F = -\frac{6n}{m} G + \frac{5n}{m} F,$$

and (for the like reasons),

$$\begin{aligned}
 -G &= +\frac{7n}{m} H - \frac{6n}{m} G, \\
 \text{and } +H &= -\frac{8n}{m} I + \frac{7n}{m} H, \\
 -I &= +\frac{9n}{m} K - \frac{8n}{m} I, \\
 +K &= -\frac{10n}{m} L + \frac{9n}{m} K, \\
 -L &= +\frac{11n}{m} M - \frac{10n}{m} L, \\
 +M &= -\frac{12n}{m} N + \frac{11n}{m} M, \\
 -N &= +\frac{13n}{m} O - \frac{12n}{m} N, \\
 +O &= -\frac{14n}{m} P + \frac{13n}{m} O, \\
 -P &= +\frac{15n}{m} Q - \frac{14n}{m} P, \\
 +Q &= -\frac{16n}{m} R + \frac{15n}{m} Q, \\
 -R &= +\frac{17n}{m} S - \frac{16n}{m} R, \\
 +S &= -\frac{18n}{m} T + \frac{17n}{m} S,
 \end{aligned}$$

&c;

&c; in all which equations the signs to be prefixed to the several co-efficients G, H, I, K, L, M, N, O, P, Q, R, S, T, &c, are alternately — and +. And the same thing, it is evident, must take place in all the following co-efficients of the powers of x in the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, &c$, which is

equal to $\overline{1 + x}^{\frac{m}{n}}$, to whatever number of terms the said series shall be continued. We may therefore conclude that the said series will, when n is greater than m , be $1 + \frac{m}{n}x - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - Ix^8 + Kx^9 - Lx^{10} + Mx^{11} - Nx^{12} + Ox^{13} - Px^{14} + Qx^{15} - Rx^{16} + Sx^{17} - Tx^{18} + &c$, in which all the terms after the two first are marked with the signs — and + alternately, or are to be alternately subtracted from, and added to, the said two first terms. Q. E. I.

And, secondly, the magnitudes of the co-efficients G, H, I, K, L, M, N, O, P, Q, R, S, T, &c, may be discovered by resolving the foregoing short simple equations, to wit,

$$\begin{aligned} + F &= -\frac{6n}{m}G + \frac{5n}{m}F, \\ - G &= +\frac{7n}{m}H - \frac{6n}{m}G, \\ + H &= -\frac{8n}{m}I + \frac{7n}{m}H, \\ - I &= +\frac{9n}{m}K - \frac{8n}{m}I, \\ + K &= -\frac{10n}{m}L + \frac{9n}{m}K, \\ - L &= +\frac{11n}{m}M - \frac{10n}{m}L, \\ + M &= -\frac{12n}{m}N + \frac{11n}{m}M, \\ - N &= +\frac{13n}{m}O - \frac{12n}{m}N, \\ + O &= -\frac{14n}{m}P + \frac{13n}{m}O, \\ - P &= +\frac{15n}{m}Q - \frac{14n}{m}P, \\ + Q &= -\frac{16n}{m}R + \frac{15n}{m}Q, \\ - R &= +\frac{17n}{m}S - \frac{16n}{m}R, \\ + S &= -\frac{18n}{m}T + \frac{17n}{m}S, &c. \end{aligned}$$

For, since $+ F$ is $= -\frac{6n}{m}G + \frac{5n}{m}F$, we shall have $F + \frac{6n}{m}G = \frac{5n}{m}F$, and $\frac{6n}{m}G = \frac{5n}{m}F - F = \left(\frac{5n}{m} - 1\right) \times F = \frac{5n-m}{m} \times F$, and consequently G ($= \frac{m}{6n} \times \frac{6n}{m}G = \frac{m}{6n} \times \frac{5n-m}{m} \times F$) $= \frac{5n-m}{6n} \times F$. Q. E. I.

And, since $-G$ is $= +\frac{7^n}{m}H - \frac{6^n}{m}G$, we shall have $\frac{6^n}{m}G - G = +\frac{7^n}{m}H$, or $\frac{6^n}{m} - 1 \times G = \frac{7^n}{m}H$, or $\frac{6^n - m}{m} \times G = \frac{7^n}{m} \times H$, and consequently H
 $(= \frac{m}{7^n} \times \frac{7^n}{m}H = \frac{m}{7^n} \times \frac{6^n - m}{m} \times G) = \frac{6^n - m}{7^n} \times G.$ Q. E. I.

And, since $+H$ is $= -\frac{8^n}{m}I + \frac{7^n}{m}H$, we shall have $H + \frac{8^n}{m}I = \frac{7^n}{m}H$,
 and $\frac{8^n}{m}I = \frac{7^n}{m}H - H = \frac{7^n}{m} - 1 \times H = \frac{7^n - m}{m} \times H$, and consequently I
 $(= \frac{m}{8^n} \times \frac{8^n}{m}I = \frac{m}{8^n} \times \frac{7^n - m}{m} \times H) = \frac{7^n - m}{8^n} \times H.$ Q. E. I.

And, since $-I$ is $= +\frac{9^n}{m}K - \frac{8^n}{m}I$, we shall have $\frac{8^n}{m}I - I = \frac{9^n}{m}K$, or
 $\frac{8^n}{m} - 1 \times I = \frac{9^n}{m}K$, or $\frac{8^n - m}{m} \times I = \frac{9^n}{m}K$, and consequently $K (= \frac{m}{9^n} \times$
 $\frac{9^n}{m}K = \frac{m}{9^n} \times \frac{8^n - m}{m} \times I) = \frac{8^n - m}{9^n} \times I.$ Q. E. I.

And, since $+K$ is $= -\frac{10^n}{m}L + \frac{9^n}{m}K$, we shall have $K + \frac{10^n}{m}L = \frac{9^n}{m}K$,
 and $\frac{10^n}{m}L = \frac{9^n}{m}K - K = \frac{9^n}{m} - 1 \times K = \frac{9^n - m}{m} \times K$, and consequently
 $L (= \frac{m}{10^n} \times \frac{10^n}{m}L = \frac{m}{10^n} \times \frac{9^n - m}{m} \times K) = \frac{9^n - m}{10^n} \times K.$ Q. E. I.

And, since $-L$ is $= +\frac{11^n}{m}M - \frac{10^n}{m}L$, we shall have $\frac{10^n}{m}L - L = +$
 $\frac{11^n}{m}M$, or $\frac{11^n}{m}M = \frac{10^n}{m}L - L = \frac{10^n}{m} - 1 \times L = \frac{10^n - m}{m} \times L$, and con-
 sequently $M (= \frac{m}{11^n} \times \frac{11^n}{m}M = \frac{m}{11^n} \times \frac{10^n - m}{m} \times L) = \frac{10^n - m}{11^n} \times L.$
 Q. E. I.

And, since $+M$ is $= -\frac{12^n}{m}N + \frac{11^n}{m}M$, we shall have $M + \frac{12^n}{m}N =$
 $\frac{11^n}{m}M$, and $\frac{12^n}{m}N = \frac{11^n}{m}M - M = \frac{11^n}{m} - 1 \times M = \frac{11^n - m}{m} \times M$, and
 consequently $N (= \frac{m}{12^n} \times \frac{12^n}{m}N = \frac{m}{12^n} \times \frac{11^n - m}{m} \times M) = \frac{11^n - m}{12^n} \times M.$
 Q. E. I.

And, since $-N$ is $= +\frac{13^n}{m}O - \frac{12^n}{m}N$, we shall have $\frac{12^n}{m}N - N =$
 $\frac{13^n}{m}O$, or $\frac{13^n}{m}O = \frac{12^n}{m}N - N = \frac{12^n}{m} - 1 \times N = \frac{12^n - m}{m} \times N$, and
 consequently $O (= \frac{m}{13^n} \times \frac{13^n}{m}O = \frac{m}{13^n} \times \frac{12^n - m}{m} \times N) = \frac{12^n - m}{13^n} \times N.$
 Q. E. I.

And, since $+O$ is $= -\frac{14^n}{m}P + \frac{13^n}{m}O$, we shall have $O + \frac{14^n}{m}P =$
 $\frac{13^n}{m}O$, and $\frac{14^n}{m}P = \frac{13^n}{m}O - O = \frac{13^n}{m} - 1 \times O = \frac{13^n - m}{m} \times O$, and
 consequently

consequently $P (= \frac{m}{14n} \times \frac{14n}{m} P = \frac{m}{14n} \times \frac{13n-m}{m} \times O = \frac{13n-m}{14n} \times O$.
Q. E. I.

And, since $-P$ is $= + \frac{15n}{m} Q - \frac{14n}{m} P$, we shall have $\frac{15n}{m} Q = \frac{14n}{m} P - P = \frac{14n}{m} - 1 \times P = \frac{14n-m}{m} \times P$, and consequently $Q (= \frac{m}{15n} \times \frac{15n}{m} Q = \frac{m}{15n} \times \frac{14n-m}{m} \times P) = \frac{14n-m}{15n} \times P$.
Q. E. I.

And, since $+Q$ is $= - \frac{16n}{m} R + \frac{15n}{m} Q$, we shall have $Q + \frac{16n}{m} R = \frac{15n}{m} Q$, and $\frac{16n}{m} R = \frac{15n}{m} Q - Q = \frac{15n}{m} - 1 \times Q = \frac{15n-m}{m} \times Q$, and consequently $R (= \frac{m}{16n} \times \frac{16n}{m} R = \frac{m}{16n} \times \frac{15n-m}{m} \times Q) = \frac{15n-m}{16n} \times Q$.
Q. E. I.

And, since $-R$ is $= + \frac{17n}{m} S - \frac{16n}{m} R$, we shall have $\frac{17n}{m} S = \frac{16n}{m} R - R = \frac{16n}{m} - 1 \times R = \frac{16n-m}{m} \times R$, and consequently $S (= \frac{m}{17n} \times \frac{17n}{m} S = \frac{m}{17n} \times \frac{16n-m}{m} \times R) = \frac{16n-m}{17n} \times R$.
Q. E. I.

And, since $+S$ is $= - \frac{18n}{m} T + \frac{17n}{m} S$, we shall have $S + \frac{18n}{m} T = \frac{17n}{m} S$, and $\frac{18n}{m} T = \frac{17n}{m} S - S = \frac{17n}{m} - 1 \times S = \frac{17n-m}{m} \times S$, and consequently $T (= \frac{m}{18n} \times \frac{18n}{m} T = \frac{m}{18n} \times \frac{17n-m}{m} \times S) = \frac{17n-m}{18n} \times S$.
Q. E. I.

Therefore the quantity $1 + x^{\frac{m}{n}}$, or the m th power of the n th root of the binomial quantity $1 + x$, will, when n is greater than m , be equal to the series
 $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 -$
 $\frac{5n-m}{6n} F x^6 + \frac{6n-m}{7n} G x^7 - \frac{7n-m}{8n} H x^8 + \frac{8n-m}{9n} I x^9 - \frac{9n-m}{10n} K x^{10} +$
 $\frac{10n-m}{11n} L x^{11} - \frac{11n-m}{12n} M x^{12} + \frac{12n-m}{13n} N x^{13} - \frac{13n-m}{14n} O x^{14} + \frac{14n-m}{15n}$
 $P x^{15} - \frac{15n-m}{16n} Q x^{16} + \frac{16n-m}{17n} R x^{17} - \frac{17n-m}{18n} S x^{18} + \&c \text{ ad infinitum,}$ in
 which all the terms, after the two first, are marked with the signs $-$ and $+$ alternately, or are alternately to be subtracted from, and added to, the two first terms, and the co-efficients of the third, fourth, fifth, sixth, and other following terms are generated from $\frac{m}{n}$, the co-efficient of the second term, and from each other, by the continual multiplication of the fractions $\frac{n-m}{2n}, \frac{2n-m}{3n}, \frac{3n-m}{4n},$
 $\frac{4n-m}{5n}, \frac{5n-m}{6n}, \frac{6n-m}{7n}, \frac{7n-m}{8n}, \frac{8n-m}{9n}, \frac{9n-m}{10n}, \&c,$ in which every new frac-
 tion

tion is derived from that which immediately preceeds it, by adding n to both the numerator and its denominator.

This series is exactly the same with that which is derived above in art. 3 from the binomial theorem, in the case of integral powers.

78. When the denominator n of the fraction $\frac{m}{n}$ (which is the index of the binomial quantity $1 + x$), is an exact multiple of the numerator m , the foregoing series $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 - \&c$, will co-incide with, or may be reduced to, the series found in the former part of this discourse for the value of $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, or the n th root of the quantity $1 + x$.

For, if n is an exact multiple of m , let it contain m exactly p times, or be equal to p times m , or pm . Then will $\sqrt[n]{1+x}$ be $= \sqrt[p]{1+x}^{\frac{m}{p}} = \sqrt[p]{1+x}^{\frac{1}{p}}$; and the series $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 - \&c$, will be $= 1 + \frac{m}{pm} A x - \frac{pm-m}{2pm} B x^2 + \frac{2pm-m}{3pm} C x^3 - \frac{3pm-m}{4pm} D x^4 + \frac{4pm-m}{5pm} E x^5 - \&c = 1 + \frac{1}{p} A x - \frac{p-1}{2p} B x^2 + \frac{2p-1}{3p} C x^3 - \frac{3p-1}{4p} D x^4 + \frac{4p-1}{5p} E x^5 - \&c$. Therefore $\sqrt[p]{1+x}$, or $\sqrt[p]{1+x}$, or the p th root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{1}{p} A x - \frac{p-1}{2p} B x^2 + \frac{2p-1}{3p} C x^3 - \frac{3p-1}{4p} D x^4 + \frac{4p-1}{5p} E x^5 - \&c$; which is the same series that was found above, in art. 47, for the value of any root of $1 + x$, excepting that the letter p is here used, instead of the letter n , to denote the index of the root that is to be extracted. Q. E. D.

79. And, if we suppose n to be exactly equal to m , and consequently $\frac{m}{n}$ to be equal to $\frac{m}{m}$, or 1 , and $\sqrt[n]{1+x}$ to be equal to $\sqrt[m]{1+x}$, or $\sqrt[1]{1+x}$, or $1 + x$, the foregoing series $1 + \frac{m}{n} A x - \frac{n-m}{2n} B x^2 + \frac{2n-m}{3n} C x^3 - \frac{3n-m}{4n} D x^4 + \frac{4n-m}{5n} E x^5 - \&c$ will be $= 1 + \frac{m}{m} A x - \frac{m-m}{2m} B x^2 + \frac{2m-m}{3m} C x^3 - \frac{3m-m}{4m} D x^4 + \frac{4m-m}{5m} E x^5 - \&c = 1 + 1 \times A x - \frac{1-1}{2} B x^2 + \frac{2-1}{3} C x^3 - \frac{3-1}{4} D x^4 + \frac{4-1}{5} E x^5 - \&c = 1 + 1 \times A x - \frac{0}{2} B x^2 + \frac{1}{3} C x^3 - \frac{2}{4} D x^4 + \frac{3}{5} E x^5 - \&c = 1 + 1 \times 1 \times x - 0 \times x^2 + \frac{1}{3} \times 0 \times x^3 - \frac{2}{4} \times 0 \times x^4 + \frac{3}{5} \times 0 \times x^5 - \&c = 1 + x - 0 + 0 - 0 + 0 - \&c = 1 + x$; as it ought to be.

80. We

80. We come now to the third case of fractional indexes, in which the numerator m of the index $\frac{m}{n}$ of the power of the binomial quantity $1 + x$, is greater than the denominator.

Of the Binomial Theorem in the case of $\sqrt[n]{1+x}^m$, or of the n th root of the m th power of the binomial quantity $1+x$, or the m th power of its n th root, when m is any whole number whatsoever, and n any other whole number less than m .

81. The methods of investigating the series that is equal to $\sqrt[n]{1+x}^m$ when the numerator m is greater than the denominator n , are the same with those which have been employed in the investigation of the series that is equal to the same quantity $\sqrt[n]{1+x}^m$ when the numerator m is less than the denominator n . But

the series that will be obtained by these methods for the value of $\sqrt[n]{1+x}^m$ will not be always exactly the same, of whatever magnitude greater than n we suppose x to be taken; as was the case with the series that was found for the value of $\sqrt[n]{1+x}^m$ when m is less than n : but it will be different for every new multiple of n that is contained in m . For, if m is greater than n , but less than $2n$, the third term of the series will be marked with the sign $+$, or added to the first term, and all the following terms will be marked with the signs $-$ and $+$ alternately: and, if m is greater than $2n$, but less than $3n$, the third and fourth terms of the series will be marked with the sign $+$, or added to the first term, and all the following terms will be marked with the signs $-$ and $+$ alternately: and, if m is greater than $3n$, but less than $4n$, the third and fourth and fifth terms of the series will be marked with the sign $+$, or added to the first term, and all the following terms will be marked with the signs $-$ and $+$ alternately: and, if m is greater than $4n$, but less than $5n$, the third, fourth, fifth and sixth terms of the series will be marked with the sign $+$, or added to the first term, and all the following terms will be marked with the signs $-$ and $+$ alternately; and in general, if m is greater than p times n , or pn , but less than $p+1$ times n , or $pn+n$, the third, fourth, fifth, sixth, and other following terms of the series, to the number of p terms, will be marked with the sign $+$, or added to the first term, and all the following terms will be marked with the signs $-$ and $+$ alternately; as was set forth in the beginning of this discourse in art. 5 and 6. Now, if we were to apply the two foregoing investigations, contained in art. 54, 55, 56, 57, 58, &c — 65, and in art. 67, 68, 69, 70, 71,

72, &c — 77, to the discovery of the series which is equal to $\sqrt[n]{1+x}^m$, when

6
 m is

m is greater than n ; and, in the resolution of each of the simple equations by which the co-efficients of the several powers of x would be to be determined, were to attend to all the different relative magnitudes of m and n , and to suppose m , first, to be greater than n , but less than $2n$, and, secondly, to be greater than $2n$, but less than $3n$, and, thirdly, to be greater than $3n$, but less than $4n$, and, fourthly, to be greater than $4n$, but less than $5n$, and so on with respect to all the co-efficients we were to investigate; we should find the examination of such a variety of cases intolerably tedious and laborious. And therefore, in applying the foregoing methods of investigation to the discovery of the

series which is equal to $\overline{1+x}^{\frac{m}{n}}$ in this second case, in which m is supposed to be greater than n , I shall make only the first of these suppositions, to wit, that m , though it is greater than n , is less than $2n$; which will reduce the investigation of this series to the same degree of difficulty, and no more, as was found in

the investigation of the series that is equal to $\overline{1+x}^{\frac{m}{n}}$ in the former case, in which m was supposed to be less n . Now upon this supposition, the series that

is equal to $\overline{1+x}^{\frac{m}{n}}$ may be investigated in the following manner.

82. The reasonings used above in art. 54, 55, 56, 57, in the three observations preparatory to the foregoing investigations, are evidently true when m , or the numerator of the index $\frac{m}{n}$ of the power of $1+x$ in the quantity $\overline{1+x}^{\frac{m}{n}}$, is of any magnitude greater than the denominator n , as well as when n is greater than m . And therefore it follows that in both cases the quantity $\overline{1+x}^{\frac{m}{n}}$ will be equal to the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, of which 1 is the first term and $\frac{m}{n}x$ is the second term, and is to be added to the first term, and consequently marked with the sign $+$, and the following terms are $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, or the second and other following powers of x in their natural order, without interruption, to wit, $x^2, x^3, x^4, x^5, \&c$, multiplied into certain fixed numeral co-efficients denoted by the capital letters $C, D, E, F, \&c$, and are to be connected with the two first terms $1 + \frac{m}{n}x$ either by addition or subtraction, and consequently are to have either the sign $+$ or the sign $-$ prefixed to them. It therefore only remains that we determine which of the said terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are to be added to the two first terms $1 + \frac{m}{n}x$, and consequently are to have the sign $+$ prefixed to them, and which of them are to be subtracted from the said two first terms, and consequently are to have the sign $-$ prefixed to them, and what are the several values, or magnitudes, of the co-efficients $C, D, E, F, \&c$, respectively.

83. Now let y be put $=$ the series $\frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (as in art. 58), or $1+y$ be $=$ the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is

is equal to $1 + x^{\frac{m}{n}}$. Then will $1 + y^n$ be = the m th power of $1 + x^{\frac{m}{n}}$, that is, to $1 + x^{\frac{m}{n}}$. And consequently the series $1 + \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c$, will be = the series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$, and consequently (subtracting 1 from both sides of the equation), the series $\frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}x^5 + \&c$, will be = the series $\frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c$, or (if, for the sake of brevity, we make $\frac{m}{1} \times \frac{m-1}{2} = Q$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} = R$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} = S$, and $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} = T$, and $\frac{n}{1} \times \frac{n-1}{2} = q$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} = r$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} = s$, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} = t$), the series $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c$, will be = the series $ny + qy^2 + ry^3 + sy^4 + ty^5 + \&c$.

84. Now let the several powers of the series $\frac{m}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c, or (putting $B = \frac{m}{n}$) of the series Bx , Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c (which is equal to y), be raised by multiplication, to the end that we may have the values of y^2, y^3, y^4, y^5 , &c, expressed in powers of x ; and let the said values of y^2, y^3, y^4, y^5 , &c, obtained by the said multiplications, be substituted instead of y^2, y^3, y^4, y^5 , &c, respectively in the last equation. And we shall then have the simple series $mx + Qx^2 + Rx^3 + Sx^4 + Tx^5 + \&c$ = the compound series

$$\begin{array}{rcl}
 n B x, & n C x^2, & n D x^3, & n E x^4, & n F x^5, & \&c \\
 + q B^2 x^2, & 2 q B C x^3, & 2 q B D x^4, & 2 q B E x^5, & \&c \\
 & q C^2 x^4, & 2 q C D x^5, & \&c \\
 + r B^3 x^3, & 3 r B^2 C x^4, & 3 r B^2 D x^5, & \&c \\
 & 3 r B C^2 x^5, & \&c \\
 + s B^4 x^4, & 4 s B^3 C x^5, & \&c \\
 + t B^5 x^5; & & & & &
 \end{array}$$

and consequently (dividing all the terms by x) the simple series $m + Qx + Rx^2 + Sx^3 + Tx^4 + \&c =$ the compound series

$$\begin{array}{rcccccc} nB, & nCx, & nDx^2, & nEx^3, & nFx^4, & \&c \\ + qB^2x, & 2qBCx^2, & 2qBDx^3, & 2qBEx^4, & \&c \\ & & qC^2x^3, & 2qCDx^4, & \&c \\ + rB^3x^2, & 3rB^2Cx^3, & 3rB^2Dx^4, & \&c \\ & & 3rBC^2x^4, & \&c \\ + sB^4x^3, & 4sB^3Cx^4, & \&c \\ + tB^5x^4, & \&c; \end{array}$$

as is shewn more at large above, in art. 58, 59, 60, and 61.

85. From this general and fundamental equation we may, by repeating the reasonings used above in art. 61, 62, 63, 64, derive the following particular equations for the determination of the magnitudes of the several co-efficients B, C, D, E, F, &c, and of the signs + and -, that are to be prefixed to them; to wit,

$$\begin{array}{l} \text{1st, } m = nB; \\ \text{2dly, } Q = nC + qB^2; \\ \text{3dly, } R = nD, 2qBC + rB^3; \\ \text{4thly, } S = nE, 2qBD, qC^2, 3rB^2C + sB^4, \\ \text{and 5thly, } T = nF, 2qBE, 2qCD, 3rB^2D, 3rBC^2, \\ \quad 4sB^3C + tB^5. \end{array}$$

86. From the first of these equations, to wit, $m = nB$, it follows that the co-efficient B is $= \frac{m}{n}$, as it has already been shewn to be in art. 56 and 57.

87. From the second of these equations, to wit, $Q = nC + qB^2$, or $Q = qB^2, nC$, we may deduce both the sign that is to be prefixed to nC and the magnitude of C, by proceeding in the manner following:

Since Q is $= m \times \frac{m-1}{2}$, and q is $n \times \frac{n-1}{2}$, and B is $= \frac{m}{n}$, we shall have

$$\begin{aligned} m \times \frac{m-1}{2} &= n \times \frac{n-1}{2} \times \frac{m}{n} B, nC, \text{ or } \frac{mm-m}{2} = \frac{mn-m}{2} B, nC, \text{ or } \frac{mm-m}{2} \\ &\times \frac{B}{B} = \frac{mn-m}{2} B, nC, \text{ or } \frac{mm-m}{2} \times \frac{n}{m} \times B = \frac{mn-m}{2} B, nC, \text{ or } \frac{m^2n-mn}{2m} B \\ &= \frac{m^2n-m^2}{2m} \times B, nC, \text{ and (adding } \frac{mn}{2m} B \text{ to both sides) } \frac{m^2n}{2m} B = \frac{m^2n+mn-m^2}{2m} \\ &B, nC, \text{ and (adding } \frac{m^2}{2m} B \text{ to both sides) } \frac{m^2n+m^2}{2m} B = \frac{m^2n+mn}{2m} B, nC, \text{ and} \\ &(\text{subtracting } \frac{m^2n}{2m} B \text{ from both sides) } \frac{m^2}{2m} B = \frac{mn}{2m} B, nC, \text{ or } \frac{m}{2} B = \frac{n}{2} B, nC; \\ &\text{and therefore (because } m \text{ is now supposed to be greater than } n, \text{ and consequently} \\ &\frac{m}{2} B \text{ is greater than } \frac{n}{2} B) \frac{m}{2} B \text{ will be } = \frac{n}{2} B + nC; \text{ and consequently } nC \\ &\text{will be } = \frac{m}{2} B - \frac{n}{2} B = \frac{m-n}{2} B, \text{ and C will be } = \frac{m-n}{2n} B = \frac{m-n}{2n} \times \frac{m}{n}, \\ &\text{or } \frac{m}{n} \times \frac{m-n}{2n}. \end{aligned}$$

Therefore Cx^2 , or the third term of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $1 + x \left| \frac{m}{n} \right|$, must be added to the two first

first terms $1 + Bx$, or $1 + \frac{m}{n}x$, and the three first terms of the said series will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2$. Q. E. I.

88. From the third of the foregoing equations, to wit, $R = nD, 2qBC + rB^3$, or (because we now know that the sign $+$ is to be prefixed to the term $2qBC$), $R = nD + 2qBC + rB^3$, we may deduce both the sign that is to be prefixed to nD , and the magnitude of D , by proceeding in the manner following:

By substituting $m \times \frac{m-1}{2} \times \frac{m-2}{3}$ in this equation instead of R , and $n \times \frac{n-1}{2}$ instead of q , and $n \times \frac{n-1}{2} \times \frac{n-2}{3}$ instead of r , and $\frac{m}{n}$ instead of B , and $\frac{m}{n} \times \frac{m-n}{2n}$ instead of C , it will become as follows, to wit, $m \times \frac{m-1}{2} \times \frac{m-2}{3} = nD + n \times \frac{n-1}{2} \times \frac{m}{n} \times \frac{m-n}{2n} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{m^3}{n^3}$, or $\frac{m^3-3m^2+2m}{6} = nD + m \times \frac{n-1}{2} \times \frac{m-n}{2n} + \frac{n^3-3n^2+2n}{6} \times \frac{m^3}{n^3}$, or $\frac{m^3n^2-3m^2n^2+2mn^2}{6nn} = nD + \frac{3m^2}{3n} \times \frac{n-1}{2} \times \frac{m-n}{2n} + m^3 \times \frac{n^3-3n^2+2n}{6nn} = nD + 3m^2 \times \frac{mn-mn-m+n}{6nn} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = nD + \frac{3m^3n-3m^2n^2-3m^3+3m^2n}{6nn} + \frac{m^3n^2-3m^3n+2m^3}{6nn} = nD - \frac{3m^2n^2-m^3+3m^2n+m^3n^2}{6nn}$; and consequently (adding $\frac{3m^2n^2}{6nn}$ to both sides) we shall have $\frac{m^3n^2+2mn^2}{6nn} = nD + \frac{3m^2n-m^3+m^3n^2}{6nn}$, and (adding $\frac{m^3}{6nn}$ to both sides) $\frac{m^3n^2+2mn^2+m^3}{6nn} = nD + \frac{3m^2n+m^3n^2}{6nn}$, and (subtracting $\frac{m^3n^2}{6nn}$ from both sides) $\frac{2mn^2+m^3}{6nn} = nD + \frac{3m^2n}{6nn}$, or $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn}, nD$; that is, $\frac{2mn^2+m^3}{6nn}$ will be equal to $\frac{3m^2n}{6nn}$ together with nD either added to it, or subtracted from it, as may be necessary to produce such equality.

89. Now, because m (though greater than n) is supposed to be less than $2n$, it follows that $m-n$ will be less than $2n-n$, or than n ; and consequently that $m \times m-n \times m-n$ will be less than $m \times m-n \times n$, or that $m \times m^2-2mn+n^2$ will be less than $mn \times m-n$, or that $m^3-2m^2n+mn^2$ will be less than m^2n-mn^2 . Therefore (adding mn^2 to both sides) $m^3-2m^2n+2mn^2$ will be less than m^2n , and (adding $2m^2n$ to both sides) m^3+2mn^2 will be less than $3m^2n$, and consequently $\frac{2mn^2+m^3}{6nn}$ will be less than $\frac{3m^2n}{6nn}$. Therefore, in order to make $\frac{3m^2n}{6nn}, nD$ be equal to $\frac{2mn^2+m^3}{6nn}$, we must subtract nD from $\frac{3m^2n}{6nn}$, and consequently must prefix the sign $-$ to nD ; so that the equation $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn}, nD$ will be $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn} - nD$. Therefore the sign $-$ must also be

prefixed to Dx^3 , or the fourth term of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\sqrt[n]{1 + x^m}$); from which fourth term the quantity nD was derived, by means of the operations of multiplication and division, in the course of the foregoing investigation. Therefore the four first terms of the said series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\sqrt[n]{1 + x^m}$, will be $1 + Bx + Cx^2 - Dx^3$, or $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - Dx^3$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - Dx^3$. Q. E. I.

90. And to determine the magnitude of D , we shall have the aforefaid equation $\frac{2mn^2+m^3}{6nn} = \frac{3m^2n}{6nn} - nD$; whence (by adding nD to both sides) we shall have $\frac{2mn^2+m^3}{6nn} + nD = \frac{3m^2n}{6nn}$, and (by subtracting $\frac{2mn^2+m^3}{6nn}$ from both sides) $nD = \frac{3m^2n-2mn^2-m^3}{6nn} = m \times \frac{3mn-2n^2-m^2}{6nn} = m \times \frac{m-n \times 2n-m}{6nn} = m \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$, and consequently (by dividing both sides by n) $D = \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$. Therefore the four first terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\sqrt[n]{1 + x^m}$), will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 -$

$\frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \left[\frac{2n-m}{3n} \right] Cx^3$. Q. E. I.

91. These four terms of the series which is equal to $\sqrt[n]{1 + x^m}$ in this case of the relative magnitudes of m and n , in which m is greater than n , but less than $2n$, are the same with the four first terms of the series given above in the beginning of this discourse, in art. 6, for the value of $\sqrt[n]{1 + x^m}$ in the same case. Therefore the said series, in art. 6 is true, at least in its four first terms.

92. I shall not attempt to find the values of the co-efficients E and F by resolving the fourth and fifth equations set down in art. 85, to wit, the equation $S = nE, 2qBD, qC^2, 3rB^2C + sB^4$, or (because it has been shewn that the co-efficient C is to have the sign $+$ prefixed to it, and the co-efficient D is to have the sign $-$ prefixed to it) $S = nE - 2qBD + qC^2 + 3rB^2C + sB^4$, and the equation $T = nF, 2qBE, 2qCD, 3rB^2D, 3rBC^2, 4sB^3C + tB^5$, or $T = nF, 2qBE - 2qCD - 3rB^2D + 3rBC^2 + 4sB^3C + tB^5$: I say, I shall not attempt to find the values of E and F by resolving these two equations, for the reasons given above in art. 65 and 66; but shall proceed to apply the other method of investigating the values of the third and fourth, and other following terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5,$

$F x^5$, &c, which is equal to $\sqrt[n]{1+x}^m$, to the present case, in which m is supposed to be greater than n , but less than $2n$. This may be done in the manner following:

93. It has been shewn in art. 82 that the two first terms of the series that is equal to $\sqrt[n]{1+x}^m$ will be 1 and $\frac{m}{n}x$ when m is greater than n , as well as when n is greater than m ; and that the second term $\frac{m}{n}x$ is to be added to the first term 1, and consequently to have the sign $+$ prefixed to it, in this case as well as in the former; and likewise that the following terms of the said series will be $C x^2$, $D x^3$, $E x^4$, $F x^5$, &c (or the square and cube and other following powers of x in their natural order, without interruption, multiplied into certain first numeral co-efficients, which may be denoted by the capital letters C, D, E, F, &c), and connected with the two first terms $1 + \frac{m}{n}x$ by either addition or subtraction: so that the series that is equal to $\sqrt[n]{1+x}^m$ is in this case, as well as in the former, $1 + \frac{m}{n}x, C x^2, D x^3, E x^4, F x^5$, &c; in which the several terms $C x^2, D x^3, E x^4, F x^5$, &c, have a comma prefixed to them, instead of either of the signs $+$ and $-$, because we do not as yet know to which of them we are to prefix the sign $+$, and to which we are to prefix the sign $-$.

94. Now, since $\sqrt[n]{1+x}^m$ is in this case, as well as in the former, = the series $1 + \frac{m}{n}x, C x^2, D x^3, E x^4, F x^5$, &c, it follows from art. 67, 68, 69, 70, 71, 72, that the simple series $\frac{m}{n}, C x, D x^2, E x^3, F x^4$, &c, will be equal to the compound series

$$\frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \frac{5n}{m} F x^3, \&c$$

$$+ 1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \&c; \text{ by}$$

the help of which equation we may discover both “which of the signs $+$ and $-$ are to be prefixed to the several co-efficients C, D, E, F, &c (and consequently to the corresponding terms $C x^2, D x^3, E x^4, F x^5$, &c, in the series

$1 + \frac{m}{n}x, C x^2, D x^3, E x^4, F x^5$, &c, which is equal to $\sqrt[n]{1+x}^m$),” and also “what are the magnitudes of the said co-efficients C, D, E, F, &c, respectively.” This may be done by proceeding in the manner following:

95. In the first place, since the simple series $\frac{m}{n} C x, D x^2, E x^3, F x^4$, &c, is equal to the compound series

$$\frac{2n}{m} C, \frac{3n}{m} D x, \frac{4n}{m} E x^2, \frac{5n}{m} F x^3, \&c$$

$$+ 1, \frac{2n}{m} C x, \frac{3n}{m} D x^2, \frac{4n}{m} E x^3, \&c; \text{ and}$$

this

this equation is always true, of how small a magnitude soever we suppose x to be taken, it follows that it will also be true when x is $= 0$. But, when x is $= 0$, all the terms in the equation that involve x will be equal to 0 likewise; that is, all the terms on the left-hand side of the equation, except the first term $\frac{m}{n}$, and all the terms on the right-hand side of the equation, except the two first terms of the two lines of terms, to wit, $\frac{2n}{m} C + 1$, will be $= 0$. Therefore $\frac{m}{n}$ will be $=$ to $\frac{2n}{m} C + 1$, or to $1, \frac{2n}{m} C$, or to 1 , together with $\frac{2n}{m} C$, either added to it, or subtracted from it, as may be necessary to produce such equality. Now, because m is supposed in the present case to be greater than n , $\frac{m}{n}$ will be greater than 1 , and consequently $\frac{2n}{m} C$ must be added to 1 , in order to make it equal to $\frac{m}{n}$. Therefore the sign $+$ must be prefixed to $\frac{2n}{m} C$ in the equation $\frac{m}{n} = \frac{2n}{m} C + 1$, or $\frac{m}{n} = 1, \frac{2n}{m} C$; and consequently the sign $+$ must also be prefixed to the third term Cx^2 of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$); from which third term the said quantity $\frac{2n}{m} C$ has been derived, by the operations of multiplication and division, in the processes of the investigation contained in art. 67, 68, — 72. Therefore the three first terms of the said series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x + Cx^2$. Q. E. I.

And the magnitude of the co-efficient C may likewise be determined by means of the equation $\frac{m}{n} = 1, \frac{2n}{m} C$, or $\frac{m}{n} = 1 + \frac{2n}{m} C$. For (by subtracting 1 from both sides) we shall have $\frac{2n}{m} C = \frac{m}{n} - 1 = \frac{m-n}{n}$; and (by dividing both sides by $\frac{2n}{m}$, or multiplying them into $\frac{m}{2n}$) we shall have $C = (\frac{m}{2n} \times \frac{m-n}{n} =) \frac{m}{n} \times \frac{m-n}{2n}$. Therefore the three first terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{m}{n}}$, will be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n} x^2$, or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2$. Q. E. I.

96. To determine the sign that is to be prefixed to the fourth term, Dx^3 , of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{m}{n}}$, and to find the magnitude of its co-efficient D , we must proceed as follows:

Since

Since the sign $+$ is to be prefixed to the co-efficient C , it must likewise be prefixed to all the terms which involve it in the grand fundamental equation set down in art. 94, to wit, the equation between the simple series $\frac{m}{n}$, Cx , Dx^2 , Ex^3 , Fx^4 , &c, and the compound series

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c \\ + 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c; \text{ which}$$

equation will therefore be as follows, to wit, the simple series $\frac{m}{n} + Cx, Dx^2, Ex^3, Fx^4, \&c =$ the compound series

$$+ \frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c \\ + 1 + \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c.$$

But it has been shewn in the last article that $\frac{m}{n}$ is $= 1 + \frac{2n}{m}C$. Therefore, if we subtract $\frac{m}{n}$ and $1 + \frac{2n}{m}C$ from the opposite sides of the last equation, the remainders will be equal to each other; that is, the simple series $Cx, Dx^2, Ex^3, Fx^4, \&c$, will be equal to the compound series

$$\frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c \\ + \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c;$$

and consequently (dividing all the terms by x) we shall have the simple series $C, Dx, Ex^2, Fx^3, \&c =$ the compound series

$$\frac{3n}{m}D, \frac{4n}{m}Ex, \frac{5n}{m}Fx^2, \&c \\ + \frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \&c.$$

And this equation will be true, of how small a magnitude soever we suppose x to be taken: and therefore it will also be true when x is $= 0$. But, when x is equal to 0, all the terms that involve x will be equal to 0 likewise, and consequently the equation will be $C = \frac{3n}{m}D + \frac{2n}{m}C$, or $C = + \frac{2n}{m}C, \frac{3n}{m}D$; that is, C will be equal to $\frac{2n}{m}C$, together with $\frac{3n}{m}D$ either added to it or subtracted from it, as may be necessary to produce the said equality. But, because m is supposed to be less than $2n$, C will be less than $\frac{2n}{m}C$; and consequently $\frac{3n}{m}D$ must be subtracted from $\frac{2n}{m}C$, in order to make it equal to C . Therefore the sign $-$ must be prefixed to the quantity $\frac{3n}{m}D$; and consequently the same sign must also be prefixed to the fourth term Dx^3 , of the series $1 + \frac{m}{n}x, Cx^2, Dx^3,$

$Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$); from which fourth term the said quantity $\frac{3^n}{m} D$ has been derived, by the operations of multiplication and division, in the course of the investigation by which the fundamental equation, set down in art. 94, was obtained. Therefore the four first terms of the series

$$1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c, \text{ which is equal to } \overline{1+x}^{\frac{m}{n}}, \text{ will be}$$

$$1 + \frac{m}{n}x + Cx^2 - Dx^3, \text{ or } 1 + \frac{m}{n}x + \times \frac{m-n}{2n}x^2 - Dx^3, \text{ or } 1 + \frac{m}{n}$$

$$Ax + \frac{m-n}{2n}Bx^2 - Dx^3. \quad \text{Q. E. I.}$$

And, to determine the magnitude of D , we shall have the equation $C = \frac{2n}{m}$
 $C - \frac{3^n}{m}D$; whence (adding $\frac{3^n}{m}D$ to both sides) we shall have $C + \frac{3^n}{m}D =$
 $\frac{2n}{m}C$, and (subtracting C from both sides) $\frac{3^n}{m}D = \frac{2n}{m}C - C = \left[\frac{2n}{m} - 1\right] \times C$
 $= \frac{2n-m}{m} \times C$, and (multiplying both sides by $\frac{m}{3^n}$) $D = \frac{2n-m}{3^n} \times C = \frac{2n-m}{3^n}$
 $\times \frac{m}{n} \times \frac{m-n}{2n}$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3^n}$. Therefore the four first terms of the

$$\text{series } 1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c, \text{ which is equal to } \overline{1+x}^{\frac{m}{n}}, \text{ will}$$

$$\text{be } 1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3^n}x^3, \text{ or } 1 + \frac{m}{n}Ax +$$

$$\frac{m-n}{2n}Bx^2 - \left[\frac{2n-m}{3^n}\right]Cx^3. \quad \text{Q. E. I.}$$

97. To determine the sign that is to be prefixed to the fifth term, Ex^4 , of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, which is equal to $\overline{1+x}^{\frac{m}{n}}$, and to find the magnitude of the co-efficient E , we must proceed as follows:

It has been shewn in the last article that the simple series $C, Dx, Ex^2, Fx^3, \&c$, is equal to the compound series

$$\frac{3^n}{m}D, \frac{4^n}{m}Ex, \frac{5^n}{m}Fx^2, \&c$$

$$+ \frac{2^n}{m}C, \frac{3^n}{m}Dx, \frac{4^n}{m}Ex^2, \&c;$$

and that the sign $-$ is to be prefixed to the terms that involve the co-efficient D . It therefore follows that the simple series $C - Dx, Ex^2, Fx^3, \&c$, will be equal to the compound series

$$- \frac{3^n}{m}D, \frac{4^n}{m}Ex, \frac{5^n}{m}Fx^2, \&c$$

$$+ \frac{2^n}{m}C - \frac{3^n}{m}Dx, \frac{4^n}{m}Ex^2, \&c.$$

Therefore, if we add $2Dx$ to both sides, we shall have the simple series $C + Dx, Ex^2, Fx^3, \&c =$ the compound series

$$- \frac{3^n}{m}D,$$

$$\begin{aligned}
 & - \frac{3^n}{m} D, \quad \frac{4^n}{m} E x, \quad \frac{5^n}{m} F x^2, \quad \&c \\
 & + \frac{2^n}{m} C - \frac{3^n}{m} D x, \quad \frac{4^n}{m} E x^2, \quad \&c \\
 & \quad + 2 D x.
 \end{aligned}$$

Further, it has also been shewn, in the last article, that C is $= \frac{2^n}{m} C - \frac{3^n}{m} D$.

Therefore, if we subtract C and $\frac{2^n}{m} C - \frac{3^n}{m} D$ from the opposite sides of the last equation, the remainders will be equal; that is, the simple series $D x, E x^2, F x^3, \&c$, will be equal to the compound series

$$\begin{aligned}
 & \frac{4^n}{m} E x, \quad \frac{5^n}{m} F x^2, \quad \&c \\
 & - \frac{3^n}{m} D x, \quad \frac{4^n}{m} E x^2, \quad \&c
 \end{aligned}$$

+ $2 D x$. Therefore (dividing all the terms by x) we shall have the simple series $+ D, E x, F x^2, \&c =$ the compound series

$$\begin{aligned}
 & \frac{4^n}{m} E, \quad \frac{5^n}{m} F x, \quad \&c \\
 & - \frac{3^n}{m} D, \quad \frac{4^n}{m} E x, \quad \&c \\
 & \quad + 2 D.
 \end{aligned}$$

And this equation will be true, of how small a magnitude soever we suppose x to be taken: and therefore it will also be true when x is $= 0$. But when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will then be $+ D = \frac{4^n}{m} E - \frac{3^n}{m} D + 2 D$. Therefore (adding $\frac{3^n}{m} D$ to both sides) we shall have $D + \frac{3^n}{m} D = \frac{4^n}{m} E + 2 D$, and (subtracting D from both sides) $\frac{3^n}{m} D = \frac{4^n}{m} E + D$, or $\frac{3^n}{m} D = D, \frac{4^n}{m} E$, or $\frac{3^n}{m} D = D$, together with $\frac{4^n}{m} E$, either added to it, or subtracted from it, as may be necessary to produce such equality. But, because m is supposed to be less than $2n$, $\frac{3^n}{m}$ will be greater than 1, and $\frac{3^n}{m} D$ will be greater than D ; and consequently $\frac{4^n}{m} E$ must be added to D , in order to make it equal to $\frac{3^n}{m} D$. Therefore the sign $+$ must be prefixed to the quantity $\frac{4^n}{m} E$ in the equation $\frac{3^n}{m} D = D, \frac{4^n}{m} E$, and therefore it must also be prefixed to the fifth term, $E x^4$, of the series $1 + \frac{m}{n} x, C x^2, D x^3,$

$E x^4, F x^5, \&c$ (which is equal to $\overline{1 + x}^{\frac{m}{n}}$), from which fifth term the said quantity $\frac{4^n}{m} E$ has been derived above in art. 67, 68, 72, by the operations of multiplication and division. Therefore the five first terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\overline{1 + x}^{\frac{m}{n}}$, will be $1 + \frac{m}{n} x$

$$+ \frac{m}{n} \times \frac{m-n}{2n} x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} x^3 + E x^4, \text{ or } 1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 - \sqrt{\frac{2n-m}{3n}} C x^3 + E x^4. \quad Q. E. I.$$

And to determine the magnitude of the co-efficient E, we shall have the equation $\frac{3n}{m} D = D + \frac{4n}{m} E$; whence (subtracting D from both sides) $\frac{4n}{m} E$ will be $= \frac{3n}{m} D - D = \left[\frac{3n}{m} - 1 \right] \times D = \frac{3n-m}{m} \times D$, and consequently (multiplying both sides into $\frac{m}{4n}$) E will be $= \frac{3n-m}{4n} \times D = \frac{3n-m}{4n} \times \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}$, or $\frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}$. Therefore the five first terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1+x}$, will be $1 + \frac{m}{n} x + \frac{m}{n} \times \frac{m-n}{2n} x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4$, or $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 - \sqrt{\frac{2n-m}{3n}} C x^3 + \frac{3n-m}{4n} D x^4. \quad Q. E. I.$

98. To determine the sign that is to be prefixed to the sixth term, $F x^5$, of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$, which is equal to $\sqrt[n]{1+x}$, and to find the magnitude of the co-efficient F, we must proceed as follows:
It has been shewn, in the last article, that the simple series D, $E x, F x^2, \&c$, is equal to the compound series

$$\begin{aligned} & \frac{4n}{m} E, \frac{5n}{m} F x, \&c \\ & - \frac{3n}{m} D, \frac{4n}{m} E x, \\ & + 2 D, \quad \&c, \text{ and also that the sign } + \text{ is to} \\ & \text{be prefixed to the terms } \frac{4n}{m} E \text{ and } \frac{4n}{m} E x, \text{ and } E x. \text{ Therefore the said equation will, when these three terms have their proper signs prefixed to them, be as follows; to wit, the simple series } D + E x, F x^2, \&c = \text{the compound series} \\ & + \frac{4n}{m} E, \frac{5n}{m} F x, \frac{6n}{m} G x^2, \&c \\ & - \frac{3n}{m} D + \frac{4n}{m} E x, \frac{5n}{m} F x^2, \&c \\ & + 2 D. \end{aligned}$$

Further, it has been shewn, in the last article, that D is equal to $+\frac{4n}{m} E - \frac{3n}{m} D + 2 D$.

Therefore, if we subtract D and $+\frac{4n}{m} E - \frac{3n}{m} D + 2 D$ from the opposite sides of the foregoing equation, the remainders will be equal, that is, the simple series $+ E x, F x^2, \&c$, will be equal to the compound series

$\frac{5^n}{m} F x, \frac{6^n}{m} G x^2, \&c$
 $+ \frac{4^n}{m} E x, \frac{5^n}{m} F x^2, \&c$; and consequently (dividing all the terms on both sides by x) the simple series $+ E, F x, \&c$, will be equal to the compound series

$$\frac{5^n}{m} F, \frac{6^n}{m} G x, \&c$$

$$+ \frac{4^n}{m} E, \frac{5^n}{m} F x, \&c.$$

And this equation will be true, of how small a magnitude soever we may suppose x to be taken; and therefore it will also be true when x is $= 0$. But, when x is $= 0$, all the terms that involve x will be equal to 0 likewise, and consequently the equation will then be $+ E = \frac{5^n}{m} F + \frac{4^n}{m} E$, or $+ E = + \frac{4^n}{m} E, \frac{5^n}{m} F$; that is, E will be equal to $\frac{4^n}{m} E$, together with $\frac{5^n}{m} F$, either added to it or subtracted from it, as may be necessary to produce such equality. But, because $2n$ is supposed to be greater than m , $4n$ will, *à fortiori*, be greater than m , and consequently $\frac{4^n}{m} E$ will be greater than E . Therefore, in order to make $\frac{4^n}{m} E, \frac{5^n}{m} F$ be equal to E , we must subtract $\frac{5^n}{m} F$ from $\frac{4^n}{m} E$, or prefix the sign $-$ to the term $\frac{5^n}{m} F$. Therefore the same sign $-$ must also be prefixed to the sixth term, $F x^5$, of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$ (which is equal to $1 + x^{\frac{m}{n}}$), from which sixth term the said quantity $\frac{5^n}{m} F$ has been derived by the operations of multiplication and division in the course of the foregoing investigation. Therefore the first six terms of the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, \&c$ (which is equal to $1 + x^{\frac{m}{n}}$), will in this case be $1 + \frac{m}{n} x + \frac{m}{n} \times \frac{m-n}{2n} x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 - F x^5$, or $1 + \frac{m}{n} A x + \frac{m-n}{2n} B x^2 - \sqrt{\frac{2n-m}{3n}} C x^3 + \frac{3n-m}{4n} D x^4 - F x^5$.
Q. E. I.

And to determine the magnitude of the co-efficient F , we shall have the equation $E = \frac{4^n}{m} E - \frac{5^n}{m} F$; whence (adding $\frac{5^n}{m} F$ to both sides) we shall have $E + \frac{5^n}{m} F = \frac{4^n}{m} E$, and (subtracting E from both sides) $\frac{5^n}{m} F (= \frac{4^n}{m} E - E = \frac{4^n}{m} - 1 \times E) = \frac{4^n - m}{m} \times E$, and (multiplying both sides into $\frac{m}{5^n}$) $F = \frac{4^n - m}{5^n} \times E = \frac{4^n - m}{5^n} \times \frac{3n - m}{4n} \times \frac{2n - m}{3n} \times \frac{m - n}{2n} \times \frac{m}{n}$. Therefore the first six
2 O 2 terms

terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$), will (upon the supposition here made, that m is greater than n , but less than $2n$) be $1 + \frac{m}{n}x + \frac{m}{n} \times \frac{m-n}{2n}x^2 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n}x^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}x^4 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}x^5$,
 or $1 + \frac{m}{n}Ax + \frac{m-n}{2n}Bx^2 - \left[\frac{2n-m}{3n}Cx^3 + \frac{3n-m}{4n}Dx^4 - \left[\frac{4n-m}{5n}Ex^5\right.\right.$
Q. E. I.

99. In like manner we may derive from the general equation in art. 94, to wit, the equation between the simple series $\frac{m}{n}, Cx, Dx^2, Ex^3, Fx^4, Gx^5, \&c$, and the compound series

$$\frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \frac{6n}{m}Gx^4, \frac{7n}{m}Hx^5, \&c$$

$$+ 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \frac{5n}{m}Fx^4, \frac{6n}{m}Gx^5, \&c,$$

the following particular equations for determining both the signs that are to be prefixed to the following terms, $Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$, of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11},$

$\&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$), and the values of the co-efficients $G, H, I, K, L, M, \&c$; to wit, the equations

$$F = \frac{6n}{m}G, \frac{5n}{m}F,$$

$$G = \frac{7n}{m}H, \frac{6n}{m}G,$$

$$H = \frac{8n}{m}I, \frac{7n}{m}H,$$

$$I = \frac{9n}{m}K, \frac{8n}{m}I,$$

$$K = \frac{10n}{m}L, \frac{9n}{m}K,$$

$$L = \frac{11n}{m}M, \frac{10n}{m}L,$$

$\&c,$

or $F = \frac{5n}{m}F, \frac{6n}{m}G,$

$$G = \frac{6n}{m}G, \frac{7n}{m}H,$$

$$H = \frac{7n}{m}H, \frac{8n}{m}I,$$

$$I = \frac{8n}{m}I, \frac{9n}{m}K,$$

$K =$

$$K = \frac{9^n}{m} K, \frac{10^n}{m} L,$$

$$L = \frac{10^n}{m} L, \frac{11^n}{m} M,$$

&c, as far as we please to continue them.

100. And, in all these equations, the single term which forms the left-hand side of the equation, will always have the same sign + or - prefixed to it as is prefixed to the term on the right-hand side of the equation that involves in it the same letter. Thus, the two terms F and $\frac{5^n}{m} F$, in the first equation, will have the same sign prefixed to them; and the two terms G and $\frac{6^n}{m} G$, in the second equation, will have the same sign prefixed to them; and the two terms H and $\frac{7^n}{m} H$, in the third equation, will have the same sign prefixed to them; and the following letters, I, K, and L, in the fourth, fifth, and sixth equations, will, in like manner, have the same signs prefixed to them as are prefixed to the terms $\frac{8^n}{m} I$, $\frac{9^n}{m} K$, $\frac{10^n}{m} L$, respectively. This follows evidently from the observation that has been often repeated in the foregoing articles, to wit, that the terms that involve the same letters in these latter, or particular equations, are all derived from the same terms of the same original series, to wit, the series $1 + \frac{m}{n} x, C x^2, D x^3, E x^4, F x^5, G x^6, H x^7, I x^8, K x^9, L x^{10}, M x^{11}, \&c$ (which is equal to $\overline{1 + x}^{\frac{m}{n}}$), by the operations of multiplication and division in the course of the foregoing investigation; by which operations the signs + and -, that are to be prefixed to them, cannot be affected.

101. Further, the terms denoted by the single letters F, G, H, I, K, L, &c, which form the left-hand sides of these equations, are uniformly less than the terms $\frac{5^n}{m} F, \frac{6^n}{m} G, \frac{7^n}{m} H, \frac{8^n}{m} I, \frac{9^n}{m} K, \frac{10^n}{m} L, \&c$, on the right-hand sides of the same equations which involve the same letters respectively; because m is supposed to be less than $2n$, and, *à fortiori*, less than $5n, 6n, 7n, 8n, 9n, 10n, \&c$. It follows, therefore, that, in order to make the right-hand sides of these several equations be equal to the left-hand sides of them respectively, it is necessary that the sign + or -, that is to be prefixed to the second term on the right-hand side of each of these equations, should be contrary to the sign which is prefixed to the first term. And hence we may determine the several signs that are to be prefixed to the several terms of the said particular equations, in the manner following.

102. It having been shewn, in art. 98, that the sign - is to be prefixed to the co-efficient F, and to the term $\frac{5^n}{m} F$, which involves it, it follows, from the last article, that the sign + must be prefixed to the remaining term $\frac{6^n}{m} G$, in the first
of

of the foregoing particular equations, to wit, the equation $F = \frac{5^n}{m} F, \frac{6^n}{m} G$; and consequently the said equation, when its terms have their proper signs prefixed to them, will be $- F = - \frac{5^n}{m} F + \frac{6^n}{m} G$.

And, secondly, since $\frac{6^n}{m} G$ has the sign $+$ prefixed to it, it follows, from art. 100, that the term G in the second equation, $G = \frac{6^n}{m} G, \frac{7^n}{m} H$, must also have the sign $+$ prefixed to it; and consequently, by art. 101, that the remaining term, $\frac{7^n}{m} H$, must have the sign $-$ prefixed to it; and consequently the said second equation, when its terms have their proper signs prefixed to them, will be $+ G = + \frac{6^n}{m} G - \frac{7^n}{m} H$.

And, thirdly, since $\frac{7^n}{m} H$ has the sign $-$ prefixed to it, it follows, from art. 100, that the term H in the third equation, $H = \frac{7^n}{m} H, \frac{8^n}{m} I$, must also have the sign $-$ prefixed to it, and consequently (by art. 101) that the remaining term, $\frac{8^n}{m} I$, must have the sign $+$ prefixed to it; and consequently the said third equation, when its terms have their proper signs prefixed to them, will be $- H = - \frac{7^n}{m} H + \frac{8^n}{m} I$.

And in like manner it may be shewn that the fourth equation, $I = \frac{8^n}{m} I, \frac{9^n}{m} K$, when the proper signs are prefixed to its terms, will be $+ I = + \frac{8^n}{m} I - \frac{9^n}{m} K$; and that the fifth equation, $K = \frac{9^n}{m} K, \frac{10^n}{m} L$, when the proper signs are prefixed to its terms, will be $- K = - \frac{9^n}{m} K + \frac{10^n}{m} L$; and that the sixth equation, $L = \frac{10^n}{m} L, \frac{11^n}{m} M$, when the proper signs are prefixed to its terms, will be $+ L = + \frac{10^n}{m} L - \frac{11^n}{m} M$.

Therefore the six foregoing equations, when their proper signs $+$ and $-$ are prefixed to their terms, will be as follows; to wit,

$$\begin{aligned} - F &= - \frac{5^n}{m} F + \frac{6^n}{m} G, \\ + G &= + \frac{6^n}{m} G - \frac{7^n}{m} H, \\ - H &= - \frac{7^n}{m} H + \frac{8^n}{m} I, \\ + I &= + \frac{8^n}{m} I - \frac{9^n}{m} K, \\ - K &= - \frac{9^n}{m} K + \frac{10^n}{m} L, \\ + L &= + \frac{10^n}{m} L - \frac{11^n}{m} M. \end{aligned}$$

In these equations the several co-efficients F, G, H, I, K, and L, have the signs — and + prefixed to them alternately; and it is easy to see that the same thing must take place in all the following co-efficients, M, N, O, P, Q, R, S, T, &c, to whatever number of terms the series be continued. We may therefore conclude that in the series $1 + \frac{m}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , Ix^8 , Kx^9 , Lx^{10} , Mx^{11} , Nx^{12} , Ox^{13} , Px^{14} , Qx^{15} , Rx^{16} , Sx^{17} , Tx^{18} , &c

(which is equal to $1 + x^{\frac{m}{n}}$), when m is greater than n , but less than $2n$, the third term, Cx^2 , is to be added to the two first $1 + \frac{m}{n}x$, and consequently marked with the sign +, and all the following terms Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , Ix^8 , Kx^9 , Lx^{10} , Mx^{11} , Nx^{12} , Ox^{13} , Px^{14} , Qx^{15} , Rx^{16} , Sx^{17} , Tx^{18} , &c, to whatever number of terms the series be continued, are to be marked with the signs — and + alternately; and consequently that, with respect to the signs of its terms, the said series will be as follows, to wit, $1 + \frac{m}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + Rx^{16} - Sx^{17} + Tx^{18} - \&c.$ Q. E. I.

103. It remains that we determine the magnitudes of the several co-efficients F, G, H, I, K, L, &c. Now these may be easily found by resolving the short simple equations obtained in the foregoing article; to wit, the equations

$$\begin{aligned} -F &= -\frac{5n}{m}F + \frac{6n}{m}G, \\ +G &= +\frac{6n}{m}G - \frac{7n}{m}H, \\ -H &= -\frac{7n}{m}H + \frac{8n}{m}I, \\ +I &= +\frac{8n}{m}I - \frac{9n}{m}K, \\ -K &= -\frac{9n}{m}K + \frac{10n}{m}L, \\ +L &= +\frac{10n}{m}L - \frac{11n}{m}M, \end{aligned}$$

&c; which may be done in the manner following:

To resolve the first equation, $-F = -\frac{5n}{m}F + \frac{6n}{m}G$, let $\frac{5n}{m}F$ be added to both sides; and we shall have $\frac{6n}{m}G = \frac{5n}{m}F - F = \left(\frac{5n}{m} - 1\right) \times F = \frac{5n-m}{m} \times F$, and consequently (multiplying both sides into $\frac{m}{6n}$) $G = \frac{5n-m}{6n} \times F$.

Q. E. I.

To resolve the second equation $+G = \frac{6n}{m}G - \frac{7n}{m}H$, let $\frac{7n}{m}H$ be added to both sides; and we shall have $+G + \frac{7n}{m}H = \frac{6n}{m}G$, and consequently (subtracting G from both sides) $\frac{7n}{m}H = \frac{6n}{m}G - G = \left(\frac{6n}{m} - 1\right) \times G = \frac{6n-m}{m} \times G$, and (multiplying both sides by $\frac{m}{7n}$) $H = \frac{6n-m}{7n} \times G$. Q. E. I.

To

To resolve the third equation $-H = -\frac{7^n}{m}H + \frac{8^n}{m}I$, let $\frac{7^n}{m}H$ be added to both sides; and we shall have $\frac{8^n}{m}I = \frac{7^n}{m}H - H = \frac{7^n}{m} - 1 \times H = \frac{7^n - m}{m} \times H$, and consequently (multiplying both sides by $\frac{m}{8^n}$) $I = \frac{7^n - m}{8^n} \times H$.
Q. E. I.

To resolve the fourth equation $+I = \frac{8^n}{m}I - \frac{9^n}{m}K$, let $\frac{9^n}{m}K$ be added to both sides; and we shall have $I + \frac{9^n}{m}K = \frac{8^n}{m}I$, and consequently (subtracting I from both sides) $\frac{9^n}{m}K = \frac{8^n}{m}I - I = \frac{8^n}{m} - 1 \times I = \frac{8^n - m}{m} \times I$, and (multiplying both sides by $\frac{m}{9^n}$) $K = \frac{8^n - m}{9^n} \times I$.
Q. E. I.

To resolve the fifth equation $-K = -\frac{9^n}{m}K + \frac{10^n}{m}L$, let $\frac{9^n}{m}K$ be added to both sides; and we shall have $\frac{10^n}{m}L = \frac{9^n}{m}K - K = \frac{9^n}{m} - 1 \times K = \frac{9^n - m}{m} \times K$, and consequently (multiplying both sides by $\frac{m}{10^n}$) $L = \frac{9^n - m}{10^n} \times K$.
Q. E. I.

And, to resolve the sixth equation, $+L = +\frac{10^n}{m}L - \frac{11^n}{m}M$, let $\frac{11^n}{m}M$ be added to both sides; and we shall have $L + \frac{11^n}{m}M = \frac{10^n}{m}L$, and consequently (subtracting L from both sides) $\frac{11^n}{m}M = \frac{10^n}{m}L - L = \frac{10^n}{m} - 1 \times L = \frac{10^n - m}{m} \times L$, and (multiplying both sides by $\frac{m}{11^n}$) $M = \frac{10^n - m}{11^n} \times L$.
Q. E. I.

It appears therefore that

$$G \text{ will be } = \frac{5^n - m}{6^n} \times F,$$

$$\text{And that } H \text{ will be } = \frac{6^n - m}{7^n} \times G,$$

$$\text{and } I = \frac{7^n - m}{8^n} \times H,$$

$$\text{and } K = \frac{8^n - m}{9^n} \times I,$$

$$\text{and } L = \frac{9^n - m}{10^n} \times K,$$

$$\text{and } M = \frac{10^n - m}{11^n} \times L.$$

And it is easy to see that, if we were to investigate the values of the following co-efficients, N, O, P, Q, R, S, T, &c, by the resolution of the following short, simple, equations which relate to them respectively, we should find

N to

$$N \text{ to be } = \frac{11n-m}{12n} \times M,$$

$$\text{and } O = \frac{12n-m}{13n} \times N,$$

$$\text{and } P = \frac{13n-m}{14n} \times O,$$

$$\text{and } Q = \frac{14n-m}{15n} \times P,$$

$$\text{and } R = \frac{15n-m}{16n} \times Q,$$

$$\text{and } S = \frac{16n-m}{17n} \times R,$$

$$\text{and } T = \frac{17n-m}{18n} \times S,$$

and so on *ad infinitum*, every new generating fraction being derived from that which immediately preceeds it by adding n to both its numerator and its denominator.

104. We may therefore now conclude that when m is greater than n , but less than $2n$, the quantity $\sqrt[n]{1+x}^{\frac{m}{n}}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 - \sqrt{\frac{2n-m}{3n}} Cx^3 + \frac{3n-m}{4n} Dx^4 - \sqrt{\frac{4n-m}{5n}} Ex^5 + \frac{5n-m}{6n} Fx^6 - \sqrt{\frac{6n-m}{7n}} Gx^7 + \frac{7n-m}{8n} Hx^8 - \sqrt{\frac{8n-m}{9n}} Ix^9 + \frac{9n-m}{10n} Kx^{10} - \sqrt{\frac{10n-m}{11n}} Lx^{11} + \frac{11n-m}{12n} Mx^{12} - \sqrt{\frac{12n-m}{13n}} Nx^{13} + \frac{13n-m}{14n} Ox^{14} - \sqrt{\frac{14n-m}{15n}} Px^{15} + \frac{15n-m}{16n} Qx^{16} - \sqrt{\frac{16n-m}{17n}} Rx^{17} - \frac{17n-m}{18n} Sx^{18} - \&c, \text{ ad infinitum};$ in which series the third term $\frac{m-n}{2n} Bx^2$ is added to the two first terms $1 + \frac{m}{n} x$, and the fourth, fifth, sixth, seventh, and other following terms are alternately subtracted from the three first terms $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2$, and added to them, and consequently marked with the signs $-$ and $+$ alternately; and the several generating fractions $\frac{2n-m}{3n}$, $\frac{3n-m}{4n}$, $\frac{4n-m}{5n}$, $\frac{5n-m}{6n}$, $\frac{6n-m}{7n}$, &c, are formed one from the other by the continual addition of n to both their numerators and their denominators.

Q. E. I.

105. We have now shewn the binomial theorem to be true in the case of $\sqrt[n]{1+x}^{\frac{m}{n}}$, or the m th power of the n th root of the binomial quantity $1+x$, when m , the numerator of the index $\frac{m}{n}$, is greater than n , its denominator, but less than $2n$. It remains that we shew it to be true also when m is greater than $2n$,

but less than $3n$, and when it is greater than $3n$ but less than $4n$, and when it is greater than $4n$ but less than $5n$, and, in general, when it is of any greater magnitude whatsoever.

106. Now for this purpose we need only observe that in all these different magnitudes of m with respect to n , as well as when m is either less than n , or greater than n , but less than $2n$, the quantity $\overline{1 + x}^{\frac{m}{n}}$ will be equal to the same series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, of which the first term is 1, and the second term is $\frac{m}{n}x$ and is added to the first term, and the third and fourth and fifth and sixth, and other following terms, $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, consist of the square, and cube, and fourth power, and fifth power and other following powers of x in their natural order without interruption, multiplied into certain fixt numbers, or numeral co-efficients, which may be denoted by the capital letters C, D, E, F, &c, and are to be connected with the two first terms $1 + \frac{m}{n}x$ either by addition or subtraction, and consequently marked with either the sign $+$ or the sign $-$. For from hence it follows that, if we apply the reasonings used above in art. 67, 68, 69, 70, 71, 72, to the investigation of the terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, in the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and of the signs $+$ and $-$, which are to be prefixed to them, in any of these new relative magnitudes of m and n , we shall always come to the same final equation which was obtained for this purpose in art. 72, to wit, the equation between the simple series $\frac{m}{n}, Cx, Dx^2, Ex^3, Fx^4, \&c$, and the compound series

$$\begin{aligned} & \frac{2n}{m}C, \frac{3n}{m}Dx, \frac{4n}{m}Ex^2, \frac{5n}{m}Fx^3, \&c \\ & + 1, \frac{2n}{m}Cx, \frac{3n}{m}Dx^2, \frac{4n}{m}Ex^3, \&c. \end{aligned}$$

And from this general and fundamental equation we may derive particular equations for the determination of the signs that are to be prefixed to the several terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, and of the magnitudes of the several co-efficients C, D, E, F, &c, which will consist of the very same terms as the particular equations derived from the said general equation in the former relative magnitudes of m and n ; to wit, the particular equations following:

$$1^{\text{st}}, \frac{m}{n} = \frac{2n}{m}C, + 1,$$

$$2^{\text{dly}}, C = \frac{3n}{m}D, \frac{2n}{m}C,$$

$$3^{\text{dly}}, D = \frac{4n}{m}E, \frac{3n}{m}D,$$

$$4^{\text{thly}}, E = \frac{5n}{m}F, \frac{4n}{m}E,$$

&c,

Or (as they may be expressed with more convenience)

1st,

$$1^{\text{st}}, \frac{m}{n} = 1, \frac{2n}{m} C,$$

$$2^{\text{dly}}, C = \frac{2n}{m} C, \frac{3n}{m} D,$$

$$3^{\text{dly}}, D = \frac{3n}{m} D, \frac{4n}{m} E,$$

$$4^{\text{thly}}, E = \frac{4n}{m} E, \frac{5n}{m} F,$$

&c. And the only difference between these particular equations in the cases of these new relative magnitudes of m and n (in which m is supposed to be greater than $2n$) and the former particular equations, consisting of the same terms, in the cases of the former relative magnitudes of m and n (in which m was supposed to be either less than n , or greater than n , but less than $2n$) will be in the signs $+$ and $-$ that are to be prefixed to their terms. This difference we must now proceed to investigate.

107. In the aforefaid particular equations, by means of which the signs to be prefixed to the terms Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c, and the values or magnitudes of the co-efficients C , D , E , F , &c, are to be determined, to wit, the equations

$$\frac{m}{n} = 1, \frac{2n}{m} C,$$

$$C = \frac{2n}{m} C, \frac{3n}{m} D,$$

$$D = \frac{3n}{m} D, \frac{4n}{m} E,$$

$$E = \frac{4n}{m} E, \frac{5n}{m} F,$$

$$F = \frac{5n}{m} F, \frac{6n}{m} G,$$

&c, the common denominator m of the several fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, &c, will, on the present supposition, be greater than the numerator, $2n$, of the first of the said fractions; and it may be greater than $3n$, $4n$, $5n$, $6n$, $7n$, and many more of the numerators of the following fractions, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c.

But, however great we may suppose m to be in comparison of n , we may always, by continuing the aforefaid particular equations to a sufficient number, come at last to an equation in which the numerator of the fraction involved in the first term on the right-hand side of the equation will be greater than the denominator m . For let pn be the greatest multiple of n that is less than m : then will $\overline{p+1} \times n$, or $pn + n$, be greater than m ; and consequently, when the fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, are continued to the term $\frac{\overline{p+1} \times n}{m}$, or $\frac{pn+n}{m}$ (to which, it is evident, they may be continued), the numerator of the said fraction $\frac{pn+n}{m}$ will be greater than its denominator. Now, so long as the numerators of

these fractions, $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, are less than their denominator m , the fractions themselves will be less than 1: but when we come to the fraction

2 P 2

$$\frac{pn+n}{m},$$

$\frac{pn+n}{m}$, of which the numerator is greater than the denominator, the said fraction will be greater than 1; and so will, *à fortiori*, all the following fractions; to wit, $\frac{p+2 \times n}{m}$, $\frac{p+3 \times n}{m}$, $\frac{p+4 \times n}{m}$, $\frac{p+5 \times n}{m}$, &c, or $\frac{pn+2n}{m}$, $\frac{pn+3n}{m}$, $\frac{pn+4n}{m}$, $\frac{pn+5n}{m}$, &c. Therefore, so long as the numerators of the said fractions $\frac{2n}{m}$, $\frac{3n}{m}$, $\frac{4n}{m}$, $\frac{5n}{m}$, $\frac{6n}{m}$, $\frac{7n}{m}$, &c, continue to be less than their common denominator m , the quantities $\frac{2n}{m} C$, $\frac{3n}{m} D$, $\frac{4n}{m} E$, $\frac{5n}{m} F$, $\frac{6n}{m} G$, $\frac{7n}{m} H$, &c (which are the first terms on the right-hand sides of the 2d, 3d, 4th, 5th, 6th, 7th, &c, particular equations above-mentioned) will be less than the single quantities C , D , E , F , G , H , &c, which form the left-hand sides of the same equations: but when we are come to the fractions $\frac{pn+n}{m}$, $\frac{pn+2n}{m}$, $\frac{pn+3n}{m}$, $\frac{pn+4n}{m}$, $\frac{pn+5n}{m}$, &c (which are greater than 1), the quantities of which these fractions will be the co-efficients, and which will be the first terms on the right-hand sides of the several particular equations in which they will appear, will be greater than the corresponding single quantities, which will form the left-hand sides of the same particular equations; or, if we suppose the said particular equations to be as follows, to wit,

$$\begin{aligned} C' &= \frac{pn+n}{m} C', \frac{pn+2n}{m} D', \\ D' &= \frac{pn+2n}{m} D', \frac{pn+3n}{m} E', \\ E' &= \frac{pn+3n}{m} E', \frac{pn+4n}{m} F', \\ \text{and } F' &= \frac{pn+4n}{m} F', \frac{pn+5n}{m} G', \end{aligned}$$

&c (in which the capital letters C , D , E , F , G , &c, have an accent ' placed over them, in order to distinguish them from the former co-efficients C , D , E , F , G , &c), the quantities $\frac{pn+n}{m} C'$, $\frac{pn+2n}{m} D'$, $\frac{pn+3n}{m} E'$, and $\frac{pn+4n}{m} F'$, &c, which are the first terms on the right-hand sides of the said particular equations, will be greater than the corresponding single terms C' , D' , E' , F' , &c, which form the left-hand sides of the same equations.

108. Further, it follows, from what has been repeatedly observed in the course of the foregoing investigations, that in all these particular equations the terms that involve the same co-efficients, or capital letters, C , D , E , F , G , &c, or C' , D' , E' , F' , G' , &c, must have the same sign $+$ or $-$ prefixed to them; because they are derived from the same original terms Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c, in the assumed series $1 + \frac{m}{n}x$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , &c, by the operations of multiplication and division, by which the signs $+$ and $-$, that are to be prefixed to them, cannot be affected. Therefore the first terms on the right-hand sides of all these particular equations must always have the same signs $+$ and $-$ prefixed to them as are to be prefixed to the single quantities which form the left-hand sides of the same equations.

109. And hence it follows that, so long as the first terms on the right-hand sides of these particular equations, to wit, the terms $\frac{2^n}{m} C$, $\frac{3^n}{m} D$, $\frac{4^n}{m} E$, $\frac{5^n}{m} F$, $\frac{6^n}{m} G$, &c, are less than the single terms that form the left-hand sides of the same equations, to wit, the terms C , D , E , F , G , &c, respectively, the second terms on the right-hand sides of the said particular equations, to wit, the terms $\frac{3^n}{m} D$, $\frac{4^n}{m} E$, $\frac{5^n}{m} F$, $\frac{6^n}{m} G$, $\frac{7^n}{m} H$, &c, must be marked with the same signs as the first terms on the same right-hand sides of the same equations, to wit, $\frac{2^n}{m} C$, $\frac{3^n}{m} D$, $\frac{4^n}{m} E$, $\frac{5^n}{m} F$, $\frac{6^n}{m} G$, &c, or must be added to the said first terms, so as thereby to increase their magnitudes. For otherwise the two quantities on the right-hand sides of the said equations cannot be equal to the single quantities on the left-hand sides of them. But, when the first terms on the right-hand sides of these particular equations become greater than the corresponding single terms that form the left-hand sides of the same equations, which happens in the equations

$$\begin{aligned} C' &= \frac{p^n + n}{m} C', \frac{p^n + 2n}{m} D', \\ D' &= \frac{p^n + 2n}{m} D', \frac{p^n + 3n}{m} E', \\ E' &= \frac{p^n + 3n}{m} E', \frac{p^n + 4n}{m} F', \\ \text{and } F' &= \frac{p^n + 4n}{m} F', \frac{p^n + 5n}{m} G', \end{aligned}$$

&c, the second terms on the right-hand sides of these equations, to wit, the terms $\frac{p^n + 2n}{m} D'$, $\frac{p^n + 3n}{m} E'$, $\frac{p^n + 4n}{m} F'$, $\frac{p^n + 5n}{m} G'$, &c, must be marked with the contrary signs to those which are to be prefixed to the first terms on the right-hand sides of those equations, to wit, the terms $\frac{p^n + n}{m} C'$, $\frac{p^n + 2n}{m} D'$, $\frac{p^n + 3n}{m} E'$, $\frac{p^n + 4n}{m} F'$, &c, respectively, so as to lessen the too great magnitudes of the said first terms, and reduce them to an equality with the single terms C' , D' , E' , F' , &c, which form the left-hand sides of the said equations.

110. From these observations it follows that, if m is greater than $2n$, but less than $3n$, or, if p is $= 2$, and consequently $\frac{p^n + n}{m}$ is $= \frac{2n + n}{m}$, or $\frac{3n}{m}$, the particular equations set down above in the beginning of art. 107, to wit, the equations

$$\begin{aligned} \frac{m}{n} &= 1, \frac{2^n}{m} C, \\ C &= \frac{2^n}{m} C, \frac{3^n}{m} D, \\ D &= \frac{3^n}{m} D, \frac{4^n}{m} E, \end{aligned}$$

$$E =$$

$$E = \frac{4^n}{m} E, \frac{5^n}{m} F,$$

$$F = \frac{5^n}{m} F, \frac{6^n}{m} G,$$

&c, will, when the proper signs + and - are prefixed to their terms, be as follows, to wit,

$$\frac{m}{n} = 1 + \frac{2^n}{m} C,$$

$$+ C = + \frac{2^n}{m} C + \frac{3^n}{m} D,$$

$$+ D = + \frac{3^n}{m} D - \frac{4^n}{m} E,$$

$$- E = - \frac{4^n}{m} E + \frac{5^n}{m} F,$$

$$+ F = + \frac{5^n}{m} F - \frac{6^n}{m} G,$$

&c; and, if m is greater than $3n$, but less than $4n$, or p is $= 3$, and consequently $\frac{pn+n}{m}$ is $= \frac{3n+n}{m} = \frac{4^n}{m}$, the said equations will, when the proper signs + and - are prefixed to their terms, be as follows, to wit,

$$\frac{m}{n} = 1 + \frac{2^n}{m} C,$$

$$+ C = + \frac{2^n}{m} C + \frac{3^n}{m} D,$$

$$+ D = + \frac{3^n}{m} D + \frac{4^n}{m} E,$$

$$+ E = + \frac{4^n}{m} E - \frac{5^n}{m} F,$$

$$- F = - \frac{5^n}{m} F + \frac{6^n}{m} G,$$

And the next equation to these will be

$$+ G = + \frac{6^n}{m} G - \frac{7^n}{m} H,$$

&c.

And, in general, all the co-efficients $C, D, E, \&c$, will be marked with the sign + till we come to the equation in which the fraction by which the co-efficient is multiplied in the first term on the right-hand side of the equation is $\frac{pn+n}{m}$, of which the numerator $pn+n$ is greater than the denominator m ; after which the next co-efficient will be marked with the sign -, and the next to that will be marked with the sign +, and the next again with the sign -, and so on to as many co-efficients as these equations shall be extended to, the signs + and - being to be prefixed to all the said following co-efficients alternately. It follows therefore that in the general series $1 + \frac{m}{n} x, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6,$

$Hx^7, Ix^8, Kx^9, Lx^{10}, \&c$, *ad infinitum*, which is $= 1 + x \sqrt[n]{\frac{m}{n}}$, all the terms $Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, \&c$, will have the sign + prefixed to them, or will be to be added to the two first terms $1 + \frac{m}{n} x$, till we come to the $\frac{pn+n}{m}$ th term,

term, $pn + n$ being the first multiple of n that is greater than m ; and the said $pn + n$ th term will be to be subtracted from the said two first terms, and marked with the sign $-$; and all the subsequent terms after the $pn + n$ th term, will be marked with the sign $+$ and the sign $-$ alternately, or will be alternately to be added to, and subtracted from, the said two first terms. Q. E. I.

Note. This conclusion is agreeable to what was asserted above in the beginning of this discourse, in art. 5 and 6.

III. It remains that we investigate the values, or magnitudes, of the co-efficients $C, D, E, F, G, \&c$, when m is greater than $2n$.

Now these magnitudes are to be determined by resolving the particular equations above-mentioned in art. 107, to wit, the equations

$$\frac{m}{n} = 1, \frac{2n}{m} C,$$

$$C = \frac{2n}{m} C, \frac{3n}{m} D,$$

$$D = \frac{3n}{m} D, \frac{4n}{m} E,$$

$$E = \frac{4n}{m} E, \frac{5n}{m} F,$$

$$F = \frac{5n}{m} F, \frac{6n}{m} G,$$

&c, and

$$C' = \frac{pn+n}{m} C', \frac{pn+2n}{m} D',$$

$$D' = \frac{pn+2n}{m} D', \frac{pn+3n}{m} E',$$

$$E' = \frac{pn+3n}{m} E', \frac{pn+4n}{m} F',$$

$$F' = \frac{pn+4n}{m} F', \frac{pn+5n}{m} G',$$

&c, which, if the several terms of them have the proper signs $+$ and $-$ prefixed to them, will (according to what has been demonstrated in the four last articles, 107, 108, 109, 110) be as follows, to wit,

$$\frac{m}{n} = 1 + \frac{2n}{m} C,$$

$$+ C = + \frac{2n}{m} C + \frac{3n}{m} D,$$

$$+ D = + \frac{3n}{m} D + \frac{4n}{m} E,$$

$$+ E = + \frac{4n}{m} E + \frac{5n}{m} F,$$

$$+ F = + \frac{5n}{m} F + \frac{6n}{m} G,$$

&c, and

$$- C' = - \left[\frac{pn+n}{m} C' + \frac{pn+2n}{m} D', \right.$$

$$+ D' = + \frac{pn+2n}{m} D' - \left[\frac{pn+3n}{m} E', \right.$$

$$- E' =$$

$$- E' = - \left[\frac{p^n + 3^n}{m} E' + \frac{p^n + 4^n}{m} F', \right. \\ \left. + F' = \frac{p^n + 4^n}{m} F' - \left[\frac{p^n + 5^n}{m} G', \right. \right.$$

&c; and these equations may be resolved in the manner following:

112. To resolve the first equation, $\frac{m}{n} = 1 + \frac{2^n}{m} C$, subtract 1 from both sides; and we shall have $\frac{2^n}{m} C = \frac{m}{n} - 1 = \frac{m-n}{n}$, and consequently (multiplying both sides by $\frac{m}{2^n}$) $C = \frac{m}{2^n} \times \frac{m-n}{n} = \frac{m}{n} \times \frac{m-n}{2^n}$. Q. E. I.

To resolve the second equation, $+ C = + \frac{2^n}{m} C + \frac{3^n}{m} D$, subtract $\frac{2^n}{m} C$ from C; and we shall have $\frac{3^n}{m} D = C - \frac{2^n}{m} C = \left[1 - \frac{2^n}{m} \right] \times C = \frac{m-2^n}{m} \times C$, and consequently (multiplying both sides by $\frac{m}{3^n}$) $D = \frac{m-2^n}{3^n} \times C$. Q. E. I.

To resolve the third equation, $+ D = + \frac{3^n}{m} D + \frac{4^n}{m} E$, subtract $\frac{3^n}{m} D$ from D; and we shall have $\frac{4^n}{m} E = D - \frac{3^n}{m} D = \left[1 - \frac{3^n}{m} \right] \times D = \frac{m-3^n}{m} \times D$, and consequently (multiplying both sides by $\frac{m}{4^n}$) $E = \frac{m-3^n}{4^n} \times D$. Q. E. I.

To resolve the fourth equation, $+ E = + \frac{4^n}{m} E + \frac{5^n}{m} F$, subtract $\frac{4^n}{m} E$ from both sides; and we shall have $\frac{5^n}{m} F = E - \frac{4^n}{m} E = \left[1 - \frac{4^n}{m} \right] \times E = \frac{m-4^n}{m} \times E$, and consequently (multiplying both sides by $\frac{m}{5^n}$) $F = \frac{m-4^n}{5^n} \times E$. Q. E. I.

To resolve the fifth equation, $+ F = + \frac{5^n}{m} F + \frac{6^n}{m} G$, subtract $\frac{5^n}{m} F$ from both sides; and we shall have $\frac{6^n}{m} G = F - \frac{5^n}{m} F = \left[1 - \frac{5^n}{m} \right] \times F = \frac{m-5^n}{m} \times F$, and consequently (multiplying both sides into $\frac{m}{6^n}$) $G = \frac{m-5^n}{6^n} \times F$. Q. E. I.

Thus it appears that, so long as the numerators of the several fractions $\frac{2^n}{m}$, $\frac{3^n}{m}$, $\frac{4^n}{m}$, $\frac{5^n}{m}$, &c, continue to be less than their common denominator m , the several terms Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , &c, of the series $1 + \frac{m}{n}x$, Cx^2 , Dx^3 , Ex^4 ,

$E x^4, F x^5, G x^6, \&c$, which is equal to $1 + x \frac{m}{n}$, will be all marked with the sign $+$, or added to the two first terms $1 + \frac{m}{n} x$, and that the co-efficient C will be $= \frac{m-n}{2n} \times \frac{m}{n}$, or $\frac{m-n}{2n} \times B$, and D will be $= \frac{m-2n}{3n} \times C$, and E will be $= \frac{m-3n}{4n} \times D$, and F will be $= \frac{m-4n}{5n} \times E$, and G will be $= \frac{m-5n}{6n} \times F$; which co-efficients are derived from $\frac{m}{n}$, or B (the co-efficient of the second term $\frac{m}{n} x$, or $B x$), by the continual multiplication of the generating fractions $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, $\frac{m-4n}{5n}$, $\frac{m-5n}{6n}$, $\&c$, of which the numerators decrease by the continual subtraction of n , and the denominators increase by the continual addition of the same quantity.

113. And, when the numerators of the fractions $\frac{2n}{m}, \frac{3n}{m}, \frac{4n}{m}, \frac{5n}{m}, \frac{6n}{m}, \frac{7n}{m}, \&c$, are greater than the common denominator m , and the equations to be resolved are consequently

$$\begin{aligned} -C' &= -\frac{pn+n}{m} C' + \frac{pn+2n}{m} D', \\ +D' &= \frac{pn+2n}{m} D' - \frac{pn+3n}{m} E', \\ -E' &= -\frac{pn+3n}{m} E' + \frac{pn+4n}{m} F', \\ +F' &= +\frac{pn+4n}{m} F' - \frac{pn+5n}{m} G', \\ &\&c, \end{aligned}$$

these equations may be resolved in the manner following:

To resolve the first equation, $-C' = -\frac{pn+n}{m} C' + \frac{pn+2n}{m} D'$, let $\frac{pn+n}{m} C'$ be added to both sides; and we shall have $\frac{pn+2n}{m} D' = \frac{pn+n}{m} C' - C' = \frac{pn+n}{m} - 1 \times C' = \frac{pn+n-m}{m} \times C'$, and consequently (multiplying both sides by $\frac{m}{pn+2n}$) $D' = \frac{pn+n-m}{pn+2n} \times C'$. Q. E. I.

To resolve the second equation, $+D' = \frac{pn+2n}{m} D' - \frac{pn+3n}{m} E'$, let $\frac{pn+3n}{m} E'$ be added to both sides; and we shall have $D' + \frac{pn+3n}{m} E' = \frac{pn+2n}{m} D'$; and (subtracting D' from both sides) $\frac{pn+3n}{m} E' = \frac{pn+2n}{m} D' - D' = \frac{pn+2n}{m} - 1 \times D' = \frac{pn+2n-m}{m} \times D'$, and (multiplying both sides into the fraction $\frac{m}{pn+3n}$) $E' = \frac{pn+2n-m}{pn+3n} \times D'$. Q. E. I.

To resolve the third equation, $-E' = -\left[\frac{pn+3n}{m}\right] \times E' + \frac{pn+4n}{m} F'$, let $\frac{pn+3n}{m} E'$ be added to both sides; and we shall have $\frac{pn+4n}{m} F' = \frac{pn+3n}{m} E' - E' = \left[\frac{pn+3n}{m} - 1\right] \times E' = \frac{pn+3n-m}{m} \times E'$, and consequently (multiplying both sides into the fraction $\frac{m}{pn+4n}$) $F' = \frac{pn+3n-m}{pn+4n} \times E'$. Q. E. I.

And, to resolve the fourth equation, $+F' = +\frac{pn+4n}{m} F' - \left[\frac{pn+5n}{m}\right] G'$, let $\frac{pn+5n}{m} \times G'$ be added to both sides; and we shall have $F' + \frac{pn+5n}{m} G' = \frac{pn+4n}{m} F'$, and (subtracting F' from both sides) $\frac{pn+5n}{m} G' = \frac{pn+4n}{m} F' - F' = \left[\frac{pn+4n}{m} - 1\right] \times F' = \frac{pn+4n-m}{m} \times F'$, and consequently (multiplying both sides by the fraction $\frac{m}{pn+5n}$) $G' = \frac{pn+4n-m}{pn+5n} \times F'$. Q. E. I.

It therefore appears that, when we are come to the equation $-C' = -\left[\frac{pn+n}{m}\right] C' + \frac{pn+2n}{m} D'$, in which the first term on the right-hand side of the equation, to wit, $\frac{pn+n}{m} C'$, is greater than the single term on the left-hand side of the same equation, to wit, C' , the several terms, D' , E' , F' , G' , &c, will be generated, or derived, from C' by the continual multiplication of the fractions $\frac{pn+n-m}{pn+2n}$, $\frac{pn+2n-m}{pn+3n}$, $\frac{pn+3n-m}{pn+4n}$, $\frac{pn+4n-m}{pn+5n}$, &c, of which both the numerators and denominators increase by the continual addition of n ; whereas, before we came to the term $\frac{pn+n-m}{pn+2n} \times C'$, the numerators of the generating fractions $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, $\frac{m-4n}{5n}$, $\frac{m-5n}{6n}$, &c, decreased by the continual subtraction of n , while their denominators increased by the continual addition of it.

114. The numerators of the first set of generating fractions, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, $\frac{m-4n}{5n}$, $\frac{m-5n}{6n}$, &c, till we come to a multiple of n that is greater than m , are the excesses of m above the several successive multiples of n , taken in their natural order, to wit, n , $2n$, $3n$, $4n$, $5n$, &c; and the numerators of the second set of generating fractions, $\frac{pn+n-m}{pn+2n}$, $\frac{pn+2n-m}{pn+3n}$, $\frac{pn+3n-m}{pn+4n}$, $\frac{pn+4n-m}{pn+5n}$, &c, after we are come to $p+1 \times n$, or $pn+n$, or the first multiple of n that is greater than m , are the excesses of the several following multiples of n , to wit, $p+1 \times n$, $p+2 \times n$, $p+3 \times n$, $p+4 \times n$, &c, or $pn+n$, $pn+2n$, $pn+3n$, $pn+4n$, &c, taken in their natural order, above m : so that in both sets of generating fractions the numerators of the said fractions are the differences of m from the several successive multiples of n , taken in their natural order, to wit,

wit, $2n, 3n, 4n, 5n, 6n, \&c$, and $p, p+1, p+2, p+3, p+4, p+5, \&c$, or $2n, 3n, 4n, 5n, 6n, \&c$, and $p, p+n, p+2n, p+3n, p+4n, p+5n, \&c$, *ad infinitum*.

115. It therefore has now been demonstrated in art. 105, 106, &c. . . . 114, that, when m is greater than $2n$, and p is the greatest multiple of n that is less than m , the quantity $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n} Ax + \frac{m-m}{2n} Bx^2 + \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \frac{m-5n}{6n} Fx^6 + \&c - \frac{pn+n-m}{pn+2n} C'x^{p+2} + \frac{pn+2n-m}{pn+3n} D'x^{p+3} - \frac{pn+3n-m}{pn+4n} E'x^{p+4} + \frac{pn+4n-m}{pn+5n} F'x^{p+5} - \frac{pn+5n-m}{pn+6n} G'x^{p+6} + \&c$, *ad infinitum*; in which series all the terms after the first term 1 are to be added to the said first term till we come to the term $\frac{pn+n-m}{pn+2n} C'x^{p+2}$, which is to be subtracted from the said first term; and all the terms after the said term $\frac{pn+n-m}{pn+2n} C'x^{p+2}$ are to be added to and subtracted from the said first term 1 alternately. All which is agreeable to what was asserted above in the beginning of this discourse, art. 5 and 6, concerning the series that was equal to $\sqrt[n]{1+x}$ upon these suppositions of the relative magnitudes of m and n .

116. We have therefore now compleated the demonstration of Sir Isaac Newton's famous binomial theorem, in all the cases of fractional powers whatsoever; to wit, 1st, in the case of the roots of a binomial quantity $1+x$, in art. 21, 22, 23, 24, 25, &c. 51, where we investigated the terms of an infinite series

that would exhibit the value of $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, when n , the index of the root, was equal to any whole number whatsoever; and, 2dly, in the first case of the powers of the roots of a binomial quantity, in art. 54, 55, 56, 57, 58, 59, 79, where we investigated the terms of an infinite series that

would exhibit the value of $\sqrt[n]{1+x}$, or of the m th power of the n th root of the binomial quantity $1+x$, when n , the index of the root, was equal to any whole number whatsoever, and m , the index of the power, was equal to any other whole number less than n ; and 3dly, in the second case of the powers of the roots of a binomial quantity, in art. 80, 81, 82, 83, 104, where we investi-

gated the terms of an infinite series that would exhibit the value of $\sqrt[n]{1+x}$, or of the m th power of the n th root of the binomial quantity $1+x$, when n , the index of the root, was any whole number whatsoever, and m , the index of the power, was any other whole number greater than n , but less than $2n$; and, 4thly, in the last case of the powers of the roots of a binomial quantity in art. 105, 106,

107, 108, &c 115, where we investigated the terms of an infinite series that would exhibit the value of $\sqrt[n]{1+x}^m$, or of the m th power of the n th root of the binomial quantity $1+x$, when n , the index of the root, was any whole number whatsoever, and m , the index of the power, was any other whole number whatsoever greater than $2n$. Here therefore we might with propriety put an end to this discourse; but as the last case above-mentioned of this theorem, in which m is greater than $2n$, is attended with rather more difficulty than the former cases of it, on account of the different numbers of terms of the series $1 + \frac{m}{n}x, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, after the two first terms $1 + \frac{m}{n}x$, which are to be marked with the sign $+$, or added to the said two first terms, according to the different magnitudes of m with respect to n , I shall now proceed to lay before the reader another demonstration of this last case of the said theorem (in which m is greater than $2n$), and likewise of the next preceeding case of it (in which m is greater than n , but less than $2n$); which is grounded on the supposition that the said theorem is true in the first case of the quantity $\sqrt[n]{1+x}^m$, or when m , the numerator of the index $\frac{m}{n}$, is less than its denominator n .

Another demonstration of the Binomial Theorem

in the case of the quantity $\sqrt[n]{1+x}^m$, or the m th power of the n th root of the binomial quantity $1+x$, when m , the index of the power, is greater than n , the index of the root; deduced from the series that is equal to

$\sqrt[n]{1+x}^m$, when m is less than n .

117. It has been shewn above, in a pretty full, and, I hope, satisfactory manner (in art. 54, 55, 56, 57, 58, 59, &c 79), that when m , the numerator of the index $\frac{m}{n}$, is less than n , its denominator, the quantity $\sqrt[n]{1+x}^m$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n}Ax - \left[\frac{n-m}{2n} Bx^2 + \frac{2n-m}{3n} Cx^3 - \left[\frac{3n-m}{4n} Dx^4 + \frac{4n-m}{5n} Ex^5 - \right. \right.$
 &c, *ad infinitum* in which all the terms after the two first terms $1 + \frac{m}{n}Ax$ are marked alternately with the signs $-$ and $+$, or are to be alternately subtracted from, and added to, the said two first terms. Now from this proposition we may deduce a proof that, when m is greater than n , but less than $2n$, the quantity

tity $\sqrt[n]{1+x}$ will be equal to the series above found for it in art. 80, 81, 82, 83, &c, 104; and that, when m is of any magnitude greater than $2n$, the quantity $\sqrt[n]{1+x}$ will be equal to the series found for it above in art. 105, 106, 107, 108, &c, 115. This may be done in the manner following:

118. In the first place let us suppose the index $\frac{m}{n}$ of the power to which the binomial quantity $1+x$ is to be raised, to be $\frac{m}{n} + 1$, or $\frac{m+n}{n}$, instead of $\frac{m}{n}$; and let p be put $= m+n$. Then will $\sqrt[n]{1+x}^{\frac{m}{n}+1}$, or $\sqrt[n]{1+x}^{\frac{m+n}{n}}$, be $= \sqrt[n]{1+x}^{\frac{p}{n}}$; in which quantity p , the numerator of the index $\frac{p}{n}$, will be greater than its denominator n , but less than $2n$. For $m+n$ (to which p is equal) is greater than n , but (because m alone is supposed to be less than n) less than $2n$.

We must therefore shew that $\sqrt[n]{1+x}^{\frac{p}{n}}$ will, upon these suppositions, be equal to the series $1 + \frac{p}{n} Ax + \frac{p-n}{2n} Bx^2 - \frac{2n-p}{3n} Cx^3 + \frac{3n-p}{4n} Dx^4 - \frac{4n-p}{5n} Ex^5 + \&c$, *ad infinitum*, in which the third term $\frac{p-n}{2n} Bx^2$ is marked with the sign $+$, or is added to the two first terms $1 + \frac{p}{n} Ax$; and all the following terms are marked with the signs $-$ and $+$ alternately, or are to be alternately subtracted from, and added to, the said two first terms, agreeably to art. 104.

119. Now $\sqrt[n]{1+x}^{\frac{p}{n}}$ is $= \sqrt[n]{1+x}^{\frac{m+n}{n}} = \sqrt[n]{1+x}^{\frac{m}{n}} + 1 = \sqrt[n]{1+x}^{\frac{m}{n}} \times \sqrt[n]{1+x}^1 = \sqrt[n]{1+x}^{\frac{m}{n}} \times (1+x)$.

But, by the supposition $\sqrt[n]{1+x}^{\frac{m}{n}}$ is $=$ the series $1 + \frac{m}{n} x - \frac{m}{n} \times \frac{n-m}{2n} x^2 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} x^3 - \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} x^4 + \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n} x^5 - \&c$, as is shewn in art. 54, 55, 56, 57, 58, 59, &c 79; or, if we denote the several co-efficients of $x, x^2, x^3, x^4, x^5, \&c$,

in this series, by the capital letters B, C, D, E, F, &c, respectively, $\sqrt[n]{1+x}^{\frac{m}{n}}$ will be $=$ the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$.

Therefore $\sqrt[n]{1+x}^{\frac{p}{n}}$ (which is $= \sqrt[n]{1+x}^{\frac{m}{n}} \times (1+x)$) will be $=$ the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c \times 1 + x =$ the compound series

$1 +$

$$1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c \\ + x + Bx^2 - Cx^3 + Dx^4 - Ex^5 + \&c.$$

But, because B is greater than C, and C is greater than D, and D than E, and E than F, and every following co-efficient of a power of x in the second line is greater than the co-efficient of the same power of x in the first line, it is evident that this compound series will be equal to the series $1 + \overline{1 + B} \times x + \overline{B - C} \times x^2 - \overline{C - D} \times x^3 + \overline{D - E} \times x^4 - \overline{E - F} \times x^5 + \&c$; in which series the third term $\overline{B - C} \times x^2$ is added to the two first terms $1 + \overline{1 + B} \times x$, and the several following terms $\overline{C - D} \times x^3$, $\overline{D - E} \times x^4$, $\overline{E - F} \times x^5$, &c, are marked with the signs $-$ and $+$ alternately, or are alternately subtracted from, and added to, the said two first terms.

Therefore $\overline{1 + x}^{\frac{p}{n}}$ will be equal to the said series $1 + \overline{1 + B} \times x + \overline{B - C} \times x^2 - \overline{C - D} \times x^3 + \overline{D - E} \times x^4 - \overline{E - F} \times x^5 + \&c$, in which the third term $\overline{B - C} \times x^2$ is added to the two first terms $1 + \overline{1 + B} \times x$, and the following terms are alternately subtracted from, and added to, the said two first terms; which is one of the properties that ought to belong to the series that is equal to $\overline{1 + x}^{\frac{p}{n}}$, according to art. 102.

120. It remains that we shew that the co-efficients $1 + B$, $B - C$, $C - D$, $D - E$, and $E - F$, &c, of the powers of x in the said series $1 + \overline{1 + B} \times x + \overline{B - C} \times x^2 - \overline{C - D} \times x^3 + \overline{D - E} \times x^4 - \overline{E - F} \times x^5$ &c (which is equal to $\overline{1 + x}^{\frac{p}{n}}$) are equal to $\frac{p}{n}$, $\frac{p}{n} \times \frac{p-n}{2n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}$ and $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}$, &c, respectively; agreeably to what is shewn in art. 104. This may be done in the manner following:

In the first place it is evident that $1 + B$ (being $= 1 + \frac{m}{n}$) will be $= \frac{n+m}{n} = \frac{p}{n}$. Q. E. D.

Secondly, since C is $= B \times \frac{n-m}{2n}$, it follows that $B - C$ will be $= B - B \times \frac{n-m}{2n} = B \times \frac{2n}{2n} - B \times \frac{n-m}{2n} = B \times \frac{2n - n + m}{2n} = B \times \frac{n+m}{2n} = B \times \frac{p}{2n}$.

Thirdly, since D is $= C \times \frac{2n-m}{3n}$, we shall have $C - D = C - C \times \frac{2n-m}{3n} = C \times \frac{3n}{3n} - C \times \frac{2n-m}{3n} = C \times \frac{3n - 2n + m}{3n} = C \times \frac{n+m}{3n} = C \times \frac{p}{3n}$.

Fourthly,

Fourthly, since E is $= D \times \frac{3n-m}{4n}$, we shall have $D - E = D - D \times \frac{3n-m}{4n} = D \times \frac{4n}{4n} - D \times \frac{3n-m}{4n} = D \times \left[\frac{4n}{4n} - \frac{3n-m}{4n} \right] = D \times \frac{4n-3n+m}{4n} = D \times \frac{n+m}{4n} = D \times \frac{p}{4n}$.

And, fifthly, since F is $= E \times \frac{4n-m}{5n}$, we shall have $E - F = E - E \times \frac{4n-m}{5n} = E \times \frac{5n}{5n} - E \times \frac{4n-m}{5n} = E \times \left[\frac{5n}{5n} - \frac{4n-m}{5n} \right] = E \times \frac{5n-4n+m}{5n} = E \times \frac{n+m}{5n} = E \times \frac{p}{5n}$.

And it is easy to see that, if we were to proceed in the same manner to examine the following differences, $F - G$, $G - H$, $H - I$, $I - K$, $K - L$, &c, continued to any number of terms, we should find them to be respectively equal to $F \times \frac{p}{6n}$, $G \times \frac{p}{7n}$, $H \times \frac{p}{8n}$, $I \times \frac{p}{9n}$, $K \times \frac{p}{10n}$, &c, in which quantities the denominators of the several fractions $\frac{p}{6n}$, $\frac{p}{7n}$, $\frac{p}{8n}$, $\frac{p}{9n}$, $\frac{p}{10n}$, &c, of which p is the numerator, increase continually by the addition of n .

It follows therefore that the series $1 + \overline{1+B} \times x + \overline{B-C} \times x^2 + \overline{C-D} \times x^3 + \overline{D-E} \times x^4 + \overline{E-F} \times x^5 + \&c$, which is $= \overline{1+x}^{\frac{p}{n}}$, will be $= 1 + \frac{p}{n} x + B \times \frac{p}{2n} x^2 - C \times \frac{p}{3n} x^3 + D \times \frac{p}{4n} x^4 - E \times \frac{p}{5n} x^5 + \&c$; and consequently this last series will be $= \overline{1+x}^{\frac{p}{n}}$. We must therefore shew that the several co-efficients $B \times \frac{p}{2n}$, $C \times \frac{p}{3n}$, $D \times \frac{p}{4n}$, and $E \times \frac{p}{5n}$, &c, of the powers of x in this last series are equal, respectively, to $\frac{p}{n} \times \frac{p-n}{2n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}$, $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}$, and $\frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}$, &c. This may be proved in the manner following:

121. Since p is $= n + m$, we shall have $2n - p (= 2n - n - m) = n - m$, and consequently (adding n continually to both sides)

$$\begin{aligned} 3n - p &= 2n - m, \\ \text{and } 4n - p &= 3n - m, \\ \text{and } 5n - p &= 4n - m, \\ \text{and } 6n - p &= 5n - m, \\ \text{and } 7n - p &= 6n - m, \\ \text{and } 8n - p &= 7n - m, \\ \text{and so on } ad\ infinitum. \end{aligned}$$

We shall therefore have

$$B (= \frac{m}{n}) = \frac{p-n}{n},$$

and

$$\begin{aligned} \text{and } C & (= \frac{m}{n} \times \frac{n-m}{2n}) = \frac{p-n}{n} \times \frac{2n-p}{2n}, \\ \text{and } D & (= \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n}) = \frac{p-n}{n} \times \frac{2n-p}{3n} \times \frac{3n-p}{3n}, \\ \text{and } E & (= \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n}) = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{4n-p}{4n}, \\ \text{and } F & (= \frac{m}{n} \times \frac{n-m}{2n} \times \frac{2n-m}{3n} \times \frac{3n-m}{4n} \times \frac{4n-m}{5n}) = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{4n-p}{4n} \times \frac{5n-p}{5n}, \end{aligned}$$

and so on *ad infinitum*, every new capital letter being equal to the next preceding capital letter multiplied into a new generating fraction (as $\frac{6n-p}{6n}$, $\frac{7n-p}{7n}$, $\frac{8n-p}{8n}$, $\frac{9n-p}{9n}$, $\frac{10n-p}{10n}$, &c), which is derived from the generating fraction next before it, by adding n to both its numerator and its denominator.

It follows therefore that the co-efficients $B \times \frac{p}{2n}$, $C \times \frac{p}{3n}$, $D \times \frac{p}{4n}$, $E \times \frac{p}{5n}$, &c, of x^2 , x^3 , x^4 , x^5 , and the following powers of x , in the third, fourth, fifth, sixth, and other following terms of the series $1 + \frac{p}{n}x + B \times \frac{p}{2n}x^2 - C \times \frac{p}{3n}x^3 + D \times \frac{p}{4n}x^4 - E \times \frac{p}{5n}x^5 + \&c$ (which has been shewn to be =

$\overline{1+x}^{\frac{p}{n}}$) will be equal to the following quantities; to wit,

$$\begin{aligned} B \times \frac{p}{2n} & \text{ will be } = \frac{p-n}{n} \times \frac{p}{2n} = \frac{p}{n} \times \frac{p-n}{2n}; \\ \text{and } C \times \frac{p}{3n} & \text{ will be } = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{p}{3n} = \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}; \\ \text{and } D \times \frac{p}{4n} & \text{ will be } = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{p}{4n} = \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}; \\ \text{and } E \times \frac{p}{5n} & \text{ will be } = \frac{p-n}{n} \times \frac{2n-p}{2n} \times \frac{3n-p}{3n} \times \frac{4n-p}{4n} \times \frac{p}{5n} = \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}. \end{aligned}$$

And it is easy to see that, if we were to compute the values of the following co-efficients $F \times \frac{p}{6n}$, $G \times \frac{p}{7n}$, $H \times \frac{p}{8n}$, $I \times \frac{p}{9n}$, $K \times \frac{p}{10n}$, &c, in the same manner by substituting in them, instead of the letters F, G, H, I, K, &c, the respective values of those letters, they would be such as would follow from the value of the last co-efficient $E \times \frac{p}{5n}$, by multiplying it continually into the fractions $\frac{5n-p}{6n}$, $\frac{6n-p}{7n}$, $\frac{7n-p}{8n}$, $\frac{8n-p}{9n}$, $\frac{9n-p}{10n}$, &c, which are formed from each other by the continual addition of n to both their numerators and their denominators.

Therefore the series $1 + \frac{p}{n}x + B \times \frac{p}{2n}x^2 - C \times \frac{p}{3n}x^3 + D \times \frac{p}{4n}x^4 - E \times \frac{p}{5n}x^5 + \&c$, which has been shewn to be equal to $\overline{1+x}^{\frac{p}{n}}$, will be

= 1 +

$$= 1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 + \&c, \text{ and}$$

consequently the quantity $\sqrt[n]{1+x}^{\frac{p}{n}}$ is = the series $1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 + \&c, \text{ ad infinitum, agreeably to what was shewn above in art. 104.}$

Q. E. D.

122. Having thus found that, when p is greater than n , but less than $2n$, the quantity $\sqrt[n]{1+x}^{\frac{p}{n}}$ is equal to the series $1 + \frac{p}{n}x + \frac{p}{n} \times \frac{p-n}{2n}x^2 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n}x^3 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n}x^4 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n}x^5 + \&c, \text{ ad infinitum, in which the third term } \frac{p}{n} \times \frac{p-n}{2n}x^2$ is marked with the sign $+$, or added to the two first terms, and all the following terms are marked with the signs $-$ and $+$ alternately, or are alternately subtracted from, and added to, the said two first terms, we must proceed to consider the

remaining cases of the quantity $\sqrt[n]{1+x}^{\frac{m}{n}}$, in which m , the index of the power to which the n th root of the binomial quantity $1+x$ is to be raised, is greater than $2n$, or twice the index of the said root. And, in order to do this with the more distinctness, I shall use the letter q , instead of the letter m , to denote the index of the said power, and shall continue to suppose m to be less than n (as it has been supposed to be in the course of the five last articles, 117, 118, 119, 120, and 121), and p to be $= m + n$, as before, and consequently $\frac{p}{n}$ to be $= \frac{m+n}{n}$, or $\frac{m}{n} + 1$, and shall suppose the new index q to be equal to $p + n$, or $m + n + n$, or $m + 2n$, and consequently $\frac{q}{n}$ to be $= \frac{p+n}{n}$, or $\frac{p}{n} + 1$; and, upon these sup-

positions, shall proceed to shew that $\sqrt[n]{1+x}^{\frac{q}{n}}$ will be = the series $1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 - \&c, \text{ in which the fourth term, as well as the third, is marked with the sign } +, \text{ or is added to the first term, agreeably to what is shewn above in art. 115, and asserted in the beginning of this discourse in art. 5 and 6.}$

123. Now, upon these suppositions, the quantity $\sqrt[n]{1+x}^{\frac{q}{n}}$, or the q th power of the n th root of the binomial quantity $1+x$, will be = $\sqrt[n]{1+x}^{\frac{p+n}{n}} =$

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$$\sqrt[n]{1+x}^{\frac{p}{n}} + 1$$

$\frac{p}{1+x} = \frac{p}{n} + \frac{p}{n} \times \frac{p-n}{2n} x + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} x^2 + \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} x^3 - \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n} x^4 + \&c \times \frac{1}{1+x} =$ (if the co-efficients of $x, x^2, x^3, x^4, x^5, \&c$, in the terms of this series, be denoted by $B', C', D', E', F', \&c$, or the capital letters $B, C, D, E, F, \&c$, with an accent placed near them at the top) the series $1 + B'x + C'x^2 - D'x^3 + E'x^4 - F'x^5 - \&c \times \frac{1}{1+x} =$ the compound series

$$1 + B'x + C'x^2 - D'x^3 + E'x^4 - F'x^5 + \&c \\ + x + B'x^2 + C'x^3 - D'x^4 + E'x^5 - \&c.$$

But, because C' is greater than D' , and D' is greater than E' , and E' is greater than F' , and every following co-efficient of a power of x in the lower line of this compound series is greater than the co-efficient of the same power of x in the upper line of the same, it is evident that the said compound series will be equal to the series $1 + \overline{1 + B'} \times x + \overline{B' + C'} \times x^2 + \overline{C' - D'} \times x^3 - \overline{D' - E'} \times$

$x^4 + \overline{E' - F'} \times x^5 - \&c$. Therefore the quantity $\frac{q}{1+x}$ will be equal to the said series $1 + \overline{1 + B'} \times x + \overline{B' + C'} \times x^2 + \overline{C' - D'} \times x^3 - \overline{D' - E'} \times x^4 + \overline{E' - F'} \times x^5 - \&c$; in which series the fourth term $\overline{C' - D'} \times x^3$, as well as the third term $\overline{B' + C'} \times x^2$, is marked with the sign $+$, or is to be added to the two first terms $1 + \overline{1 + B'} \times x$; and all the following terms are marked with the signs $-$ and $+$ alternately, or are to be alternately subtracted from, and added to, the said first terms; which is one of the properties of the series $1 + \frac{q}{n} x + \frac{q}{n} \times \frac{q-n}{2n} x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n} x^5 - \&c$, to which

we are now to shew the said quantity $\frac{q}{1+x}$ to be equal.

124. It remains that we shew that the co-efficients $1 + B', B' + C', C' - D', D' - E', E' - F', \&c$, of the several powers of x in the series $1 + \overline{1 + B'} \times x + \overline{B' + C'} \times x^2 + \overline{C' - D'} \times x^3 - \overline{D' - E'} \times x^4 + \overline{E' - F'} \times x^5 - \&c$

(which has been shewn to be equal to $\frac{q}{1+x}$), are respectively equal to the co-efficients $\frac{q}{n}, \frac{q}{n} \times \frac{q-n}{2n}, \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}, \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}$, and $\frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}, \&c$, of the same powers of x in the series $1 + \frac{q}{n} x + \frac{q}{n} \times \frac{q-n}{2n} x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n} x^5 - \&c$, to which we

are

are now to prove the quantity $\overline{1+x}^{\frac{q}{n}}$ to be equal. For then it will follow that this last series being thus shewn to be equal to the foregoing series $1 + \overline{1+B'} \times x + \overline{B'+C'} \times x^2 + \overline{C'-D'} \times x^3 - \overline{D'-E'} \times x^4 + \overline{E'-F'} \times x^5 - \&c$, which has been shewn to be equal to $\overline{1+x}^{\frac{q}{n}}$, will also be equal to $\overline{1+x}^{\frac{q}{n}}$. We must therefore endeavour to shew that

$$1 + B' \text{ will be } = \frac{q}{n},$$

$$\text{and } B' + C' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n},$$

$$\text{and } C' - D' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n},$$

$$\text{and } D' - E' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n},$$

$$\text{and } E' - F' \text{ will be } = \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n},$$

and that the following co-efficients $F' - G'$, $G' - H'$, $H' - I'$, $I' - K'$, $K' - L'$, &c, of the powers of x in the former series will, in like manner, be equal, respectively, to the following co-efficients of the same powers of x in the latter series. This may be done in the manner following.

125. Since $p + n$ is $= q$, we shall have

$$p = q - n,$$

$$\text{and } p - n (= q - n - n) = q - 2n,$$

$$\text{and } 2n - p (= 2n - [q - n] = 2n - q + n) = 3n - q,$$

and (adding n continually to both sides)

$$3n - p = 4n - q,$$

$$4n - p = 5n - q,$$

$$5n - p = 6n - q,$$

$$6n - p = 7n - q,$$

$$7n - p = 8n - q,$$

$$8n - p = 9n - q;$$

and so on *ad infinitum*.

Therefore

$$B' \text{ will be } (= \frac{p}{n}) = \frac{q-n}{n},$$

$$\text{and } C' \text{ will be } (= \frac{p}{n} \times \frac{p-n}{2n} = \frac{q-n}{n} \times \frac{q-2n}{2n}) = B' \times \frac{q-2n}{2n},$$

$$\text{and } D' \text{ will be } (= \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} = C' \times \frac{2n-p}{3n}) = C' \times \frac{3n-q}{3n},$$

$$\text{and } E' (= \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} = D' \times \frac{3n-p}{4n}) = D' \times \frac{4n-q}{4n},$$

$$\text{and } F' (= \frac{p}{n} \times \frac{p-n}{2n} \times \frac{2n-p}{3n} \times \frac{3n-p}{4n} \times \frac{4n-p}{5n} = E' \times \frac{4n-p}{5n}) = E' \times \frac{5n-q}{5n},$$

And, in like manner,

$$G' \text{ will be } (= F' \times \frac{5n-p}{6n}) = F' \times \frac{6n-q}{6n},$$

$$\text{and } H' \text{ will be } (= G' \times \frac{6n-p}{7n}) = G' \times \frac{7n-q}{7n},$$

2 R 2

and

$$\text{and } I' \text{ will be } (= H' \times \frac{7^n - p}{8^n}) = H' \times \frac{8^n - q}{8^n},$$

$$\text{and } K' \text{ will be } (= I' \times \frac{8^n - p}{9^n}) = I' \times \frac{9^n - q}{9^n};$$

and so on *ad infinitum*.

Therefore the co-efficient of x , to wit,

$$1 + B', \text{ will be } (= 1 + \frac{q-n}{n} = \frac{n}{n} + \frac{q-n}{n} = \frac{n+q-n}{n}) = \frac{q}{n};$$

$$\text{and } B' + C' \text{ will be } (= B' + B' \times \frac{q-2n}{2n} = B' \times \frac{2n}{2n} + B' \times \frac{q-2n}{2n} = B' \times \frac{2n+q-2n}{2n}) = B' \times \frac{q}{2n};$$

$$\text{and } C' - D' \text{ will be } (= C' - C' \times \frac{3^n - q}{3^n} = C' \times \frac{3^n}{3^n} - C' \times \frac{3^n - q}{3^n} = C' \times \frac{3^n - (3^n - q)}{3^n}) = C' \times \frac{q}{3^n};$$

$$\text{and } D' - E' \text{ will be } (= D' - D' \times \frac{4^n - q}{4^n} = D' \times \frac{4^n}{4^n} - D' \times \frac{4^n - q}{4^n} = D' \times \frac{4^n - (4^n - q)}{4^n}) = D' \times \frac{q}{4^n};$$

$$\text{and } E' - F' \text{ will be } (= E' - E' \times \frac{5^n - q}{5^n} = E' \times \frac{5^n}{5^n} - E' \times \frac{5^n - q}{5^n} = E' \times \frac{5^n - (5^n - q)}{5^n}) = E' \times \frac{q}{5^n};$$

and, in like manner,

$$F' - G' \text{ will be } = F' \times \frac{q}{6^n},$$

$$\text{and } G' - H' \text{ will be } = G' \times \frac{q}{7^n},$$

$$\text{and } H' - I' \text{ will be } = H' \times \frac{q}{8^n},$$

$$\text{and } I' - K' \text{ will be } = I' \times \frac{q}{9^n};$$

and so on *ad infinitum*.

Therefore the series $1 + \overline{1 + B'} \times x + \overline{B' + C'} \times x^2 + \overline{C' - D'} \times x^3 - \overline{D' - E'} \times x^4 + \overline{E' - F'} \times x^5 - \&c$ (which has been shewn to be equal to $\overline{1 + x}^{\frac{q}{n}}$) will be $= 1 + \frac{q}{n} x + B' \times \frac{q}{2n} x^2 + C' \times \frac{q}{3n} x^3 - D' \times \frac{q}{4n} x^4 + E' \times \frac{q}{5n} x^5 - \&c = 1 + \frac{q}{n} x + \frac{q-n}{n} \times \frac{q}{2n} x^2 + \frac{q-n}{n} \times \frac{q-2n}{2n} \times \frac{q}{3n} x^3 - \frac{q-n}{n} \times \frac{q-2n}{2n} \times \frac{3^n - q}{3^n} \times \frac{q}{4n} x^4 + \frac{q-n}{n} \times \frac{q-2n}{2n} \times \frac{3^n - q}{3^n} \times \frac{4^n - q}{4n} \times \frac{q}{5n} x^5 - \&c = 1 + \frac{q}{n} x + \frac{q}{n} \times \frac{q-n}{2n} x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3^n - q}{4n} x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3^n - q}{4n} \times \frac{4^n - q}{5n} x^5 - \&c.$ Therefore the quantity $\overline{1 + x}^{\frac{q}{n}}$, or the q th, or $p + n$ th, or $m + 2n$ th, power

power of the n th root of the binomial quantity $1 + x$, will be equal to the series $1 + \frac{q}{n}x + \frac{q}{n} \times \frac{q-n}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n}x^3 - \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n}x^4 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{4n-q}{5n}x^5 - \&c$, or (if we denote the co-efficients of $x, x^2, x^3, x^4, x^5, \&c$, in this series, by $B'', C'', D'', E'', F'', \&c$, or the capital letters $B, C, D, E, F, \&c$, with two accents placed near them at the top) to the series $1 + \frac{q}{n}x + \frac{q-n}{2n}B''x^2 + \frac{q-2n}{3n}C''x^3 - \frac{3n-q}{4n}D''x^4 + \frac{4n-q}{5n}E''x^5 - \&c$, *ad infinitum*; in which the fourth term $\frac{q-2n}{3n}C''x^3$, as well as the third term $\frac{q-n}{2n}B''x^2$, is marked with the sign $+$, or is to be added to the first term 1 , and all the following terms $\frac{3n-q}{4n}D''x^4, \frac{4n-q}{5n}E''x^5, \&c$, are marked with the signs $-$ and $+$ alternately; or are to be alternately subtracted from, and added to, the said first term, agreeably to what is shewn above in art. 115, and asserted in the beginning of this discourse in art. 5 and 6.

Q. E. D.

126. And, in like manner, if r be $= q + n$, or $p + n + n$, or $m + 3n$; we shall have $1 + x \Big| \frac{r}{n} = 1 + x \Big| \frac{q+n}{n} = 1 + x \Big| \frac{q}{n} + 1 = 1 + x \Big| \frac{q}{n} \times 1 + x \Big| 1 = 1 + x \Big| \frac{q}{n} \times 1 + x =$ the series $1 + B''x + C''x^2 + D''x^3 - E''x^4 + F''x^5 - \&c$, $\times 1 + x =$ the compound series

$$\begin{aligned}
 & 1 + B''x + C''x^2 + D''x^3 - E''x^4 + F''x^5 - \&c \\
 & + x + B''x^2 + C''x^3 + D''x^4 - E''x^5 + \&c \\
 = & 1 + \overbrace{1 + B''} \times x + \overbrace{B'' + C''} \times x^2 + \overbrace{C'' + D''} \times x^3 + \overbrace{D'' - E''} \times x^4 \\
 & - \overbrace{E'' - F''} \times x^5 + \&c = 1 + 1 + \frac{q}{n} \times x + \overbrace{B'' + B'' \times \frac{q-n}{2n}} \times x^2 \\
 & + \overbrace{C'' + C'' \times \frac{q-2n}{3n}} \times x^3 + \overbrace{D'' - D'' \times \frac{3n-q}{4n}} \times x^4 - \overbrace{E'' - E'' \times \frac{4n-q}{5n}} \times x^5 \\
 & + \&c = 1 + \frac{n+q}{n}x + B'' \times \frac{2n+q-n}{2n}x^2 + C'' \times \frac{3n+q-2n}{3n}x^3 + \\
 & D'' \times \frac{4n-(3n-q)}{4n}x^4 - E'' \times \frac{5n-(4n-q)}{5n}x^5 + \&c = 1 + \frac{r}{n}x + B'' \times \frac{n+q}{2n}x^2 \\
 & + C'' \times \frac{n+q}{3n}x^3 + D'' \times \frac{n+q}{4n}x^4 - E'' \times \frac{n+q}{5n}x^5 + \&c = 1 + \frac{r}{n}x + \\
 & B'' \times \frac{r}{2n}x^2 + C'' \times \frac{r}{3n}x^3 + D'' \times \frac{r}{4n}x^4 - E'' \times \frac{r}{5n}x^5 + \&c = 1 + \frac{r}{n}x \\
 & + \frac{q}{n} \times \frac{r}{2n}x^2 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{r}{3n}x^3 + \frac{q}{n} \times \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{r}{4n}x^4 - \frac{q}{n} \times \\
 & \frac{q-n}{2n} \times \frac{q-2n}{3n} \times \frac{3n-q}{4n} \times \frac{r}{5n}x^5 + \&c = 1 + \frac{r}{n}x + \frac{r-n}{n} \times \frac{r}{2n}x^2 + \frac{r-n}{n} \\
 & \times \frac{r-2n}{2n} \times \frac{r}{3n}x^3 + \frac{r-n}{n} \times \frac{r-2n}{2n} \times \frac{r-3n}{3n} \times \frac{r}{4n}x^4 - \frac{r-n}{n} \times \frac{r-2n}{2n} \times \\
 & \frac{r-3n}{3n}
 \end{aligned}$$

the terms of this series) $1 + B^{iv}x + C^{iv}x^2 + D^{iv}x^3 + E^{iv}x^4 + F^{iv}x^5 -$
 $\&c$, or $1 + \frac{s}{n}x + \frac{s-n}{2n}B^{iv}x^2 + \frac{s-2n}{3n}C^{iv}x^3 + \frac{s-3n}{4n}D^{iv}x^4 + \frac{s-4n}{5n}$
 $E^{iv}x^5 - \&c$. Q. E. D.

128. We have now shewn in the course of the foregoing articles, from art. 117 to art. 127, how from the series $1 + \frac{m}{n}Ax - \left[\frac{n-m}{2n}Bx^2 + \frac{2n-m}{3n}Cx^3 - \left[\frac{3n-m}{4n}Dx^4 + \frac{4n-m}{5n}Ex^5 - \&c\right.\right.$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$ when m is less than n), we may derive the series which is equal to $\overline{1+x}^{\frac{m}{n}+1}$, or $\overline{1+x}^{\frac{m+n}{n}}$, or $\overline{1+x}^{\frac{p}{n}}$, to wit, the series $1 + \frac{p}{n}x + \frac{p-n}{2n}B'x^2 - \left[\frac{2n-p}{3n}C'x^3 + \frac{3n-p}{4n}D'x^4 - \left[\frac{4n-p}{5n}E'x^5 + \&c\right.\right.$ and from this second series, which is equal to $\overline{1+x}^{\frac{p}{n}}$, we may derive the series which is equal to $\overline{1+x}^{\frac{m}{n}+2}$, or $\overline{1+x}^{\frac{m+2n}{n}}$, or $\overline{1+x}^{\frac{p+n}{n}}$, or $\overline{1+x}^{\frac{q}{n}}$, to wit, the series $1 + \frac{q}{n}x + \frac{q-n}{2n}B''x^2 + \frac{q-2n}{3n}C''x^3 - \left[\frac{3n-q}{4n}D''x^4 + \frac{4n-q}{5n}E''x^5 - \&c\right.$ and from this third series, which is equal to $\overline{1+x}^{\frac{q}{n}}$, we may derive the series which is equal to $\overline{1+x}^{\frac{m}{n}+3}$, or $\overline{1+x}^{\frac{m+3n}{n}}$, or $\overline{1+x}^{\frac{q+n}{n}}$, or $\overline{1+x}^{\frac{r}{n}}$, to wit, the series $1 + \frac{r}{n}x + \frac{r-n}{2n}B'''x^2 + \frac{r-2n}{3n}C'''x^3 + \frac{r-3n}{4n}D'''x^4 - \left[\frac{4n-r}{5n}E'''x^5 + \&c\right.$ and from this fourth series, which is equal to $\overline{1+x}^{\frac{r}{n}}$, we may derive the series which is equal to $\overline{1+x}^{\frac{m}{n}+4}$, or $\overline{1+x}^{\frac{m+4n}{n}}$, or $\overline{1+x}^{\frac{r+n}{n}}$, or $\overline{1+x}^{\frac{s}{n}}$, to wit, the series $1 + \frac{s}{n}x + \frac{s-n}{2n}B^{iv}x^2 + \frac{s-2n}{3n}C^{iv}x^3 + \frac{s-3n}{4n}D^{iv}x^4 + \frac{s-4n}{5n}E^{iv}x^5 - \&c$. And in each of these serieses after the first (which is equal to $\overline{1+x}^{\frac{m}{n}}$), we may observe, 1st, that there is one more term marked with the sign $+$, or added to the first term 1, than in the series next before it; and, 2dly, that, after the several terms in the beginning of the series which are thus to be added to each other, the following terms are marked with the signs $-$ and $+$ alternately, or are to be alternately subtracted from, and added to, the first term; and 3dly, that the co-efficients $B, C, D, E, F, \&c$, and $B', C', D', E', F', \&c$, and $B'', C'', D'', E'', F'', \&c$, and $B''', C''', D''', E''', F''', \&c$, and $B^{iv}, C^{iv}, D^{iv}, E^{iv}, F^{iv}, \&c$, in all these serieses, are derived

rived from the first term 1 by the same law, or by the continual multiplication of similar generating fractions, to wit, the generating fractions $\frac{m}{n}$, $\frac{n-m}{2n}$, $\frac{2n-m}{3n}$, $\frac{3n-m}{4n}$, $\frac{4n-m}{5n}$, &c, and $\frac{p}{n}$, $\frac{p-n}{2n}$, $\frac{2n-p}{3n}$, $\frac{3n-p}{4n}$, $\frac{4n-p}{5n}$, &c, and $\frac{q}{n}$, $\frac{q-n}{2n}$, $\frac{q-2n}{3n}$, $\frac{3n-q}{4n}$, $\frac{4n-q}{5n}$, &c, and $\frac{r}{n}$, $\frac{r-n}{2n}$, $\frac{r-2n}{3n}$, $\frac{r-3n}{4n}$, $\frac{4n-r}{5n}$, &c, and $\frac{s}{n}$, $\frac{s-n}{2n}$, $\frac{s-2n}{3n}$, $\frac{s-3n}{4n}$, $\frac{s-4n}{5n}$, &c; the denominators of all which generating fractions are n , $2n$, $3n$, $4n$, $5n$, &c, or n and its several successive multiples in their natural order; and the numerators of the first fractions in each set are m , p , q , r , s , or the indexes of the powers to which the n th root of $1+x$ is to be raised; and the numerators of the following fractions are the excesses of n , $2n$, $3n$, $4n$, &c, above the said indexes m , p , q , r , s , or of the said indexes above n , $2n$, $3n$, $4n$, &c, according as n , $2n$, $3n$, $4n$, &c, happen to be greater or less than the said indexes. And I apprehend that, from the manner in which

those several serieses, which are equal to $\frac{p}{1+x} \frac{1}{n}$, $\frac{q}{1+x} \frac{1}{n}$, $\frac{r}{1+x} \frac{1}{n}$, and $\frac{s}{1+x} \frac{1}{n}$, or $\frac{m+n}{1+x} \frac{1}{n}$, $\frac{m+2n}{1+x} \frac{1}{n}$, $\frac{m+3n}{1+x} \frac{1}{n}$, and $\frac{m+4n}{1+x} \frac{1}{n}$, have

been derived from the series which is equal to $\frac{m}{1+x} \frac{1}{n}$, and from each other, it will be sufficiently evident that the following serieses, that will be equal to

$\frac{m+5n}{1+x} \frac{1}{n}$, $\frac{m+6n}{1+x} \frac{1}{n}$, $\frac{m+7n}{1+x} \frac{1}{n}$, $\frac{m+8n}{1+x} \frac{1}{n}$, $\frac{m+9n}{1+x} \frac{1}{n}$, &c, *ad infinitum*, will have the same three properties which have been found to be-

long to the four above-mentioned serieses which are equal to $\frac{m+n}{1+x} \frac{1}{n}$,

$\frac{m+2n}{1+x} \frac{1}{n}$, $\frac{m+3n}{1+x} \frac{1}{n}$, and $\frac{m+4n}{1+x} \frac{1}{n}$; and consequently that, if Q be any whole number, how great soever, and M be $= m + Qn$, the quantity

$\frac{m+Qn}{1+x} \frac{1}{n}$, or $\frac{M}{1+x} \frac{1}{n}$, will be equal to the series $1 + \frac{M}{n}x + \frac{M}{n} \times \frac{M-n}{2n}x^2 +$

$\frac{M}{n} \times \frac{M-n}{2n} \times \frac{M-2n}{3n}x^3 + \frac{M}{n} \times \frac{M-n}{2n} \times \frac{M-2n}{3n} \times \frac{M-3n}{4n}x^4 + \frac{M}{n} \times \frac{M-n}{2n} \times \frac{M-2n}{3n} \times \frac{M-3n}{4n} \times \frac{M-4n}{5n}x^5 +$ &c, or (if we put B , C , D , E , F , &c,

for the several co-efficients of x , x^2 , x^3 , x^4 , x^5 , &c, in the terms of this series) to the series $1 + \frac{M}{n}x + \frac{M-n}{2n}Bx^2 + \frac{M-2n}{3n}Cx^3 + \frac{M-3n}{4n}Dx^4 + \frac{M-4n}{5n}$

$E.x^5 +$ &c, till we come to the term in which the multiple of n that enters the numerator of the generating fraction is greater than M , which multiple we will suppose to be $P+1 \times n$, or $Pn+n$. The term in which this happens will be marked with the sign $-$, or will be subtracted from the first term 1, and all the following terms will be marked with the sign $+$ and the sign $-$ alternately, or will be alternately added to, and subtracted from, the said first terms; and the numerators of the following generating fractions will be $P+2 \times n - M$,

$\overline{P+3} \times n - M$, $\overline{P+4} \times n - M$, $\overline{P+5} \times n - M$, $\overline{P+6} \times n - M$, &c, or $Pn + 2n - M$, $Pn + 3n - M$, $Pn + 4n - M$, $Pn + 5n - M$, $Pn + 6n - M$, &c *ad infinitum*. All which is agreeable to what has been shewn above in art. 105, 106, 107, &c 115, with only a small variation in the notation, which the different methods of investigation used in the two places had made necessary; the capital letters P and M being used in these latter articles instead of the small letters p and m , respectively, which were employed in art. 105, 106, 107, &c 115. And it also agrees with what was stated and asserted above in the beginning of this discourse, in art. 5 and 6.

Another Investigation of so many of the first terms of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$ (which is equal to $\overline{1+x}^{\frac{m}{n}}$) as are to be added together before any terms are subtracted from the first term 1, when m , the numerator of the index $\frac{m}{n}$, is greater than n , its denominator; deduced from the binomial theorem in the case of integral powers.

129. But there is still another method of discovering many of the first terms of the series which is equal to $\overline{1+x}^{\frac{m}{n}}$, in the case that has been last under consideration, or when m , the numerator of the index $\frac{m}{n}$, is greater than n , its denominator, which the lovers of these subjects will, I imagine, be glad to see. It is founded on a supposition that the quantity $\overline{1+x}^{\frac{m}{n}}$ will be equal to a series of the following form, to wit, $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, of which 1 is the first term, and all the following terms consist of the several successive powers of x in their natural order, to wit, $x, x^2, x^3, x^4, x^5, \&c$, multiplied into certain numeral co-efficients, which may be denoted by the capital letters B, C, D, E, F, &c, and that the second term, Bx , is to be marked with the sign +, or added to the first term 1, and all the following terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$, are to be marked either with the sign + or the sign -, or to be connected with the said first term 1, either by addition or subtraction, as shall be hereafter determined in the course of a proper investigation of the subject. That this supposition is true, or "that the quantity $\overline{1+x}^{\frac{m}{n}}$ will be equal to such a series," has been shewn in a pretty ample manner in the foregoing part of this discourse, in art. 54, 55, 56, 57, and art. 82; to which I shall therefore now refer the reader.

130. Upon this supposition, "that $\overline{1+x}^{\frac{m}{n}}$ is equal to the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c,$ " it was shewn above, in art. 58, 59, 60, and 61, and in art. 84, that, if

$$\text{we put } Q = m \times \frac{m-1}{2},$$

$$\text{and } R = m \times \frac{m-1}{2} \times \frac{m-2}{3} = \frac{m-2}{3} \times Q,$$

$$\text{and } S = m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} = \frac{m-3}{4} \times R,$$

$$\text{and } T = m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} = \frac{m-4}{5} \times S,$$

$$\text{and } q = n \times \frac{n-1}{2},$$

$$\text{and } r = n \times \frac{n-1}{2} \times \frac{n-2}{3} = \frac{n-2}{3} \times q,$$

$$\text{and } s = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} = \frac{n-3}{4} \times r,$$

$$\text{and } t = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} = \frac{n-4}{5} \times s,$$

we shall come to the following general and fundamental equation, to wit, the simple series $m + Qx + Rxx + Sx^3 + Tx^4 + \&c,$ is = the compound series

$$\begin{array}{r} nB, \quad nCx, \quad nDx^2, \quad nEx^3, \quad nFx^4, \quad \&c \\ + qB^2x, \quad 2qBCx^2, \quad 2qBDx^3, \quad 2qBEx^4, \quad \&c \\ \quad \quad \quad qC^2x^3, \quad 2qCDx^4, \quad \&c \\ + rB^3x^2, \quad 3rB^2Cx^3, \quad 3rB^2Dx^4, \quad \&c \\ \quad \quad \quad 3rBC^2x^4, \quad \&c \\ + sB^4x^3, \quad 4sB^3Cx^4, \quad \&c \\ \quad \quad \quad + tB^5x^4, \quad \&c \end{array}$$

by the help of which we may determine both the signs which are to be prefixed to the several terms $Cx^2, Dx^3, Ex^4, Fx^5, \&c$ of the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c,$ and the values, or magnitudes, of the co-efficients $B, C, D, E, F, \&c.$

131. And the manner in which these points are to be determined, is by deducing from the foregoing general equation several particular simple equations, involving the several co-efficients, $B, C, D, E, F, \&c,$ singly, or separately from the others that have not yet been discovered, and resolving the said simple equations; which simple equations will, by art. 85, be as follows; to wit,

$$\begin{array}{l} \text{1st, } m = nB; \\ \text{2dly, } Q = nC + qB^2; \\ \text{3dly, } R = nD, 2qBC + rB^3; \\ \text{4thly, } S = nE, 2qBD, qC^2, 3rB^2C + sB^4; \\ \text{and 5thly, } T = nF, 2qBE, 2qCD, 3rB^2D, 3rBC^2, 4sB^3C + tB^5. \end{array}$$

And, by resolving the first of these simple equations, to wit, $m = nB,$ it was found, in art. 86, that $B = \frac{m}{n}$; and by resolving the second of these equations,

equations, to wit, $Q = nC + qB^2$, it was found, in art. 87, that, if m was greater than n (as we have here supposed it to be), the term nC must have the sign $+$ prefixed to it, or must be added to qB^2 , and consequently that the term Cx^2 , in the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, (which is $= \overline{1+x}^{\frac{m}{n}}$), must also have the sign $+$ prefixed to it, or be added to the first term 1 ; and it was likewise found that the co-efficient C would be equal to $\frac{m-n}{2n} \times B$, or $\frac{m}{n} \times \frac{m-n}{2n}$.

And, in like manner, it would be possible, by resolving the third and fourth and fifth of these equations (if we would go through the labour of doing so), to determine which of the signs $+$ and $-$ was to be prefixed to each of the following terms, $Dx^3, Ex^4, Fx^5, \&c$, in the series $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c$, (which is equal to $\overline{1+x}^{\frac{m}{n}}$), and what would be the values, or magnitudes, of the co-efficients D, E , and $F, \&c$.

And, further, it is evident that all the values of the said co-efficients $B, C, D, E, F, \&c$, which would be obtained by the resolutions of these simple equations, would be derived from m and n , the numerator and denominator of the index $\frac{m}{n}$, by various additions, subtractions, multiplications, and divisions; because all the known quantities that enter those equations, to wit, the quantities $Q, R, S, T, \&c$, and $q, r, s, t, \&c$, are only different multiples, or parts, or sums, or differences, or, in general, different combinations, of the said original quantities m and n . And therefore all the values of the co-efficients $B, C, D, E, F, \&c$, must themselves also consist of certain combinations of the same quantities m and n ; as has been found to be the case with the co-efficients B and C , which are equal to $\frac{m}{n}$ and $\frac{m}{n} \times \frac{m-n}{2n}$ respectively.

132. These things being premised, it will, I think, be evident, “that, if m be greater than p times n , or pn , but less than $p + 1$ times n , or $pn + n$ (p being any whole number whatsoever), the several values of the co-efficients $D, E, F, G, H, \&c$, will consist of the very same combinations of the original quantities m and n , when m is of any one magnitude greater than pn , but less than $pn + n$, as when m is of any other magnitude less than its former magnitude, but yet greater than pn , and consequently as when the excess of m above pn becomes equal to 0 , or m is exactly equal to pn .” And consequently, if we can discover to what combinations of m and n the said co-efficients, $D, E, F, G, H, \&c$, will be equal when m is exactly equal to pn , we may conclude that the said co-efficients will be equal to the same combinations of m and n when m is of any other magnitude greater than pn , but less than $pn + n$. This is the principle of the present investigation, which may be easily deduced from it, by the help of the binomial theorem in the case of integral powers, in the manner following.

133. Let us then suppose m to be exactly equal to p times n , or pn , and try to discover what will be the values of the co-efficients $B, C, D, E, F, G, H, \&c$, upon this supposition.

Now upon this supposition $\frac{m}{n}$ will be $= \frac{pn}{n} = p$; and consequently $\sqrt[n]{1+x}^{\frac{m}{n}}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to $\sqrt[n]{1+x}^p$, or that integral power of the same binomial quantity $1+x$, of which the whole number p is the index. But, by the binomial theorem in the case of integral powers (which has been demonstrated above in the tract contained in pages 153, 154, 155, 156, &c. . . . 169) $\sqrt[n]{1+x}^p$ is equal the series $1 + \frac{p}{1}x + \frac{p}{1} \times \frac{p-1}{2}x^2 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}x^3 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}x^4 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}x^5 + \&c$, continued to $p+1$

terms. Therefore the quantity $\sqrt[n]{1+x}^{\frac{m}{n}}$ will, when m is $= pn$, be equal to the series $1 + \frac{p}{1}x + \frac{p}{1} \times \frac{p-1}{2}x^2 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}x^3 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4}x^4 + \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} \times \frac{p-4}{5}x^5 + \&c$, continued to $p+1$ terms, and consequently (if we substitute $\frac{m}{n}$ in these terms instead of p , to which

it is equal), to the series $1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$, continued to $p+1$ terms, or the series $1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$, continued to $p+1$ terms, or the series $1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$, continued to $p+1$ terms. Therefore by the principle laid down in the last article, when m is of any magnitude greater than pn , but less than $(p+1)n$, or $pn+n$, it will also be true that the first $p+1$ terms of the

series that is equal to $\sqrt[n]{1+x}^{\frac{m}{n}}$ will be $1 + \frac{\frac{m}{n}}{1}x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2}x^2 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3}x^3 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4}x^4 + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} \times \frac{\frac{m}{n}-2}{3} \times \frac{\frac{m}{n}-3}{4} \times \frac{\frac{m}{n}-4}{5}x^5 + \&c$

$\times \frac{m-3n}{4n} \times \frac{m-4n}{5n} x^5$ continued to $p+1$ terms, or $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2$
 $+ \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \&c$, continued to $p+1$
 terms. Q. E. I.

Thus, for example, if n be 7, and m be 373, which is greater than 371, or 53 times 7, but less than 378, or 54 times 7, the first 54 terms of the series that is equal to $\sqrt[n]{1+x}$, or $\sqrt[7]{1+x}$ or to the 373d power of the 7th root of the binomial quantity $1+x$, will be $1 + \frac{373}{7} Ax + \frac{373-7}{14} Bx^2 + \frac{373-14}{21} Cx^3 + \frac{373-21}{28} Dx^4 + \frac{373-28}{35} Ex^5 + \frac{373-35}{42} Fx^6 + \frac{373-42}{49} Gx^7 + \frac{373-49}{56} Hx^8 + \frac{373-56}{63} Ix^9 + \frac{373-63}{70} Kx^{10} + \&c$, continued to 54 terms.

134. This investigation of the values of the co-efficients B, C, D, E, F, G, H, &c, of $x, x^2, x^3, x^4, x^5, x^6, x^7$, and the following powers of x in the series

$1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to $\sqrt[n]{1+x}$, is shorter and easier than the foregoing ones. But it relates only to the $p+1$ first terms of the said series, or those terms of it which are all to be added together. For to so many terms only will the series that is equal to $\sqrt[n]{1+x}$

(from which series the series that is equal to $\sqrt[n]{1+x}$ has been here derived) extend. We cannot therefore discover by it what will be the co-efficients of the terms of the series in question that come after the $p+1$ th term (and of which the number will be infinite), nor whether the said terms are to be added to, or subtracted from, the first term 1 of the series. But, when p is a very great number, or m (which is greater than p times n , or pn) is very much greater than n (as in the foregoing example, in which m is supposed to be 373 and n to be 7), the first $p+1$ terms of the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 + \frac{m-2n}{3n}$

$Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \&c$, (which is equal to $1 + x \sqrt[n]{m}$), will be very nearly equal to the whole series, and consequently very nearly equal to

$\sqrt[n]{1+x}$; unless when x is very nearly equal to 1, in which case it would be necessary to take in more than the first $p+1$ terms of the series, and consequently to have recourse to some of the former investigations of it.

An example of the binomial theorem in raising the mth power of the nth root of a binomial quantity, when m and n are very great whole numbers.

135. Let m be $= 970,877$, and n be $= 10,000$. And let it be required to raise the binomial quantity $1 + \frac{24}{1000}$, or $1 + \frac{3}{125}$, to the $\frac{m}{n}$ th, or $\frac{970,877}{10,000}$ th, or 97.0877 th power, or to find the $970,877$ th power of the $10,000$ th root of the said binomial quantity.

136. Now it has been shewn in several different ways in the course of the foregoing articles, that, if x be any quantity not greater than 1, and n be any whole number whatsoever, and m any other whole number greater than n , and m be greater than pn , or p times n , but less than $(p+1) \times n$, or $p+1$ times n , the quantity $(1+x)^{\frac{m}{n}}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 + \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \frac{m-5n}{6n} Fx^6 + \&c$, to be continued till we come to the term $\frac{m-pn}{(p+1) \times n} \times P'x^{p+1}$, in which P' , or P with an accent placed over it, represents the co-efficient of the next preceeding term; the capital letter A being put for the first term 1, and the capital letters $B, C, D, E, F, \&c$, for the numeral co-efficients of x, x^2, x^3, x^4, x^5 , and the other following powers of x in the second, third, fourth, fifth, sixth, and other following terms of the said series. And after the term $\frac{m-pn}{(p+1) \times n} \times P'x^{p+1}$, the following terms of the series will be $-\frac{pn+n-m}{pn+2n} \times Q'x^{p+2} + \frac{pn+2n-m}{pn+3n} \times R'x^{p+3} - \frac{pn+3n-m}{pn+4n} \times S'x^{p+4} + \frac{pn+4n-m}{pn+5n} \times T'x^{p+5} - \frac{pn+5n-m}{pn+6n} \times V'x^{p+6} + \&c$, which are marked with the signs $-$ and $+$ alternately, or are alternately to be subtracted from, and added to, the first term 1.

137. This is in general the best method of expressing this series; more especially when we are considering the law of the generation of its terms, and the methods by which the said law may be investigated: which is the subject of the foregoing articles. But, when we have occasion actually to compute a considerable number of the first terms of the series, in order to find some high fractional power of the binomial quantity $1+x$ in some particular example (as is the case at present), it will be more convenient to make the capital letters $B, C, D, E, F, \&c$, stand for the *whole* second, third, fourth, fifth, and sixth, and other following

following terms, including the powers of x , instead of representing only the numeral co-efficients of the powers of x in the said terms. And then the foregoing

theorem will be as follows; to wit, $1 + x^{\frac{m}{n}} =$ the series $1 + \frac{m}{n} A x + \frac{m-m}{2n} B x + \frac{m-2n}{3n} C x + \frac{m-3n}{4n} D x + \frac{m-4n}{5n} E x + \frac{m-5n}{6n} F x + \&c.$ Now, if we substitute 970,877 in this equation instead of m , and 10,000 instead of n ,

and $\frac{24}{1000}$, or $\frac{3}{125}$, instead of x , we shall have $1 + \frac{24}{1000} \sqrt[970,877]{10,000} =$ the series

$$1 + \frac{970,877}{10,000} A \times \frac{24}{1000} + \frac{970,877 - 10,000}{2 \times 10,000} B \times \frac{24}{1000} + \frac{970,877 - 2 \times 10,000}{3 \times 10,000} C \times \frac{24}{1000} + \frac{970,877 - 3 \times 10,000}{4 \times 10,000} D \times \frac{24}{1000} + \frac{970,877 - 4 \times 10,000}{5 \times 10,000} E \times \frac{24}{1000} + \frac{970,877 - 5 \times 10,000}{6 \times 10,000} F \times \frac{24}{1000} + \&c =$$

the series $1 + 97.0877 A \times \frac{24}{1000} + \frac{97.0877 - 1}{2} B \times \frac{24}{1000} + \frac{97.0877 - 2}{3} C \times \frac{24}{1000} + \frac{97.0877 - 3}{4} D \times \frac{24}{1000} + \frac{97.0877 - 4}{5} E \times \frac{24}{1000} + \frac{97.0877 - 5}{6} F \times \frac{24}{1000} + \&c =$

the series $1 + 97.0877 A \times \frac{24}{1000} + \frac{96.0877}{2} B \times \frac{24}{1000} + \frac{95.0877}{3} C \times \frac{24}{1000} + \frac{94.0877}{4} D \times \frac{24}{1000} + \frac{93.0877}{5} E \times \frac{24}{1000} + \frac{92.0877}{6} F \times \frac{24}{1000} + \frac{91.0877}{7} G \times \frac{24}{1000} + \frac{90.0877}{8} H \times \frac{24}{1000} + \frac{89.0877}{9} I \times \frac{24}{1000} + \frac{88.0877}{10} K \times \frac{24}{1000} + \frac{87.0877}{11} L \times \frac{24}{1000} + \frac{86.0877}{12} M \times \frac{24}{1000} + \frac{85.0877}{13} N \times \frac{24}{1000} + \frac{84.0877}{14} O \times \frac{24}{1000} + \frac{83.0877}{15} P \times \frac{24}{1000} + \frac{82.0877}{16} Q \times \frac{24}{1000} + \frac{81.0877}{17} R \times \frac{24}{1000} + \frac{80.0877}{18} S \times \frac{24}{1000} + \frac{79.0877}{19} T \times \frac{24}{1000} + \frac{78.0877}{20} V \times \frac{24}{1000} + \frac{77.0877}{21} W \times \frac{24}{1000} + \frac{76.0877}{22} X \times \frac{24}{1000} + \frac{75.0877}{23} Y \times \frac{24}{1000} + \frac{74.0877}{24} Z \times \frac{24}{1000} + \frac{73.0877}{25} A' \times \frac{24}{1000} + \frac{72.0877}{26} B' \times \frac{24}{1000} + \frac{71.0877}{27} C' \times \frac{24}{1000} + \frac{70.0877}{28} D' \times \frac{24}{1000} + \frac{69.0877}{29} E' \times \frac{24}{1000} + \frac{68.0877}{30} F' \times \frac{24}{1000} + \&c =$

$$\begin{aligned}
 &1.000,000,000,000,000,000,000, A, \\
 &+ 2.330,104,800,000,000,000,000, B, \\
 &+ 2.686,732,931,891,520,000,000, C, \\
 &+ 2.043,802,040,062,570,290,432, D, \\
 &+ 1.153,779,799,228,770,568,290, E, \\
 &+ 0.515,532,997,520,006,524,943, F, \\
 &+ 0.189,896,992,062,892,419,468, G, \\
 &+ 0.059,304,960,836,321,574,297, H, \\
 &+ 0.016,027,942,561,002,861,266, I, \\
 &+ 0.003,807,713,435,978,278,943, K, \\
 &+ 0.000,804,990,525,202,617,220, L, \\
 &+ 0.000,152,955,869,152,773,747, M,
 \end{aligned}$$

+ 0.000,

+ 0.000,026,335,237,953,726,481, N,
 + 0.000,004,136,870,448,803,617, O,
 + 0.000,000,596,331,293,550,623, P,
 + 0.000,000,079,276,472,990,633, Q,
 + 0.000,000,009,761,435,007,869, R,
 + 0.000,000,001,117,457,383,747, S,
 + 0.000,000,000,119,326,122,283, T,
 + 0.000,000,000,011,920,709,761, V,
 + 0.000,000,000,001,117,032,969, W,
 + 0.000,000,000,000,098,410,859, X,
 + 0.000,000,000,000,008,168,570, Y,
 + 0.000,000,000,000,000,640,026, Z,
 + 0.000,000,000,000,000,047,741, A',
 + 0.000,000,000,000,000,003,327, B',
 + 0.000,000,000,000,000,000,221, C',
 + 0.000,000,000,000,000,000,013, D',
 + 0.000,000,000,000,000,000,000, E',

+ &c = 9.999,979,282,720,950,507,346, &c. Therefore (if no mistake has been made in computing these numbers) the quantity $1 + \frac{24}{1000} \sqrt[970,877]{\frac{970,877}{10000}}$, or the 970,877th power of the 10,000th root of the binomial quantity $1 + \frac{24}{1000}$, will be equal to the mixt number 9.999,979,282,720,950,507,346, &c.

Q. E. I.

137. By the help of this number we may find the logarithm of 2, or of the ratio of 2 to 1, in Briggs's System of Logarithms, to a considerable degree of exactness, in the manner above explained, in the tract intitled, "*An Appendix to Dr. Halley's Discourse concerning Logarithms*," in pages 123, 124, 125, &c, to 152, without having recourse to the doctrine of fluxions, or infinite serieses, or the arithmetic of infinites in any of its modifications. This may be done in the manner following.

An application of the foregoing example to the investigation of the Logarithm of the ratio of 2 to 1, in Brigg's System of Logarithms.

138. The tenth power of 2 is 1024. Therefore the ratio of 2 to 1 is one tenth part of the ratio of 1024 to 1. But the ratio of 1024 to 1 is equal to the ratio of 1024 to 1000, together with the ratio of 1000 to 1. Therefore the ratio of 2 to 1 is equal to one tenth part of the ratio of 1024 to 1000, together with one tenth part of the ratio of 1000 to 1. Therefore the logarithm of the ratio of 2 to 1 is equal to one tenth part of the logarithm of the ratio of 1024 to 1000, together with one tenth part of the logarithm of the ratio of 1000 to 1. But the logarithm of the ratio of 1000 to 1 in Briggs's System of Logarithms is 3. Therefore

Therefore the logarithm of the ratio of 2 to 1 in Briggs's System of Logarithms is equal to one tenth part of the logarithm of the ratio of 1024 to 1000, together with one tenth part of 3, or with $\frac{3}{10}$, or 0.300,000,000,000,000,000,000, &c.

Therefore, if we can discover the logarithm of the ratio of 1024 to 1000, we may easily deduce from it the logarithm of the ratio of 2 to 1, by first dividing it by 10, and then adding the quotient to 0.300,000,000,000,000,000,000, &c, which is the tenth part of the logarithm of the ratio of 1000 to 1. We must therefore endeavour to find the logarithm of the ratio of 1024 to 1000. This may be done in the manner following.

139. The logarithm of the ratio of 1024 to 1000 is the same with that of the ratio of $\frac{1024}{1000}$ to $\frac{1000}{1000}$, or of $1 + \frac{24}{1000}$ to 1. We must therefore endeavour to find the logarithm of the ratio of $1 + \frac{24}{1000}$ to 1, or its proportion to the logarithm of the greater ratio of 10 to 1, or (which comes to the same thing) the proportion of the ratio of $1 + \frac{24}{1000}$ to 1 to the greater ratio of 10 to 1.

140. Let this proportion be that of 1 to x . Then, since the ratio of $1 + \frac{24}{1000}$ to 1 is to the ratio of $1 + \frac{24}{1000}$ to 1 as 1 is to x ; and the said ratio of $1 + \frac{24}{1000}$ to 1 is to the ratio of 10 to 1 in the same proportion of 1 to x , it follows that the ratio of $1 + \frac{24}{1000}$ to 1 must be equal to the ratio of 10 to 1, and consequently that $1 + \frac{24}{1000}$ will be equal to 10. We must therefore endeavour to resolve the equation $1 + \frac{24}{1000} = 10$.

141. By the binomial theorem the quantity $1 + \frac{24}{1000}$ is equal to the series

$$1 + \frac{x}{1} \times \frac{24}{1000} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{24}{1000}^2 + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{24}{1000}^3 + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{24}{1000}^4 + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3} \times \frac{x-3}{4} \times \frac{x-4}{5} \times \frac{24}{1000}^5 + \&c = 1 + x \times \frac{24}{1000} + \frac{xx-x}{2} \times \frac{24}{1000}^2 + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3 + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{24}{1000}^4 + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \frac{24}{1000}^5 + \&c.$$

Therefore the series $1 + x \times \frac{24}{1000} + \frac{xx-x}{2} \times \frac{24}{1000}^2 + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3 + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{24}{1000}^4 + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \frac{24}{1000}^5 + \&c$, will be = 10, and consequently (subtracting 1 from both sides) the series $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \frac{24}{1000}^2 + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3 + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{24}{1000}^4 + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \frac{24}{1000}^5 + \&c$, will be

$$+ \frac{xx-x}{2} \times \left[\frac{24}{1000} \right]^2 + \frac{x^3-3x^2+2x}{6} \times \left[\frac{24}{1000} \right]^3 + \frac{x^4-6x^3+11xx-6x}{24} \times \left[\frac{24}{1000} \right]^4 + \frac{x^5-10x^4+35x^3-50x^2+24x}{120} \times \left[\frac{24}{1000} \right]^5 + \&c, \text{ will be } = 9.$$

142. Now, if we take any finite number of terms of this series, and suppose them to be equal to the whole series, and consequently to the absolute term 9, and resolve the equation resulting from such supposition, it is evident that the value of x , that will be obtained by such resolution, will always be greater than the true value of x in the foregoing equation between the whole of the said series and the absolute term 9; but that, the more terms of the said series are retained in such finite equation, the nearer will the root of such finite equation approach to the true value of x in the said infinite equation between the said infinite series and the absolute term 9, or to the value of x in the original equation

$$1 + \left[\frac{24}{1000} \right]^x = 10. \text{ In order therefore to approach gradually to the true value}$$

of x in the said original equation $1 + \left[\frac{24}{1000} \right]^x = 10$, we will, first, suppose one term of the foregoing series to be equal to 9; and, 2dly, two terms of it to be equal to the same quantity; and, 3dly, three terms of it, and, 4thly, four terms of it, to be equal to the same quantity, and will resolve the several equations resulting from these suppositions. This may be done in the manner following.

143. If we suppose the first term $x \times \frac{24}{1000}$ of the foregoing series to be equal to the whole series, and consequently to the absolute term 9, we shall have $x \times 24 = 9000$, and $x = \frac{9000}{24} = 375$. Therefore 375 is the first approximation to the value of x in the equation $1 + \left[\frac{24}{1000} \right]^x = 10$. But this approximation is very much too great, the true value of x being (as we shall presently see) somewhat greater than 97, but less than 98.

144. In the second place, we will suppose the two first terms, $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \left[\frac{24}{1000} \right]^2$, of the foregoing series, to be equal to the whole series, and consequently to the absolute term 9. And we shall then have $\frac{24x}{1000} \times \frac{2 \times 1000}{2 \times 1000} + \frac{xx-x}{2 \times 1000 \times 1000} \times \left[\frac{24}{1000} \right]^2 = 9$, or $\frac{48,000x}{2 \times 1000,000} + \frac{xx-x}{2 \times 1000,000} \times 576 = 9$, or $\frac{48,000 + 576xx - 576x}{2 \times 1000,000} = 9$, or $\frac{47,424x + 576xx}{2 \times 1000,000} = 9$, or $\frac{23,712x + 288xx}{1000,000} = 9$, and consequently $23,712x + 288xx = 9,000,000$, and (dividing all the terms by 288), $xx + 82.333,333, \&c x = 31,250$. This quadratick equation we must now resolve.

Add

Add $41.166,666, \&c)^2$ to both sides; and we shall have $xx + 82.333,333, \&c x + 41.166,666, \&c)^2 = 31,250 + 41.166,666, \&c)^2 = 31250 + 1694.1456 = 32,944.1456$, and consequently $x + 41.166,666, \&c (= \sqrt{32,944.1456}) = 181.50$, and $x (= 181.50 - 41.16) = 140.34$. Therefore 140.34, or (dropping the decimal fraction .34) 140, is a second approximation to the true value of x in the equation $1 + \frac{24}{1000}x = 10$.

Q. E. I.

This approximation is much nearer than the first approximation, 375, to the true value of x in this equation, but yet is much too great.

145. In the third place let us suppose the three first terms, $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \frac{24}{1000}^2 + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3$, of the foregoing series, to be equal to the whole series, and consequently to the absolute term 9.

Then, since the two first terms of this series have been shewn to be equal to $\frac{47,424x+576xx}{2 \times 1000,000}$, or $\frac{23,712x+288xx}{1000,000}$, it follows that $\frac{23,712x+288xx}{1000,000} + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3$ will, upon this supposition, be = 9, or that $\frac{23,712x+288xx}{1000,000} + \frac{x^3-3xx+2x}{6} \times \frac{13824}{1000^3}$ will be = 9, or that $\frac{23,712x+288xx}{1000,000} + \frac{x^3-3xx+2x}{6} \times \frac{2304}{1000^3}$ will be = 9, or that $\frac{23,712x+288xx}{1000,000} + \frac{2304x^3-6912xx+4608x}{1000,000,000}$ will be = 9, or that $\frac{23,712,000x+288,000xx}{1000,000,000} + \frac{2304x^3-6912xx+4608x}{1000,000,000}$ will be = 9, or that $\frac{2304x^3+281,088xx+23,716,608x}{1000,000,000}$ will be = 9, and consequently that $2304x^3 + 281,088xx + 23,716,608x$ will be = 9000,000,000, and (dividing all the terms by 2304) that $x^3 + 122xx + 10293.666 \&c x$ will be = 3,906,250. This cubick equation we must now endeavour to resolve.

Now, since we know that the value of x in this equation must be less than the root of the foregoing quadratick equation, and that root was found to be nearly equal to 140, it seems reasonable to conjecture that the root of the present cubick equation will be pretty nearly equal to 100. We will therefore suppose x to be = 100, and will substitute 100 instead of it in the compound quantity $x^3 + 122xx + 10293.666 \&c x$, in order to discover whether the result will be nearly equal to the absolute term 3,906,250, and which of them is greater than the other.

Now, if we suppose x to be = 100, we shall have $xx = 10,000$, and $x^3 = 1,000,000$, and consequently $122xx (= 122 \times 10,000) = 1,220,000$, and $10,293.666x (= 10,293.666 \times 100) = 1,029,366.6$, and $x^3 + 122xx + 10,293.666x (= 1,000,000 + 1,220,000 + 1,029,366.6) = 3,249,366.6$;
2 T 2
which

which is less than the absolute term 3,906,250. Therefore 100 is less than the true value of x in the said cubick equation $x^3 + 122xx + 10,293.666x = 3,906,250$.

We will therefore form a second conjecture concerning the root of this cubick equation, and will suppose it to be $= 110$.

Now, if x is $= 110$, we shall have $xx = 12,100$, and $x^3 = 1,331,000$, and $122xx (= 122 \times 12,100) = 1,476,200$, and $10,293.666x (= 10,293.666 \times 110) = 1,132,303.26$, and consequently $x^3 + 122xx + 10,293.666x (= 1,000,000 + 1,476,200 + 1,132,303.26) = 3,939,503.26$; which is but little greater than the absolute term 3,906,250. We may therefore conclude that the root of the cubick equation $x^3 + 122xx + 10,293.666x = 3,906,250$ is somewhat, but not much, less than 110, and consequently that 110 will be

a third approximation to the value of the index x in the equation $1 + \frac{24}{1000}x = 10$. Q. E. I.

146. In the 4th place let us suppose the four first terms of the foregoing series, to wit, the terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \frac{24}{1000}^2 + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3 + \frac{x^4-6x^3+11x^2-6x}{24} \times \frac{24}{1000}^4$, to be equal to the whole series, and consequently to the absolute term 9.

Then, since it has been seen in the last article, that the three first terms of this series are equal to $\frac{2304x^3+281,088xx+23,716,608x}{1000,000,000}$ it follows that the four first

$$\begin{aligned} \text{terms of this series will be } &= \frac{2304x^3+281,088xx+23,716,608x}{1000,000,000} + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{24}{1000}^4 \\ &= \frac{2304x^3+281,088xx+23,716,608x}{1000,000,000} + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{24^4}{1000^4} = \\ &= \frac{2304x^3+281,088xx+23,716,608x}{1000,000,000} + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{24^3}{1000^4} = \\ &= \frac{2304x^3+281,088xx+23,716,608x}{1000,000,000} + \frac{x^4-6x^3+11xx-6x}{24} \times \frac{13824}{1000^4} = \\ &= \frac{2304x^3+281,088xx+23,716,608x}{1000,000,000} + \frac{13,824x^4-82,944x^3+152,064xx-82,944x}{1000,000,000,000} = \\ &= \frac{2,304,000x^3+281,088,000xx+23,716,608,000x}{1000,000,000,000} + \frac{13,824x^4-82,944x^3+152,064x^2-82,944x}{1000,000,000,000} \\ &= \frac{13,824x^4+2,221,056x^3+281,240,064xx+23,716,525,056x}{1000,000,000,000}. \end{aligned}$$

Therefore $\frac{13,824x^4+2,221,056x^3+281,240,064xx+23,716,525,056x}{1000,000,000,000}$ will be $= 9$, and consequently

$13,824x^4 + 2,221,056x^3 + 281,240,064xx + 23,716,525,056x$ will be $= 9,000,000,000,000$, and (dividing all the terms by 13,824) $x^4 + 160.666,666, \&c x^3 + 20,344.333,333, \&c xx + 1,715,605.111,111, \&c x$ will be $= 651,041,666.666,666, \&c$. This biquadratick equation we must now endeavour to resolve.

Now,

Now, since we know that the root of this biquadratic equation must be less than the root of the foregoing cubic equation, and that root was found to be nearly equal to 110, we may reasonably conjecture that the root of this biquadratic equation will be nearly equal to 100. And so, upon trial, we shall find it to be. For, if x be $= 100$, we shall have $xx = 10,000$, and $x^3 = 1,000,000$, and $x^4 = 100,000,000$, and consequently $160.666,666, \&c \times x^3 (= 160.666,666, \&c \times 1,000,000 =) 160,666,666.666, \&c$, and $20,344.333,333, \&c \times xx (= 20,344.333,333, \&c \times 10,000 =) 203,443,333.333, \&c$, and $1,715,605.111,111, \&c \times x (= 1,715,605.111,111, \&c \times 100 =) 171,560,511.111,111, \&c$, and $x^4 + 160.666,666, \&c \times x^3 + 20,344.333,333, \&c \times xx + 1,715,605.111,111, \&c \times x (= 100,000,000 + 160,666,666.666, \&c + 203,443,333.333, \&c + 171,560,511.111, \&c) = 635,670,511.110, \&c$; which is not a great deal less than the absolute term $651,041,666.666, \&c$. And consequently 100 is not a great deal less than the true value of x in the aforesaid biquadratic equation $x^4 + 160.666,666, \&c \times x^3 + 20,344.333,333, \&c \times xx + 1,715,605.111,111, \&c \times x = 651,041,666.666, \&c$.

We may therefore consider the number 100 as a fourth approximation to the true value of x in the equation $1 + \frac{24}{1000}x = 10$. Q. E. I.

147. By the resolution of these four equations we have obtained the four numbers 375, 140, 110, and 100 for so many successive approximations to the true value of x in the equation $1 + \frac{24}{1000}x = 10$. And, as the difference between the second and third of these values, to wit, 140 and 110, is 30, and the difference between the third and fourth, to wit, 110 and 100, is only 10, (which is but a third part of 30) it seems reasonable to conjecture that the difference between 100, the fourth, or last, approximation to the true value of x in the equation $1 + \frac{24}{1000}x = 10$, and the said true value will be less than a third part of the former difference 10, or will be nearly equal to 3, and consequently that the said true value of x in the equation $1 + \frac{24}{1000}x = 10$ will be nearly equal to $100 - 3$, or 97. We will therefore suppose the true value of x in this equation $1 + \frac{24}{1000}x = 10$ to be equal to 97, and will now proceed to try whether it is so, or not, and whether it is greater, or less, than the said true value, by raising the binomial quantity $1 + \frac{24}{1000}$ to the 97th power. This may be done most conveniently by means of the binomial theorem in the manner following.

148. By the binomial theorem we shall have $1 + \frac{24}{1000}x^{97} =$ the series $1 + 97 A \times \frac{24}{1000} + \frac{96}{2} B \times \frac{24}{1000}^2 + \frac{95}{3} C \times \frac{24}{1000}^3 + \frac{94}{4} D \times \frac{24}{1000}^4 + \frac{93}{5} E \times \frac{24}{1000}^5$

$$\begin{aligned}
& \left(\frac{24}{1000}\right)^5 + \frac{92}{6} F \times \left(\frac{24}{1000}\right)^6 + \frac{91}{7} G \times \left(\frac{24}{1000}\right)^7 + \frac{90}{8} H \times \left(\frac{24}{1000}\right)^8 + \frac{89}{9} I \times \left(\frac{24}{1000}\right)^9 \\
& + \frac{88}{10} K \times \left(\frac{24}{1000}\right)^{10} + \frac{87}{11} L \times \left(\frac{24}{1000}\right)^{11} + \frac{86}{12} M \times \left(\frac{24}{1000}\right)^{12} + \frac{85}{13} N \times \left(\frac{24}{1000}\right)^{13} + \\
& \frac{84}{14} O \times \left(\frac{24}{1000}\right)^{14} + \frac{83}{15} P \times \left(\frac{24}{1000}\right)^{15} + \frac{82}{16} Q \times \left(\frac{24}{1000}\right)^{16} + \frac{81}{17} R \times \left(\frac{24}{1000}\right)^{17} + \frac{80}{18} \\
& S \times \left(\frac{24}{1000}\right)^{18} + \frac{79}{19} T \times \left(\frac{24}{1000}\right)^{19} + \frac{78}{20} V \times \left(\frac{24}{1000}\right)^{20} + \frac{77}{21} W \times \left(\frac{24}{1000}\right)^{21} + \frac{76}{22} X \\
& + \left(\frac{24}{1000}\right)^{22} + \frac{75}{23} Y \times \left(\frac{24}{1000}\right)^{23} + \frac{74}{24} Z \times \left(\frac{24}{1000}\right)^{24} + \frac{73}{25} A' \times \left(\frac{24}{1000}\right)^{25} + \frac{72}{26} B' \times \\
& \left(\frac{24}{1000}\right)^{26} + \frac{71}{27} C' \times \left(\frac{24}{1000}\right)^{27} + \frac{70}{28} D' \times \left(\frac{24}{1000}\right)^{28} + \frac{69}{29} E' \times \left(\frac{24}{1000}\right)^{29} + \&c; \text{ in which} \\
& \text{series the capital letter A stands for the first term 1, and the following letters} \\
& \text{B, C, D, E, F, \&c, stand for the co-efficients of the powers of } \frac{24}{1000} \text{ in the second,} \\
& \text{third, fourth, fifth, sixth, and other following terms of the series. But, for the} \\
& \text{purpose of computing these several terms, it will be more convenient to make} \\
& \text{the letters B, C, D, E, F, \&c, stand for the whole second, third, fourth, fifth,} \\
& \text{sixth, and other following terms of the series respectively, including the powers} \\
& \text{of } \frac{24}{1000}; \text{ and then we shall have } 1 + \left(\frac{24}{1000}\right)^{97} = \text{the series } 1 + 97 A \times \frac{24}{1000} \\
& + \frac{96}{2} B \times \frac{24}{1000} + \frac{95}{3} C \times \frac{24}{1000} + \frac{94}{4} D \times \frac{24}{1000} + \frac{93}{5} E \times \frac{24}{1000} + \frac{92}{6} F \times \\
& \frac{24}{1000} + \frac{91}{7} G \times \frac{24}{1000} + \frac{90}{8} H \times \frac{24}{1000} + \frac{89}{9} I \times \frac{24}{1000} + \frac{88}{10} K \times \frac{24}{1000} + \frac{87}{11} L \\
& \times \frac{24}{1000} + \frac{86}{12} M \times \frac{24}{1000} + \frac{85}{13} N \times \frac{24}{1000} + \frac{84}{14} O \times \frac{24}{1000} + \frac{83}{15} P \times \frac{24}{1000} + \frac{82}{16} \\
& Q \times \frac{24}{1000} + \frac{81}{17} R \times \frac{24}{1000} + \frac{80}{18} S \times \frac{24}{1000} + \frac{79}{19} T \times \frac{24}{1000} + \frac{78}{20} V \times \frac{24}{1000} + \\
& \frac{77}{21} W \times \frac{24}{1000} + \frac{76}{22} X \times \frac{24}{1000} + \frac{75}{23} Y \times \frac{24}{1000} + \frac{74}{24} Z \times \frac{24}{1000} + \frac{73}{25} A' \times \frac{24}{1000} \\
& + \frac{72}{26} B' \times \frac{24}{1000} + \frac{71}{27} C' \times \frac{24}{1000} + \frac{70}{28} D' \times \frac{24}{1000} + \frac{69}{29} E' \times \frac{24}{1000} + \&c =
\end{aligned}$$

$$\begin{aligned}
& 1.000,000,000,000,000,000, \\
& + 2.328,000,000,000,000,000, \\
& + 2.681,856,000,000,000,000, \\
& + 2.038,210,560,000,000,000, \\
& + 1.149,550,755,840,000,000, \\
& + 0.513,159,457,406,976,000, \\
& + 0.188,842,680,325,767,168, \\
& + 0.058,918,916,261,639,356, \\
& + 0.015,908,107,390,642,626, \\
& + 0.003,775,524,154,045,849, \\
& + 0.000,797,390,701,334,483, \\
& + 0.000,151,359,253,126,036, \\
& + 0.000,026,033,791,537,678,
\end{aligned}$$

$$+ 0,000,$$

$$\begin{aligned}
&+ 0.000,004,085,302,672,066, \\
&+ 0.000,000,588,283,584,777, \\
&+ 0.000,000,078,124,060,058, \\
&+ 0.000,000,009,609,259,387, \\
&+ 0.000,000,001,098,847,073, \\
&+ 0.000,000,000,117,210,354, \\
&+ 0.000,000,000,011,696,359, \\
&+ 0.000,000,000,001,094,779, \\
&+ 0.000,000,000,000,096,340, \\
&+ 0.000,000,000,000,007,987, \\
&+ 0.000,000,000,000,000,625, \\
&+ 0.000,000,000,000,000,046, \\
&+ 0.000,000,000,000,000,003, \\
&+ 0.000,000,000,000,000,000, \\
&+ \&c
\end{aligned}$$

$= 9.979,201,547,673,599,050, \&c$; which is somewhat less than 10. Therefore 97 is somewhat less than the true value of the index x in the equation

$1 + \frac{24}{1000}^x = 10$. But it differs from it by less than an unit. For, if we multiply the number just now found for the value of $1 + \frac{24}{1000}^{97}$, to wit, 9.979,201,

547,673,599,050, into $1 + \frac{24}{1000}$, or 1.024, we shall find the product to be $=$

10.218,702,384,817,765,427,200, which is therefore the value of $1 + \frac{24}{1000}^{98}$.

Therefore $1 + \frac{24}{1000}^{98}$ is greater than 10, or than $1 + \frac{24}{1000}^x$; and $1 + \frac{24}{1000}^{97}$ is

less than 10, or than $1 + \frac{24}{1000}^x$; and consequently x must be less than 98, but greater than 97, or the difference between 97 and the true value of x in the

equation $1 + \frac{24}{1000}^x = 10$ must be less than 1. We therefore now know with

certainty that the two first figures of the true value of x in the equation

$1 + \frac{24}{1000}^x = 10$ are 97; which is a considerable step towards a more accurate determination of it.

149. Let the excess of the true value of x in the equation $1 + \frac{24}{1000}^x = 10$ above 97 be called z . And we shall then have $1 + \frac{24}{1000}^{97+z} (= 1 + \frac{24}{1000}^x)$

$= 10$. But $1 + \frac{24}{1000}^{97+z}$ is $= 1 + \frac{24}{1000}^{97} \times 1 + \frac{24}{1000}^z = 9.979,201,547,673,$

599,050, $\times 1 + \frac{24}{1000}^z$. Therefore $9.979,201,547,673,599,050 \times 1 + \frac{24}{1000}^z$ is

is = 10; and consequently $1 + \frac{24}{1000}z$ is = $\frac{10,000,000,000,000,000,000}{9,979,201,547,673,599,050} = 1.002,084,180,004,486,389$. But, by the binomial theorem, $1 + \frac{24}{1000}z$ is = the series $1 + z \times \frac{24}{1000} + z \times \frac{z-1}{2} \times \left(\frac{24}{1000}\right)^2 + z \times \frac{z-1}{2} \times \frac{z-2}{3} \times \left(\frac{24}{1000}\right)^3 + z \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \left(\frac{24}{1000}\right)^4 + \&c = 1 + z \times \frac{24}{1000} + \frac{zz-z}{2} \times \left(\frac{24}{1000}\right)^2 + \frac{z^3-3zz+2z}{6} \times \left(\frac{24}{1000}\right)^3 + \frac{z^4-6z^3+11zz-6z}{24} \times \left(\frac{24}{1000}\right)^4 + \&c =$ (because z is less than 1, and consequently zz is less than z , and z^3 than zz , and z^4 than z^3) $1 + z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \times \left(\frac{24}{1000}\right)^2 + \frac{2z-3zz+z^3}{6} \times \left(\frac{24}{1000}\right)^3 - \left[\frac{6z-11zz+6z^3-z^4}{24} \times \left(\frac{24}{1000}\right)^4 + \&c\right.\right.$ Therefore the said series $1 + z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \times \left(\frac{24}{1000}\right)^2 + \frac{2z-3zz+z^3}{6} \times \left(\frac{24}{1000}\right)^3 - \left[\frac{6z-11zz+6z^3-z^4}{24} \times \left(\frac{24}{1000}\right)^4 + \&c\right.\right.$ will be = 1.002,084,180,004,486,389, and consequently (subtracting 1 from both sides) the series $z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \times \left(\frac{24}{1000}\right)^2 + \frac{2z-3zz+z^3}{6} \times \left(\frac{24}{1000}\right)^3 - \left[\frac{6z-11zz+6z^3-z^4}{24} \times \left(\frac{24}{1000}\right)^4 + \&c\right.\right.$ will be = 0.002,084,180,004,486,389, &c. This equation we must now endeavour to resolve.

150. To find the value of z in this equation we will, first, suppose the first term $z \times \frac{24}{1000}$ of the last-mentioned series to be, alone, equal to the whole series, and consequently to the absolute term 0.002,084,180,004,486,389, &c; and we shall then have $z \times 24 = 1000 \times 0.002,084,180,004,486,389, \&c = 2.084,180,004,486,389, \&c$, and consequently $z = \frac{2.084,180,004,486,389, \&c}{24} = 0.086,840, \&c$. Q. E. I.

Therefore x , or $97 + z$, will be = $97 + 0.086,840, \&c$, or $97.086,840, \&c$; of which number the first four figures 97.08 are exact, the more accurate value of x being 97.087,787,353,856,001,437, as I have found by dividing 1 by 0.010,299,956,639,811,952,13, which is the logarithm of the ratio of 1024 to 1000. And of these four figures, 97.08, which are exact, the two last, to wit, .08, have been obtained by the resolution of this very easy simple equation.

151. This first value of z (which is obtained by resolving the simple equation $z \times \frac{24}{1000} = 0.002,084,180,004,486,389$) must be less than the true value of z in the infinite equation set down in the latter part of art. 149; because the second and other following terms of the infinite series contained in that equation are marked alternately with the sign — and the sign +, or are alternately subtracted

tracted from, and added to, the first term of it, the consequence of which is (as the terms form a decreasing progression) that the first term alone must be greater than the whole series, and the first and second together, that is, the excess of the first above the second, must be less than the whole series, and, in like manner, that the three first terms must be greater, and the four first terms must be less, and every following odd number of terms must be greater, and every following even number of terms must be less, than the whole series. Therefore, when the first term is supposed to be equal to the whole series, or to the absolute term 0.002,084,180,004,486,389, it is supposed to be less than it really is, and the value of it derived from such supposition will be less than its true value. Therefore 0.086,840 is less than the true value of z . Q. E. D.

152. In the next place we will suppose the two first terms, $z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \times \frac{24}{1000} \right]^2$, of the foregoing series to be equal to the whole series, and consequently to the absolute term 0.002,084,180,004,486,389, and will resolve the quadratick equation resulting from such supposition, which (from what has been shewn in the foregoing article) will give us a value of z somewhat greater than the truth, but which will come nearer to it than the last value 0.086,840.

Now these two terms $z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \times \frac{24}{1000} \right]^2$ are $= z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \right]^2 \times \frac{24^2}{1000^2} = z \times \frac{24}{1000} - \left[\frac{z-zz}{2} \times \frac{576}{1000,000} \right] = z \times \frac{24}{1000} - \frac{576z-576zz}{2 \times 1000,000} = z \times \frac{24}{1000} \times \frac{2 \times 1000}{2 \times 1000} - \frac{576z-576zz}{2 \times 1000,000} = \frac{48,000z-576z+576zz}{2 \times 1000,000} = \frac{47,424z+576zz}{2 \times 1000,000} = \frac{23,712z+288zz}{1000,000}$. Therefore $\frac{23,712z+288zz}{1000,000}$ will be $= 0.002,084,180,004,486,389$, and consequently $23,712z + 288zz$ will be $(= 1000,000 \times 0.002,084,180,004,486,389) = 2,084.180,004,486,389$, and (dividing all the terms by 288) $zz + 82.333,333, \&c, \times z$ will be $= 7.236,736,126,688$. Therefore (adding $41.166,666, \&c)^2$ to both sides) we shall have $zz + 82.333,333, \&c, \times z + 41.166,666, \&c)^2 (= 7.236,736,126,688 + 41.166,666, \&c)^2 = 7.236,736,126,688 + 1694.694,389,555,556) = 1701.931,125,682,244$, and consequently $z + 41.166,666, \&c (= \sqrt{1701.931,125,682,244}) = 41.254,467$, and $z (= 41.254,467 - 41.166,666) = 0.087,801$. Therefore 0.087,801 is a second approximation to the true value of z in the infinite equation set down at the end of art. 149, or in the equation $1 + \frac{24}{1000}z = 1.002,084,180,004,486,389$. Q. E. I.

153. The arithmetical mean between 0.086,840, and 0.087,801 is 0.087,320. But the true value of z is much nearer to 0.087,801 than to 0.086,840. Therefore it must be greater than the said arithmetical mean between them, to wit, 0.087,320, and probably not much less than 0.087,801, the latter and greater of the two foregoing approximations to it. It seems reasonable there-

fore to conjecture that it is equal to 0.087,7. We will therefore suppose it to be equal to 0.087,7, and consequently suppose x , or $97 + x$, to be equal to 97.0877, and shall proceed to make a trial whether this number is greater or less than the true value of x in the equation $1 + \frac{24}{1000}^x = 10$, by raising the binomial quantity $1 + \frac{24}{1000}$ to the 97.0877th, or $\frac{970,877}{10,000}$ th power by means of the binomial theorem.

154. Now this has been done above in art. 136, and it has appeared that the said power of the binomial quantity $1 + \frac{24}{1000}$ is equal to the mixt number 9.999,979,282,720,950,507,346, &c, which is very little less than 10. Therefore the number 97.0877 must be very little less than the true value of the index x in the equation $1 + \frac{24}{1000}^x = 10$.

Let y be the excess whereby the true value of x exceeds 97.0877. Then will $97.0877 + y$ be $= x$, and $1 + \frac{24}{1000}^x$ will be $= 1 + \frac{24}{1000}^{97.0877+y}$ and consequently $1 + \frac{24}{1000}^{97.0877+y}$ will be $= 10$. But $1 + \frac{24}{1000}^{97.0877+y}$ is $= 1 + \frac{24}{1000}^{97.0877} \times 1 + \frac{24}{1000}^y = 9.999,979,282,720,950,507,346, \&c \times 1 + \frac{24}{1000}^y$. Therefore $9.999,979,282,720,950,507,346, \&c \times 1 + \frac{24}{1000}^y$ will be $= 10$, and consequently $1 + \frac{24}{1000}^y$ will be $(= \frac{10,000,000,000,000,000,000,000,000}{9.999,979,282,720,950,507,346,}) = 1.000,002,071,732,197,014, \&c$. But, by the binomial theorem, $1 + \frac{24}{1000}^y$ is $=$ the infinite series $1 + y \times \frac{24}{1000} + y \times \frac{y-1}{2} \times \frac{24}{1000}^2 + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{24}{1000}^3 + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{24}{1000}^4 + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{y-4}{5} \times \frac{24}{1000}^5 + y \times \frac{y-1}{2} \times \frac{y-2}{3} \times \frac{y-3}{4} \times \frac{y-4}{5} \times \frac{y-5}{6} \times \frac{24}{1000}^6 + \&c = 1 + y \times \frac{24}{1000} + \frac{yy-y}{2} \times \frac{24}{1000}^2 + \frac{y^3-3yy+2y}{6} \times \frac{24}{1000}^3 + \frac{y^4-6y^3+11yy-6y}{24} \times \frac{24}{1000}^4 + \frac{y^5-10y^4+35y^3-50yy+24y}{120} \times \frac{24}{1000}^5 + \frac{y^6-15y^5+85y^4-235y^3+274yy-120y}{720} \times \frac{24}{1000}^6 + \&c =$ (because y is less than 1, and consequently yy is less than y , and y^3 than yy , and y^4 than y^3 , and every following power of y less than the next preceeding power of it), $1 + y \times \frac{24}{1000} - \left[\frac{y-yy}{2} \times \frac{24}{1000}^2 + \frac{2y-3yy+y^3}{6} \times \frac{24}{1000}^3 - \left[\frac{6y-11yy+6y^3-y^4}{24} \times \frac{24}{1000}^4 + \frac{24y-50yy+35y^3-10y^4+y^5}{120} \times \frac{24}{1000}^5 - \right. \right.$

$\left[\frac{120y - 274yy + 235y^3 - 8y^4 + 5y^5 - y^6}{720} \times \frac{24}{1000} \right]^6 + \&c.$ Therefore this series $1 + y \times \frac{24}{1000} - \left[\frac{y-yy}{2} \times \frac{24}{1000} \right]^2 + \frac{2y-3yy+y^3}{6} \times \frac{24}{1000}^3 - \left[\frac{6y-11yy+6y^3-y^4}{24} \times \frac{24}{1000} \right]^4 + \&c$ will be $= 1.000,002,071,732,197,014$; and consequently (subtracting 1 from both sides) the series $y \times \frac{24}{1000} - \left[\frac{y-yy}{2} \times \frac{24}{1000} \right]^2 + \frac{2y-3yy+y^3}{6} \times \frac{24}{1000}^3 - \left[\frac{6y-11yy+6y^3-y^4}{24} \times \frac{24}{1000} \right]^4 + \&c$ will be $= 0.000,002,071,732,197,014$. This equation we must now endeavour to resolve.

155. Now, if we suppose the first term, $y \times \frac{24}{1000}$, of this series to be equal to the whole series, and consequently to the absolute term $0.000,002,071,732,197,014$, we shall have $y \times 24 (= 1000 \times 0.000,002,071,732,197,014) = 0.002,071,732,197,014$, and consequently $y (= \frac{0.002,071,732,197,014}{24}) = 0.000,086,322,1$. Therefore x , or $97.0877 + y$, will be $= 97.087,786,322,1$, of which the first seven figures $97.087,78$ are exact, the more accurate value of x being (as we before observed in art. 150) $97.087,787,353,856,001,437$.

156. In the next place we will suppose the two first terms, $y \times \frac{24}{1000} - \left[\frac{y-yy}{2} \times \frac{24}{1000} \right]^2$, of the foregoing series, to be equal to the whole series, and consequently to the absolute term $0.000,002,071,732,197,014$, and will resolve the quadratick equation thence resulting.

Now $y \times \frac{24}{1000} - \left[\frac{y-yy}{2} \times \frac{24}{1000} \right]^2$ is $= y \times \frac{24}{1000} \times \frac{2 \times 1000}{2 \times 1000} - \left[\frac{y-yy}{2} \times \frac{24}{1000} \right]^2$
 $\frac{576}{1000,000} = \frac{48,000y}{2 \times 1000,000} - \frac{576y - 576yy}{2 \times 1000,000} = \frac{48,000y - 576y + 576yy}{2 \times 1000,000} = \frac{47,424y + 576yy}{2 \times 1000,000} = \frac{23,712y + 288yy}{1000,000}$. Therefore $\frac{23,712y + 288yy}{1000,000}$ will be $= 0.000,002,071,732,197,014$, and consequently $23,712y + 288yy$ will be $(= 1000,000 \times 0.000,002,071,732,197,014) = 2.071,732,197,014$, and (dividing all the terms by 288) $82.333,333, \&c \times y + yy$ will be $= 0.007,193,514,572$. Therefore, if we add $41.166,666, \&c]^2$, or $41 + \frac{1}{6}]^2$, to both sides, we shall have $41.166,666, \&c]^2 + 82.333,333, \&c \times y + yy (= 0.007,193,514,572 + 41 + \frac{1}{6})^2 = 0.007,193,514,572 + 41]^2 + 2 \times 41 \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = 0.007,193,514,572 + 1681 + 41 \times \frac{1}{3} + 0.166,666,666,666, \&c \times \frac{1}{6} = 0.007,193,514,572 + 1681 + 13.666,666,666,666, + \frac{0.166,666,666,666}{6} = 0.007,193,514,572 + 1694.666,666,666,666, + 0.027,777,777,777 = 0.007,193,514,572 + 1694.694,444,444,444) = 1694.701,637,959,015$. Therefore $41.166,666,666,$

2 U 2

666, &c + y will be $(= \sqrt{1694.701,637,959,015}) = 41.166,754,037$, and y will be $(= 41.166,754,037 - 41.166,666,666) = 0.000,087,371$, and consequently x , or $97.0877 + y$, will be $= 97.087,787,371$; of which number the first nine figures $97.087,787,3$ are exact, the more accurate value of x being $97.087,787,353,856,001,437$. This is a very considerable degree of exactness.

157. In order to obtain the value of y to a still greater degree of exactness, it will be convenient to set down the infinite equation that is obtained in art. 154, in a different manner, as follows; to wit,

$$\begin{aligned}
 y \times \frac{24}{1000} - \frac{y}{2} \times \frac{24}{1000}^2 + \frac{2y}{6} \times \frac{24}{1000}^3 - \frac{6y}{24} \times \frac{24}{1000}^4 + \&c \\
 + \frac{yy}{2} \times \frac{24}{1000}^2 - \frac{3yy}{6} \times \frac{24}{1000}^3 + \frac{11yy}{24} \times \frac{24}{1000}^4 - \&c \\
 + \frac{y^3}{6} \times \frac{24}{1000}^3 - \frac{6y^3}{24} \times \frac{24}{1000}^4 + \&c \\
 + \frac{y^4}{24} \times \frac{24}{1000}^4 - \&c
 \end{aligned}$$

$$= 0.000,002,071,732,197,014.$$

Now, since y is a very small quantity (being less than $0.000,088$, and, *a fortiori*, less than 0.0001 , or $\frac{1}{10,000}$), it follows that yy and y^3 and y^4 and all the following powers of y will be very small in comparison of y ; and consequently, if we omit all the terms in the foregoing equation which involve yy , or any higher power of y , the series which forms the left-hand side of the said equation will not be much affected thereby, but the remaining terms of it (which are those that involve only the simple power of y) will be very nearly equal to the absolute term $0.000,002,071,732,197,014$; that is, the upper line of terms alone,

to wit, $y \times \frac{24}{1000} - \frac{y}{2} \times \frac{24}{1000}^2 + \frac{2y}{6} \times \frac{24}{1000}^3 - \frac{6y}{24} \times \frac{24}{1000}^4 + \frac{24y}{120} \times \frac{24}{1000}^5 - \frac{120y}{720} \times \frac{24}{1000}^6 + \&c$, will be very nearly equal to $0.000,002,071,732,197,$

014 . But this line of terms is $= y \times \frac{24}{1000} - \frac{y}{2} \times \frac{24}{1000}^2 + \frac{y}{3} \times \frac{24}{1000}^3 - \frac{y}{4} \times \frac{24}{1000}^4 + \frac{y}{5} \times \frac{24}{1000}^5 - \frac{y}{6} \times \frac{24}{1000}^6 + \&c = y \times$ the series $\frac{24}{1000} - \frac{1}{2} \times \frac{24}{1000}^2 + \frac{1}{3} \times \frac{24}{1000}^3 - \frac{1}{4} \times \frac{24}{1000}^4 + \frac{1}{5} \times \frac{24}{1000}^5 - \frac{1}{6} \times \frac{24}{1000}^6 + \&c =$ (if we put A for the first term $\frac{24}{1000}$ of the said series, and B for its second term $\frac{1}{2} \times \frac{24}{1000}^2$, and C for its third term $\frac{1}{3} \times \frac{24}{1000}^3$, and $D, E, F, G, H, \&c$, for its fourth, fifth, sixth, seventh, eighth, and other following terms respectively) $y \times$ the series $\frac{24}{1000} - \frac{1}{2} A \times \frac{24}{1000} + \frac{2}{3} B \times \frac{24}{1000} - \frac{3}{4} C \times \frac{24}{1000} + \frac{4}{5} D \times \frac{24}{1000} - \frac{5}{6} E \times \frac{24}{1000} + \frac{6}{7} F \times \frac{24}{1000} - \frac{7}{8} G \times \frac{24}{1000} + \frac{8}{9} H \times \frac{24}{1000} - \frac{9}{10} I \times \frac{24}{1000} + \frac{10}{11} K \times \frac{24}{1000} - \frac{11}{12} L \times \frac{24}{1000} + \&c = y \times$ the series

$$\begin{aligned}
& 0.024,000,000,000,000,000,000, \quad A, \\
& - 0.000,288,000,000,000,000,000, \quad B, \\
& + 0.000,004,608,000,000,000,000, \quad C, \\
& - 0.000,000,082,944,000,000,000, \quad D, \\
& + 0.000,000,001,592,524,800,000, \quad E, \\
& - 0.000,000,000,031,850,496,000, \quad F, \\
& + 0.000,000,000,000,655,210,203, \quad G, \\
& - 0.000,000,000,000,013,759,414, \quad H, \\
& + 0.000,000,000,000,000,293,534, \quad I, \\
& - 0.000,000,000,000,000,006,222, \quad K, \\
& + 0.000,000,000,000,000,000,135, \quad L, \\
& - 0.000,000,000,000,000,000,002, \quad M, \\
& + \&c = \\
& 0.024,004,609,593,180,303,872 \times y \\
& - 0.000,288,082,031,864,261,638 \times y \\
& = 0.023,716,527,561,316,042,234 \times y.
\end{aligned}$$

Therefore $0.023,716,527,561,316,042,234 \times y$ will be very nearly equal to the absolute term $0.000,002,071,732,197,014$; and consequently y will be very nearly equal to $\frac{0.000,002,071,732,197,014}{0.023,716,527,561,316,042,234}$, or $0.000,087,353,943,011$. Therefore x , or $97.0877 + y$, will be $= 97.087,787,353,943,011$; of which number the first eleven figures, $97.087,787,353$, are exact, the more accurate value of x in the equation $1 + \frac{24}{1000}^x = 10$ being $97.087,787,353,856,001,437$.

158. We have now obtained the value of the index x in the equation $1 + \frac{24}{1000}^x = 10$ exact to eleven places of figures. And, if we wish to obtain it to a still greater degree of exactness, we need only retain a few of those terms of the compound infinite series set down in art. 157 which involve the square of y , and which constitute the second horizontal row of terms in the said compound series. Now the first five terms of the said second horizontal row are $+\frac{yy}{2} \times \frac{24}{1000}^2 - \frac{3yy}{6} \times \frac{24}{1000}^3 + \frac{11yy}{24} \times \frac{24}{1000}^4 - \frac{50yy}{120} \times \frac{24}{1000}^5 + \frac{274yy}{720} \times \frac{24}{1000}^6$, which are equal to $yy \times$ the series $\frac{1}{2} \times \frac{24}{1000}^2 - \frac{3}{6} \times \frac{24}{1000}^3 + \frac{11}{24} \times \frac{24}{1000}^4 - \frac{50}{120} \times \frac{24}{1000}^5 + \frac{274}{720} \times \frac{24}{1000}^6 = yy \times$ the series $\frac{1}{2} \times 0.024^2 - \frac{1}{2} \times 0.024^3 + \frac{11}{24} \times 0.024^4 - \frac{5}{12} \times 0.024^5 + \frac{137}{360} \times 0.024^6 = yy \times$ the series $\frac{1}{2} \times 0.000,576 - \frac{1}{2} \times 0.000,013,824 + \frac{11}{24} \times 0.000,000,331,776 - \frac{5}{12} \times 0.000,000,007,962,624 + \frac{137}{360} \times 0.000,000,000,031,850,496 = yy \times$ the series

0.000,288,

$$\left. \begin{array}{l} 0.000,288,000,000,000,000, \\ - 0.000,006,912,000,000,000, \\ + 0.000,000,152,064,000,000,000, \\ - 0.000,000,003,317,760,000,000, \\ + 0.000,000,000,012,120,883,200, \end{array} \right\} = yy \times$$

$0.000,288,152,076,120,883,200 - yy \times 0.000,006,915,317,760,000,000 = yy \times 0.000,281,236,758,360,883,200$. Therefore the two upper horizontal lines of the compound infinite series set down in art. 157 are $= 0.023,716,527,561,316,042,234 \times y + 0.000,281,236,758,360,883,200 \times yy$; and consequently these two terms $0.023,716,527,561,316,042,234 \times y + 0.000,281,236,758,360,883,200 \times yy$ will be very nearly equal to the absolute term $0.000,002,071,732,197,014$. This quadratick equation we must now endeavour to resolve.

This equation may be most conveniently resolved by approximation, by substituting, instead of y , in the quantity $0.000,281,236,758,360,883,200 \times yy$ the value of y derived from the simple equation $0.023,716,527,561,316,042,234 \times y = 0.000,002,071,732,197,014$, which is $= \left(\frac{0.000,002,071,732,197,014}{0.023,716,527,561,316,042,234} \right)$ or $0.000,087,353,943,011$. We shall then have $yy = \frac{0.000,087,353,943,011^2}{0.000,000,007,630,710,833,523,6}$, or (dropping the last seven figures), $0.000,000,007,630,710$, and consequently $0.000,281,236,758,360,883,200 \times yy (= 0.000,281,236,758,360,883,200 \times 0.000,000,007,630,710) = 0.000,000,002,146,036,144,391,975,043,072$. Therefore $0.023,716,527,561,316,042,234 \times y + 0.000,000,000,002,146,036,144,391,975,043,072$ will be $(= 0.023,716,527,561,316,042,234 \times y + 0.000,281,236,758,360,883,200 \times yy) = 0.000,002,071,732,197,014$, and consequently (dividing all the terms by $0.023,716,527,561,316,042,234$) $y + 0.000,000,000,090,487$ will be $= 0.000,087,353,943,011$, and y will be $=$

$\frac{0.000,087,353,943,011}{0.000,000,000,090,487} = 0.000,087,353,852,524$. Therefore x , or $97.0877 + y$ will be $= 97.087,787,353,852,524$; of which number the first thirteen figures $97.087,787,353,85$ are exact, the more accurate value of x being (as we have before observed) $97.087,787,353,856,001,437$. We have therefore now obtained the value of x , or the index of the power of the binomial quantity $1 + \frac{24}{1000}$ in the original equation $1 + \frac{24}{1000}^x = 10$, exact to thirteen places of figures.

Q. E. I.

159. The ratio of $1 + \frac{24}{1000}^x$, or 10, to 1 is to the ratio of $1 + \frac{24}{1000}$ to 1 as x is to 1, and consequently as 1 is to the fraction $\frac{1}{x}$; so that, if 1 be taken for the representative of the ratio of 10 to 1 (as it is in Briggs's System of Logarithms), the fraction $\frac{1}{x}$ will be the representative of the ratio of $1 + \frac{24}{1000}$ to 1, that

that is, in other words, the fraction $\frac{1}{x}$ will be Briggs's logarithm of the ratio of $1 + \frac{24}{1000}$ to 1. Therefore Briggs's logarithm of the ratio of $1 + \frac{24}{1000}$ to 1, or of the ratio of 1024 to 1000, will be $(= \frac{1.000,000,000,000,000,000}{97.087,787,353,852,524}) = 0.010,299,956,639,812$. Therefore the logarithm of the ratio of 1024 to 1 will be $(= \log. \frac{1024}{1000} + \log. \frac{1000}{1} = 0.010,299,956,639,812 + \log. \frac{1000}{1} = 0.010,299,956,639,812 + 3) = 3.010,299,956,639,812$; and consequently the logarithm of the ratio of 2 (which is the 10th root of 1024) to 1 will be $(= \frac{3.010,299,956,639,812}{10}) = 0.301,029,995,663,981,2$; or, according to the common mode of expression, the logarithm of the number 2 will be $= 0.301,029,995,663,981,2$. Q. E. I.

This value of the logarithm of 2 is exact to fifteen places of figures, and errs only in the 16th, or last, figure, which should be an unit instead of a 2, the more accurate value of this logarithm (according to Mr. Abraham Sharp's computation) being 0.301,029,995,663,981,195,213,738,894,724,493.

160. As the solution of the foregoing problem consists of a great number of steps, which, for the ease of the reader, have been set forth distinctly and at considerable length, it will not be amiss to take a short review of all the foregoing processes, and of the several gradual approximations, to the value of x , (or of the index of the power of the binomial quantity $1 + \frac{24}{1000}$ in the equation $1 + \frac{24}{1000}^x = 10$) which have been obtained by means of them.

A review of the several steps of the foregoing resolution of the equation $1 + \frac{24}{1000}^x = 10$, and computation of the logarithm of 2.

161. The first step towards finding the value of the index x in the equation $1 + \frac{24}{1000}^x = 10$ was to expand the quantity $1 + \frac{24}{1000}^x$ into an infinite series by means of Sir Isaac Newton's binomial theorem, by which we obtained the equation $1 + x \times \frac{24}{1000} + \frac{x(x-1)}{2} \times \frac{24}{1000}^2 + \frac{x^3 - 3xx + 2x}{6} \times \frac{24}{1000}^3 + \frac{x^4 - 6x^3 + 11x^2 - 6x}{24} \times \frac{24}{1000}^4 + \frac{x^5 - 10x^4 + 35x^3 - 50x^2 + 24x}{120} \times \frac{24}{1000}^5 + \&c = 10$, and consequently (by subtracting 1 from both sides) the equation $x \times \frac{24}{1000} + \frac{x(x-1)}{2} \times \frac{24}{1000}^2 + \frac{x^3 - 3xx + 2x}{6} \times \frac{24}{1000}^3 + \frac{x^4 - 6x^3 + 11x^2 - 6x}{24} \times \frac{24}{1000}^4 + \frac{x^5 - 10x^4 + 35x^3 - 50x^2 + 24x}{120} \times \frac{24}{1000}^5 + \&c = 9$.

$$\frac{xx-x}{2} \times \left[\frac{24}{1000}\right]^2 + \frac{x^3-3xx+2x}{6} \times \left[\frac{24}{1000}\right]^3 + \frac{x^4-6x^3+11x^2-6x}{24} \times \left[\frac{24}{1000}\right]^4 + \frac{x^5-10x^4+35x^3-50xx+24x}{120} \times \left[\frac{24}{1000}\right]^5 + \&c = 9.$$

162. We then supposed, first, one term, then two terms, then three terms, and lastly, four terms, of the series which forms the left-hand side of this last equation, to be equal to the whole series, and consequently to the absolute term 9, and we resolved the several equations resulting from those suppositions.

By resolving the simple equation resulting from the first of these suppositions, to wit, the simple equation $x \times \frac{24}{1000} = 9$, we found x to be equal to 375; which was therefore our first approximation to the value of x in the original equation $1 + \left[\frac{24}{1000}\right]^x = 10$. This approximation is very wide of the true value of x , being more than three times as great.

From the second supposition, to wit, that the two terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \left[\frac{24}{1000}\right]^2$ were equal to the whole series, and consequently to 9, there resulted the quadratick equation $xx + 82.333,333, \&c \times x = 31,250$; by the resolution of which x appeared to be $= 140.34$; which was therefore our second approximation to the value of x in the original equation $1 + \left[\frac{24}{1000}\right]^x = 10$. This approximation is much nearer to the truth than the former, but yet is considerably too large.

From the third supposition, to wit, that the three terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \left[\frac{24}{1000}\right]^2 + \frac{x^3-3xx+2x}{6} \times \left[\frac{24}{1000}\right]^3$ were equal to the whole series, and consequently to 9, there resulted the cubick equation $x^3 + 122xx + 10,293.666,666, \&c \times x = 3,906,250$; by the resolution of which in a gross manner, by conjecturing x to be equal, first, to 100, and then (upon finding it to be greater than 100), to 110, we found it to be somewhat less than 110; which was therefore our third approximation to the value of x in the original equation $1 + \left[\frac{24}{1000}\right]^x = 10$.

And from the fourth supposition, to wit, that the four terms $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \left[\frac{24}{1000}\right]^2 + \frac{x^3-3xx+2x}{6} \times \left[\frac{24}{1000}\right]^3 + \frac{x^4-6x^3+11xx-6x}{24} \times \left[\frac{24}{1000}\right]^4$ are equal to the whole series, and consequently to 9, there resulted the biquadratick equation $x^4 + 160.666,666, \&c \times x^3 + 20,344.333,333, \&c \times xx + 1,715,605.111,111, \&c \times x = 651,041,666.666,666, \&c$; by the resolution of which in a gross manner, by a conjecture and trial, we found x to be somewhat greater than 100; which was therefore our fourth approximation to the value of x in the original equation $1 + \left[\frac{24}{1000}\right]^x = 10$.

163. Having

163. Having thus obtained the numbers 375, 140, 110, and 100 for our four first approximations to the value of x in the equation $1 + \frac{24}{1000}x = 10$, we observed that the difference between the second and third approximations, to wit, 140 and 110, was 30, and that the difference between the third and fourth approximations, to wit, 110 and 100, was only 10, or one third part of the preceding difference; and we were thereby led to conjecture that the difference between the fourth approximation 100 and the true value of x in the equation $1 + \frac{24}{1000}x = 10$ would probably be less than a third part of the last difference 10, or would be nearly equal to 3, and consequently that the true value of x in the equation $1 + \frac{24}{1000}x = 10$ would be nearly equal to $100 - 3$, or 97, or that 97 would be a fifth approximation to the said true value of x . And we then tried whether the said true value of x was greater or less than 97, by raising the binomial quantity $1 + \frac{24}{1000}$ to the 97th power by means of the binomial theorem; and we found the said 97th power of $1 + \frac{24}{1000}$ to be equal to 9.979,201,547,673,599,050, which is somewhat less than 10; whence it followed that 97 must be somewhat less than the true value of x in the equation $1 + \frac{24}{1000}x = 10$. We then multiplied the number 9.979,201,547,673,599,050 (which is equal to the 97th power of $1 + \frac{24}{1000}$) into $1 + \frac{24}{1000}$, or 1.024, and found the product to be $= 10.218,702,384,817,765,427,200$, which is greater than 10, and we therefore concluded that the 98th power of $1 + \frac{24}{1000}$ was greater than 10, and consequently that the true value of x in the equation $1 + \frac{24}{1000}x = 10$ was greater than 97, but less than 98.

164. We then put z for the unknown difference by which the true value of x in the equation $1 + \frac{24}{1000}x = 10$ exceeds 97, so that x was $= 97 + z$. And we thereby had $1 + \frac{24}{1000}^{97+z} (= 1 + \frac{24}{1000}^x) = 10$.

But $1 + \frac{24}{1000}^{97+z}$ is $= 1 + \frac{24}{1000}^{97} \times 1 + \frac{24}{1000}^z$; and $1 + \frac{24}{1000}^{97}$ had been found to be $= 9.979,201,547,673,599,050$, &c. Therefore $9.979,201,547,673,599,050 \times 1 + \frac{24}{1000}^z$ is $(= 1 + \frac{24}{1000}^{97} \times 1 + \frac{24}{1000}^z = 1 + \frac{24}{1000}^{97+z}) = 10$, and consequently $1 + \frac{24}{1000}^z$ is $(= \frac{10.000,000,000,000,000,000}{9.979,201,547,673,599,050, \&c.}) = 1.002,084,180,004,486,389$. And thus we obtained a new equation, in which the unknown index z of the power of the binomial

nomial quantity $1 + \frac{24}{1000}$ is less than an unit, instead of the original equation $1 + \frac{24}{1000}^x = 10$, in which the index x of the power of the same quantity is greater than 97. In consequence of this change of equations the subsequent approximations to the true value of x , or $97 + z$, became much swifter than they were before.

165. We then expanded the quantity $1 + \frac{24}{1000}^z$ into an infinite series by the binomial theorem, and thereby obtained the equation $1 + z \times \frac{24}{1000} - \frac{z-zz}{2} \times \frac{24}{1000}^2 + \frac{2z-3zz+z^3}{6} \times \frac{24}{1000}^3 - \frac{6z-11zz+6z^3-z^4}{24} \times \frac{24}{1000}^4 + \&c = 1.002, 084, 180, 004, 486, 389$, and (by subtracting 1 from both sides) the equation $z \times \frac{24}{1000} - \frac{z-zz}{2} \times \frac{24}{1000}^2 + \frac{2z-3zz+z^3}{6} \times \frac{24}{1000}^3 - \frac{6z-11zz+6z^3-z^4}{24} \times \frac{24}{1000}^4 + \&c = 0.002, 084, 180, 004, 486, 389, \&c$.

166. We then proceeded to find approximations to the value of z in this new equation in the same manner as we had before found approximations to the value of x in the former equation $x \times \frac{24}{1000} + \frac{xx-x}{2} \times \frac{24}{1000}^2 + \frac{x^3-3xx+2x}{6} \times \frac{24}{1000}^3 + \frac{x^4-6x^3+11x^2-6x}{24} \times \frac{24}{1000}^4 + \&c = 9$, by, first, supposing the first term, $z \times \frac{24}{1000}$, alone, of the series which forms the left-hand side of the equation, to be equal to the whole series, and consequently to the absolute term 0.002, 084, 180, 004, 486, 389, &c, and then supposing the two first terms, $z \times \frac{24}{1000} - \frac{z-zz}{2} \times \frac{24}{1000}^2$, of the same series to be equal to the same quantity, and by resolving the equations resulting from those suppositions.

From the first of those suppositions we had the simple equation $z \times 24 = 2.084, 180, 004, 486, 389, \&c$; by the resolution of which we had $z = 0.086, 840$, and consequently x , or $97 + z$, $= 97.086, 840$; of which number the four first figures, 97.08, are exact; the more accurate value of x in the original equation $1 + \frac{24}{1000}^x = 10$ being 97.087, 787, 353, 856, 001, 437, as I have found by dividing 1 by 0.010, 299, 956, 639, 811, 952, 13, which is the logarithm of the ratio of 1024 to 1000. And of these four figures, 97.08, which are exact, the two last, to wit, .08, were obtained by the resolution of the foregoing very easy simple equation.

This number, 97.086, 840, is therefore the sixth approximation to the true value of x in the original equation $1 + \frac{24}{1000}^x = 10$.

From the second supposition there resulted the quadratic equation $zz + 82.$

333,333,

333,333, &c $\times z = 7.236,736,126,688$; by the resolution of which we had $z = 0.087,801$, and consequently x , or $97 + z$, $= 97.087,801$, of which number the five first figures, 97.087 , are exact. Therefore $97.087,801$ was a seventh approximation to the value of x in the equation $1 + \frac{24}{1000}^x = 10$.

167. We then found the arithmetical mean between the two last values of z , to wit, $0.086,840$, and $0.087,801$, which was $0.087,320$; and we observed that this mean must be less than the truth, because the second value of z , to wit, $0.087,801$, must be much nearer to its true value than the first value of it, to wit, $0.086,840$. And therefore we conjectured that the true value of z (being greater than $0.087,320$, and, probably not much less than $0.087,801$) might be very nearly equal to $0.087,7$, and consequently that the true value of x , or $97 + z$, might be very nearly equal to $97.087,7$. And thus we obtained 97.0877 for an eighth approximation to the true value of x in the equation $1 + \frac{24}{1000}^x = 10$.

168. We then dropped all further consideration of the equation $z \times \frac{24}{1000} - \frac{z-zz}{2} \times \frac{24}{1000}^2 + \frac{2z-3zz+z^3}{6} \times \frac{24}{1000}^3 - \frac{6z-11zz+6z^3-z^4}{24} \times \frac{24}{1000}^4 + \&c = 0.002,084,180,004,486,389$, &c, and made a trial of the exactness of the last value of x obtained by the foregoing processes, to wit, 97.0877 , by raising the binomial quantity $1 + \frac{24}{1000}$ to the power of which 97.0877 is the index; which was done by the help of Sir Isaac Newton's binomial theorem. And we found that the said power of $1 + \frac{24}{1000}$ was $= 9.999,979,282,720,950,507,346$, which is very little less than 10 ; and we thence concluded that 97.0877 must be a very little less than the true value of x in the equation $1 + \frac{24}{1000}^x = 10$.

169. We then supposed x to be $= 97.0877 + y$, and consequently $1 + \frac{24}{1000}^{97.0877+y}$ to be $= 10$.

Then, since $1 + \frac{24}{1000}^{97.0877+y}$ is $= 1 + \frac{24}{1000}^{97.0877} \times 1 + \frac{24}{1000}^y$, we had $1 + \frac{24}{1000}^{97.0877} \times 1 + \frac{24}{1000}^y = 10$, and consequently (because $1 + \frac{24}{1000}^{97.0877}$ has been found to be $= 9.999,979,282,720,950,507,346$) $9.999,979,282,720,950,507,346 \times 1 + \frac{24}{1000}^y = 10$, and $1 + \frac{24}{1000}^y = \frac{10.000,000,000,000,000,000,000}{9.999,979,282,720,950,507,346} = 1.000,002,071,732,197,014$, &c.

170. Having thus obtained a third equation $1 + \frac{24}{1000}^y = 1.000,002,071,732,197,014$, &c, in which y is much smaller than z in the former equation,

we proceeded to expand $1 + \frac{24}{1000}y$ into an infinite series by means of Sir Isaac Newton's binomial theorem, and thereby obtained the equation $1 + y \times \frac{24}{1000}$
 $- \frac{\left[\frac{y-yy}{2}\right] \times \frac{24}{1000}}{2} + \frac{2y-3yy+y^3}{6} \times \frac{24}{1000} - \frac{\left[\frac{6y-11yy+6y^3-y^4}{24}\right] \times \frac{24}{1000}}{24}$
 $+ \frac{24y-50yy+35y^3-10y^4+y^5}{120} \times \frac{24}{1000} - \frac{\left[\frac{120y-274yy+235y^3-85y^4+15y^5-y^6}{720}\right] \times \frac{24}{1000}}{720}$
 $+ \&c = 1.000,002,071,732,197,014, \&c$, and (by subtracting 1 from both
 sides) the equation $y \times \frac{24}{1000} - \frac{\left[\frac{y-yy}{2}\right] \times \frac{24}{1000}}{2} + \frac{2y-3yy+y^3}{6} \times \frac{24}{1000}$
 $- \frac{\left[\frac{6y-11yy+6y^3-y^4}{24}\right] \times \frac{24}{1000}}{24} + \frac{24y-50yy+35y^3-10y^4+y^5}{120} \times \frac{24}{1000}$
 $- \frac{\left[\frac{120y-274yy+235y^3-85y^4+15y^5-y^6}{720}\right] \times \frac{24}{1000}}{720} + \&c = 0.000,002,071,732,197,$
 $014, \&c.$

171. We then proceeded to approximate to the value of y in this equation, by first supposing the first term, $y \times \frac{24}{1000}$, alone, of the series $y \times \frac{24}{1000} - \frac{\left[\frac{y-yy}{2}\right] \times \frac{24}{1000}}{2} + \&c$ (which forms the left-hand side of this equation), to be equal to the whole series, and consequently to the absolute term 0.000,002,071,732,197,014, &c; and, secondly, by supposing the two first terms, $y \times \frac{24}{1000} - \frac{\left[\frac{y-yy}{2}\right] \times \frac{24}{1000}}{2}$, of the said series to be equal to the same quantity, and resolving the equations resulting from these suppositions.

From the first of these suppositions we had the simple equation $y \times 24 = 0.002,071,732,197,014, \&c$; by the resolution of which we had $y = 0.000,086,322,1$, and consequently $x (= 97.0877 + y) = 97.087,786,322,1$; of which number the first seven figures 97.087,78 are exact, the more accurate value of x being (as we have before observed) 97.087,787,353,856,001,437.

Therefore the number 97.087,786,322,1 is a ninth approximation to the true value of x in the equation $1 + \frac{24}{1000}x = 10$.

And from the second of these suppositions there resulted the quadratick equation $yy + 82.333,333, \&c \times y = 0.007,193,514,572$; by the resolution of which we had $y = 0.000,087,371$, and consequently $x (= 97.0877 + y) = 97.087,787,371$; of which number the first nine figures, 97.087,787,3, are exact.

This number 97.087,787,371, is therefore a tenth approximation to the true value of x in the equation $1 + \frac{24}{1000}x = 10$.

172. We then, in order to obtain the value of y to a still greater degree of exactness, had recourse to a different method of resolving the equation $y \times \frac{24}{1000}$

$$- \frac{y-yy}{2} \times \frac{24}{1000}^2 + \frac{2y-3yy+y^3}{6} \times \frac{24}{1000}^3 - \frac{6y-11yy+6y^3-y^4}{24} \times \frac{24}{1000}^4 + \&c =$$

$$0.000,002,071,732,197,014, \&c;$$
 which was grounded on the omission of all the members in each term of the series that involved either yy , or y^3 , or y^4 , or any other power of y , except the simple power, or y itself: by which means the said equation was changed into the following simple equation, to wit, $y \times \frac{24}{1000}$

$$- \frac{y}{2} \times \frac{24}{1000}^2 + \frac{2y}{6} \times \frac{24}{1000}^3 - \frac{6y}{24} \times \frac{24}{1000}^4 + \frac{24y}{120} \times \frac{24}{1000}^5 - \frac{120y}{720} \times \frac{24}{1000}^6$$

$$+ \&c = 0.000,002,071,732,197,014, \&c,$$
 or $y \times \frac{24}{1000} - \frac{y}{2} \times \frac{24}{1000}^2 + \frac{y}{3} \times \frac{24}{1000}^3 - \frac{y}{4} \times \frac{24}{1000}^4 + \frac{y}{5} \times \frac{24}{1000}^5 - \frac{y}{6} \times \frac{24}{1000}^6 + \&c = 0.000,002,071,732,197,014, \&c,$
 or $y \times$ the infinite series $1 \times \frac{24}{1000} - \frac{1}{2} \times \frac{24}{1000}^2 + \frac{1}{3} \times \frac{24}{1000}^3 - \frac{1}{4} \times \frac{24}{1000}^4 + \frac{1}{5} \times \frac{24}{1000}^5 - \frac{1}{6} \times \frac{24}{1000}^6 + \&c = 0.000,002,071,732,197,014, \&c,$
 or (if we put A for the first term $1 \times \frac{24}{1000}$ of this series, and B for its second term $\frac{1}{2} \times \frac{24}{1000}^2$, and $C, D, E, F, \&c$ for its third, fourth, fifth, sixth, and other following terms, respectively), $y \times$ the series $1 \times \frac{24}{1000} - \frac{1}{2} A \times \frac{24}{1000} + \frac{2}{3} B \times \frac{24}{1000} - \frac{3}{4} C \times \frac{24}{1000} + \frac{4}{5} D \times \frac{24}{1000} - \frac{5}{6} E \times \frac{24}{1000} + \frac{6}{7} F \times \frac{24}{1000} - \frac{7}{8} G \times \frac{24}{1000} + \frac{8}{9} H \times \frac{24}{1000} - \frac{9}{10} I \times \frac{24}{1000} + \frac{10}{11} K \times \frac{24}{1000} - \frac{11}{12} L \times \frac{24}{1000} + \&c = 0.000,002,071,732,197,014, \&c.$
 We then computed the value of the said infinite series $1 \times \frac{24}{1000} - \frac{1}{2} A \times \frac{24}{1000} + \frac{2}{3} B \times \frac{24}{1000} - \frac{3}{4} C \times \frac{24}{1000} + \frac{4}{5} D \times \frac{24}{1000} - \frac{5}{6} E \times \frac{24}{1000} + \&c$, and found it to be $= 0.023,716,527,561,316,042,234$; which gave us the simple equation $y \times 0.023,716,527,561,316,042,234 = 0.000,002,071,732,197,014, \&c$, by the resolution of which we had $y (= \frac{0.000,002,071,732,197,014}{0.023,716,527,561,316,042,234}) = 0.000,087,353,943,011$, and consequently $x (= 97.0877 + y) = 97.087,787,353,943,011$; of which number the first eleven figures, $97.087,787,353$, are exact, the more accurate value of x in the equation $1 + \frac{24}{1000}^x = 10$ being (as we have before observed) $97.087,787,353,856,001,437$.

This number $97.087,787,353,943,011$, is therefore the eleventh approximation to the true value of x in the equation $1 + \frac{24}{1000}^x = 10$.

173. And, lastly, to obtain the value of y to a still greater degree of exactness, we retained in art. 158 the five first terms of the infinite series set down in

art. 157 and 170, that involved the square of y , to wit, the five terms $\frac{yy}{2} \times \left[\frac{24}{1000} \right]^2 - \frac{3yy}{6} \times \left[\frac{24}{1000} \right]^3 + \frac{11yy}{24} \times \left[\frac{24}{1000} \right]^4 - \frac{50yy}{120} \times \left[\frac{24}{1000} \right]^5 + \frac{274yy}{720} \times \left[\frac{24}{1000} \right]^6$, which are equal to $yy \times$ the series $\frac{1}{2} \times \left[\frac{24}{1000} \right]^2 - \frac{3}{6} \times \left[\frac{24}{1000} \right]^3 + \frac{11}{24} \times \left[\frac{24}{1000} \right]^4 - \frac{50}{120} \times \left[\frac{24}{1000} \right]^5 + \frac{274}{720} \times \left[\frac{24}{1000} \right]^6 = yy \times 0.000,281,236,758,360,883,200$. And hence we obtained the quadratick equation $0.023,716,527,561,316,042,234 \times y + 0.000,281,236,758,360,883,200 \times yy = 0.000,002,071,732,197,014$, &c; which (being resolved by approximation by substituting, instead of y , in the quantity $0.000,281,236,758,360,883,200 \times yy$ the value of y before obtained by the resolution of the simple equation, to wit, $0.000,087,353,943,011$) gave us $y = 0.000,087,353,852,524$, and consequently $x (= 97.0877 + y) 97.087,787,353,852,524$; of which number the first thirteen figures, $97.087,787,353,85$, are exact.

This number $97.087,787,353,852,524$ is therefore the twelfth approximation to the true value of x in the equation $1 + \frac{24}{1000} x^x = 10$.

174. We then divided 1 by this last, or twelfth, near value of x , to wit, $97.087,787,353,852,524$, in order to obtain the value of $\frac{1}{x}$, or the logarithm of the ratio of $1 + \frac{24}{1000}$ to 1 in Briggs's system; and we found the quotient to be $= 0.010,299,956,639,812$. And hence it followed that the logarithm of 2, or of the ratio of 2 to 1 (being $= \frac{1}{10} \log. \frac{1024}{1} = \frac{1}{10} \log. \frac{1024}{1000} + \frac{1}{10} \log. \frac{1000}{1} = \frac{1}{10} \log. \frac{1024}{1000} + \frac{1}{10} \times 3 = \frac{1}{10} \log. \frac{1024}{1000} + 0.3 = \frac{1}{10} \log. \frac{1.024}{1} + 0.3 = \frac{1}{10} \log. 1 + \frac{24}{1000} + 0.3 = \frac{1}{10} \times 0.010,299,956,639,812 + 0.3$) would be $= 0.3 + 0.001,029,995,663,981,2$, or $0.301,029,995,663,981,2$; which is exact to 15 places of figures, the more accurate value of that logarithm (according to Mr. Abraham Sharp's computations) being $0.301,029,995,663,981,195,213,738,894,724,493$.

End of the review of the several steps of the foregoing resolution of the

equation $1 + \frac{24}{1000} x^x = 10$, and computation of the logarithm of 2.

A SCHOLIUM.

175. The foregoing method of computing Briggs's logarithm of 2 is certainly somewhat laborious, but much less so than the methods used for the same purpose by Mr. Briggs himself, which required many very long extractions of the square.

square-root. For the difficulty of performing the operations that were necessary in those methods was so great, that (according to what Mr. Euclid Speidall informs us (see above, page 73) he had been told) *it was the work of eight persons for a whole year to compute the logarithm of 2 by those methods exact to 15 places of figures*, or to the degree of exactness to which it has been obtained in the foregoing articles. This assertion of Mr. Speidall seems, I confess, a little strange. Yet, as he published his tract on logarithms (which has been printed above in the former part of this volume) so long ago as in the year 1688, it seems probable that in his youth (perhaps, about the year 1660) he might have conversed with some old men who had been acquainted with Mr. Briggs himself, who published his *Arithmetica Logarithmica* in the year 1624, which was less than 40 years before that time; and this seems the more likely to have been the case, as his father, Mr. John Speidall, was an eminent mathematician, and had very much cultivated the, at that time, new invention of logarithms; which must have given both him and his son an opportunity of hearing many remarkable particulars relating to them.

We may further observe that the foregoing method of computing logarithms by the help of the binomial theorem, and Mr. Briggs's methods of computing them by repeated extractions of the square-root, are equally founded on the pure and genuine principles of arithmetick, without any reference to the hyperbola, or the logarithmick curve, or any other geometrical figure, and also without any recourse to the doctrine of infinitesimals, or of fluxions, or of the limits of ratios, or in general, of the arithmetick of infinites in any of its modifications; which is, in Dr. Halley's opinion, the proper way of treating this subject, and the way in which he boasts (though without being sufficiently authorized in his pretensions), that he himself has treated it in the foregoing discourse reprinted above in this volume in pages 84, 85, 86, 87, 88, 90, and 91.

CONCLUSION.

176. I have now completed the investigation of the famous binomial theorem in all the cases of fractional powers; which was the proposed subject of this discourse. This, however, is but a part of that important and most comprehensive proposition. For it is found to be true likewise in the cases of negative powers, both integral and fractional, that is, in the cases of $\overline{1+x}^{-m}$,

and $\overline{1+x}^{-\frac{m}{n}}$, or of $\frac{1}{1+x^m}$ and $\frac{1}{1+x^{\frac{m}{n}}}$. But, as the knowledge of these

cases is not necessary to the understanding any of the foregoing methods of computing logarithms, I shall not on this occasion enter into any inquiries concerning them. And therefore I here conclude what I meant to offer to the reader's consideration concerning the binomial theorem *properly so called*. But, as the theorem concerning the fractional powers of a residual quantity, such as

$1-x$,

$1 - x$, is very nearly related to the foregoing theorem concerning the fractional powers of the binomial quantity $1 + x$, inasmuch that it is usually considered as a branch of it;—and, as the said *residual theorem*, in the first case of it, or the case of the n th root of the residual quantity $1 - x$, is made use of (as well as the binomial theorem) in the investigations of some of the foregoing methods of computing logarithms;—I shall now proceed to shew how we may derive from the theorems above demonstrated concerning the roots, and the powers of the roots, of the binomial quantity $1 + x$, the like theorems concerning the roots, and the powers of the roots, of the residual quantity $1 - x$. But this shall be the subject of a separate tract. And therefore I here conclude this discourse concerning the roots, and the powers of the roots, of the binomial quantity $1 + x$.

End of the Discourse concerning the Binomial Theorem in the case of Fractional Powers.

A DISCOURSE

CONCERNING

SIR ISAAC NEWTON'S RESIDUAL THEOREM,

OR THEOREM FOR RAISING THE POWERS OF THE RESIDUAL
QUANTITY $1 - x$, IN THE CASE OF FRACTIONAL POWERS, OR
POWERS OF WHICH THE INDEXES ARE FRACTIONS.

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CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

ART. I. **I**N the foregoing discourse we investigated the serieses that were equal to the n th root of the binomial quantity $1 + x$ and to the m th power of its n th root. In the present discourse we are to investigate the serieses that are equal to the n th root of the residual quantity $1 - x$, and to the m th power of its n th root. Now these serieses may be investigated by the same methods which were employed in the foregoing discourse to investigate the serieses which are equal to $\sqrt[n]{1 + x}$ and $\sqrt[n]{1 + x}^m$: but they may likewise be derived from those former serieses (obtained in the foregoing discourse), by a just and legitimate train of reasoning, with much less trouble than would be necessary to the discovery of them by a new application of all the methods of investigation used in the foregoing tract. And therefore I shall, for brevity's sake, have recourse to this derivative method of obtaining them, rather than to the methods employed in the foregoing discourse. And, first, I shall consider the n th root of the residual quantity $1 - x$, and endeavour to shew that it is equal to an infinite series consisting of the very same terms as the series which is equal to the n th root of the binomial quantity $1 + x$, but with the sign $-$ prefixed to all the terms after the first term 1 , instead of only the third, and fifth, and seventh, and other following odd terms of the series, as in the series which is equal to

$$\sqrt[n]{1 + x}, \text{ or } \sqrt[n]{1 + x}^{\frac{1}{n}}.$$

2. Now it has been shewn in the foregoing tract, that, if n be any whole number whatsoever, the quantity $\sqrt[n]{1 + x}^{\frac{1}{n}}$, or $\sqrt[n]{1 + x}$, will be equal to the

infinite series $1 + \frac{1}{n} Ax - \sqrt[n]{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \sqrt[n]{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \sqrt[n]{\frac{5n-1}{6n}} Fx^6 + \&c \text{ ad infinitum}$. We will therefore now proceed

to shew that the quantity $\sqrt[n]{1-x}$, or $\overline{1-x}^{\frac{1}{n}}$, will be equal to the infinite series $1 - \frac{1}{n} Ax - \sqrt[n]{\frac{n-1}{2n}} Bx^2 - \sqrt[n]{\frac{2n-1}{3n}} Cx^3 - \sqrt[n]{\frac{3n-1}{4n}} Dx^4 - \sqrt[n]{\frac{4n-1}{5n}} Ex^5 - \sqrt[n]{\frac{5n-1}{6n}} Fx^6 - \&c \text{ ad infinitum}$, which consists of the very same terms as the

former series (which is equal to $\overline{1+x}^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$), but with the sign $-$ prefixed to every term after the first term, instead of every other term. This may be shewn in the manner following.

3. Since the series $1 + \frac{1}{n} Ax - \sqrt[n]{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \sqrt[n]{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$ is equal to $\overline{1+x}^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, it follows that, if we raise the said series to the n th power, or multiply it $n-1$ times into itself, the product of these multiplications will be equal to $1+x$; that is, the product of the series $1 + Bx - Cxx + Dx^3 - Ex^4 + Fx^5 - \&c \text{ ad infinitum}$ multiplied $n-1$ times into itself (upon the supposition that the co-efficient B is $= \frac{1}{n} A = \frac{1}{n} \times 1$, or $\frac{1}{n}$, and that the co-efficient C is $= \frac{n-1}{2n} B$, and that the co-efficients $D, E, F, \&c$, are equal, respectively, to $\frac{2n-1}{3n} C, \frac{3n-1}{4n} D, \frac{4n-1}{5n} E, \&c$) will be equal to $1+x$; and consequently the co-efficient of x will be equal to 1, and the compound quantities which will be the co-efficients of x^2, x^3, x^4, x^5 , and of all the following powers of x in the terms of the said product, will be, each of them, equal to 0, or will consist of several members, of which some will be marked with the sign $+$, and others with the sign $-$, and which will be of such magnitudes that the sum of the terms, or members, which are marked with the sign $-$ will be equal to the sum of the terms, or members, which are marked with the sign $+$. Some of these multiplications will be as follow.

$$\begin{array}{rcl}
 1 + Bx - Cx^2 + Dx^3 - \&c & = & \overline{1+x}^{\frac{1}{n}}. \\
 1 + Bx - Cx^2 + Dx^3 - \&c & = & \overline{1+x}^{\frac{1}{n}}. \\
 \hline
 1 + Bx - Cx^2 + Dx^3 - \&c & & \\
 + Bx + B^2x^2 - BCx^3 + \&c & & \\
 - Cx^2 - BCx^3 + \&c & & \\
 + Dx^3 + \&c & & \\
 \hline
 1 + 2Bx - 2Cx^2 + 2Dx^3 \&c & & \\
 + B^2x^2 - 2BCx^3 + \&c & & \\
 \hline
 & & \left. \vphantom{\begin{array}{l} 1 + 2Bx - 2Cx^2 + 2Dx^3 \&c \\ + B^2x^2 - 2BCx^3 + \&c \end{array}} \right\} = \overline{1+x}^{\frac{2}{n}}.
 \end{array}$$

$1 + Bx$

$$\begin{array}{r}
 1 + Bx - Cx^2 + Dx^3 - \&c \\
 \hline
 1 + 2Bx - 2Cx^2 + 2Dx^3 \quad \&c \\
 \quad + B^2x^2 - 2BCx^3 \quad \&c \\
 \quad + Bx + 2B^2x^2 - 2BCx^3 + \&c \\
 \quad \quad + B^3x^3 - \&c \\
 \quad - Cx^2 - 2BCx^3 + \&c \\
 \quad \quad + Dx^3 + \&c
 \end{array}
 = \overline{1+x}^{\frac{1}{n}}.$$

$$\begin{array}{r}
 1 + 3Bx - 3Cx^2 + 3Dx^3 \quad \&c \\
 \quad + 3B^2x^2 - 6BCx^3 \quad \&c \\
 \quad \quad + B^3x^3 \quad \&c
 \end{array}
 \left. \vphantom{\begin{array}{r} 1 + 3Bx - 3Cx^2 + 3Dx^3 \\ \quad + 3B^2x^2 - 6BCx^3 \\ \quad \quad + B^3x^3 \end{array}} \right\} = \overline{1+x}^{\frac{3}{n}}.$$

$$\begin{array}{r}
 1 + Bx - Cx^2 + Dx^3 - \&c \\
 \hline
 1 + 3Bx - 3Cx^2 + 3Dx^3 \quad \&c \\
 \quad + 3B^2x^2 - 6BCx^3 \quad \&c \\
 \quad \quad + B^3x^3 \quad \&c \\
 \quad + Bx + 3B^2x^2 - 3BCx^3 + \&c \\
 \quad \quad + 3B^3x^3 - \&c \\
 \quad - Cx^2 - 3BCx^3 \quad \&c \\
 \quad \quad + Dx^3 + \&c
 \end{array}
 = \overline{1+x}^{\frac{1}{n}}.$$

$$\begin{array}{r}
 1 + 4Bx - 4Cx^2 + 4Dx^3 \quad \&c \\
 \quad + 6B^2x^2 - 12BCx^3 \quad \&c \\
 \quad \quad + 4B^3x^3 \quad \&c
 \end{array}
 \left. \vphantom{\begin{array}{r} 1 + 4Bx - 4Cx^2 + 4Dx^3 \\ \quad + 6B^2x^2 - 12BCx^3 \\ \quad \quad + 4B^3x^3 \end{array}} \right\} = \overline{1+x}^{\frac{4}{n}}.$$

$$\begin{array}{r}
 1 + Bx - Cx^2 + Dx^3 - \&c \\
 \hline
 1 + 4Bx - 4Cx^2 + 4Dx^3 \quad \&c \\
 \quad + 6B^2x^2 - 12BCx^3 \quad \&c \\
 \quad \quad + 4B^3x^3 \quad \&c \\
 \quad + Bx + 4B^2x^2 - 4BCx^3 + \&c \\
 \quad \quad + 6B^3x^3 - \&c \\
 \quad - Cx^2 - 4BCx^3 + \&c \\
 \quad \quad + Dx^3 + \&c
 \end{array}
 = \overline{1+x}^{\frac{1}{n}}.$$

$$\begin{array}{r}
 1 + 5Bx - 5Cx^2 + 5Dx^3 \quad \&c \\
 \quad + 10B^2x^2 - 20BCx^3 \quad \&c \\
 \quad \quad + 10B^3x^3 \quad \&c
 \end{array}
 \left. \vphantom{\begin{array}{r} 1 + 5Bx - 5Cx^2 + 5Dx^3 \\ \quad + 10B^2x^2 - 20BCx^3 \\ \quad \quad + 10B^3x^3 \end{array}} \right\} = \overline{1+x}^{\frac{5}{n}}.$$

4. By these multiplications it appears that the square of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (carried only to the third power of x) is equal to the compound series

$$\begin{array}{r}
 1 + 2Bx - 2Cx^2 + 2Dx^3 \quad \&c \\
 \quad + B^2x^2 - 2BCx^3 \quad \&c;
 \end{array}$$

and that its cube is equal to the compound series

$$\begin{array}{r}
 1 + 3Bx - 3Cx^2 + 3Dx^3 - \&c \\
 \quad + 3B^2x^2 - 6BCx^3 \quad \&c \\
 \quad \quad + B^3x^3 \quad \&c;
 \end{array}$$

2 Y 2

and

and that its fourth power is equal to the compound series

$$\begin{aligned} 1 + 4 Bx - 4 Cx^2 + 4 Dx^3 & \&c \\ + 6 B^2x^2 - 12 BCx^3 & \&c \\ + 4 B^3x^3 & \&c; \end{aligned}$$

and that its fifth power is equal to the compound series

$$\begin{aligned} 1 + 5 Bx - 5 Cx^2 + 5 Dx^3 & \&c \\ + 10 B^2x^2 - 20 BCx^3 & \&c \\ + 10 B^3x^3 & \&c. \end{aligned}$$

5. Now, when n is $= 2$, the first of these compound serieses (which is equal to the square of the series $1 + Bx - Cx^2 + Dx^3 - \&c$), to wit, the series

$$\begin{aligned} 1 + 2 Bx - 2 Cx^2 + 2 Dx^3 & \&c \\ + B^2x^2 - 2 BCx^3 & \&c, \end{aligned}$$

must be equal to $1 + x$, and consequently $2 B$ (the co-efficient of x) must be $= 1$, and $B^2 - 2 C$ (the compound co-efficient of x^2) must be $= 0$, and $2 D - 2 BC$ (the compound co-efficient of x^3) must also be $= 0$, and every following co-efficient of one of the powers of x in the said series must, in like manner, be $= 0$, if the co-efficients of the terms of the series $1 + Bx - Cx^2 + Dx^3 - \&c$, or $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \&c$, have been

rightly assigned. For otherwise the said compound series cannot be equal to $1 + x$, as it ought to be upon the present supposition that n is $= 2$, because

upon this supposition the series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \&c$

(which is universally equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, will be $= \sqrt[n]{1+x}$, or, $\sqrt[n]{1+x}$, or the square-root of $1+x$, and consequently the square of the said series must be equal to $1+x$.

And accordingly we shall find that, if n be supposed to be $= 2$, the co-efficient $2 B$, of the second term Bx , of this compound series, will be $= 1$, and $B^2 - 2 C$ and $2 D - 2 BC$ (the compound co-efficients of the two following powers of x , to wit, x^2 and x^3 in the said series) will each of them be equal to 0 . For, if n is $= 2$, we shall have $B (= \frac{1}{n} \times A = \frac{1}{2} \times A = \frac{1}{2} \times 1) = \frac{1}{2}$, and $C (= \frac{n-1}{2n} B = \frac{2-1}{2 \times 2} \times \frac{1}{2} = \frac{1}{4} \times \frac{1}{2}) = \frac{1}{8}$, and $D (= \frac{2n-1}{3n} C = \frac{2 \times 2 - 1}{3 \times 2} \times \frac{1}{8} = \frac{4-1}{6} \times \frac{1}{8} = \frac{3}{6} \times \frac{1}{8} = \frac{1}{2} \times \frac{1}{8}) = \frac{1}{16}$; and consequently $2 B (= 2 \times \frac{1}{2}) = 1$, and $B^2 - 2 C (= \frac{1}{4} - 2 \times \frac{1}{8} = \frac{1}{4} - \frac{2}{8} = \frac{1}{4} - \frac{1}{4}) = 0$, and $2 D - 2 BC (= 2 \times \frac{1}{16} - 2 \times \frac{1}{2} \times \frac{1}{8} = \frac{2}{16} - \frac{2}{16}) = 0$.

6. In like manner, when n is $= 3$, the second of these compound serieses (which is equal to the cube of the series $1 + Bx - Cx^2 + Dx^3 - \&c$), to wit, the compound series

$$1 + 3 Bx$$

$$1 + 3Bx - 3Cx^2 + 3Dx^3 \&c \\ + 3B^2x^2 - 6BCx^3 \&c \\ + B^3x^3 \&c$$

must be $=$ to $1 + x$, and consequently $3B$ must be $= 1$, and $3B^2 - 3C$ must be $= 0$, and $3D - 6BC + B^3$ must be $= 0$, or $3C$ must be $= 3B^2$, and $6BC$ must be $= B^3 + 3D$.

And so we shall find these several quantities to be. For when n is $= 3$, we shall have $B (= \frac{1}{n} A = \frac{1}{3} \times A = \frac{1}{3} \times 1) = \frac{1}{3}$, and $C (= \frac{n-1}{2n} B = \frac{3-1}{2 \times 3} \times \frac{1}{3} = \frac{2}{2 \times 3} \times \frac{1}{3} = \frac{1}{3} \times \frac{1}{3}) = \frac{1}{9}$, and $D (= \frac{2n-1}{3n} C = \frac{2 \times 3-1}{3 \times 3} \times \frac{1}{9} = \frac{6-1}{9} \times \frac{1}{9} = \frac{5}{9} \times \frac{1}{9}) = \frac{5}{81}$; and consequently $3B (= 3 \times \frac{1}{3}) = 1$, and $3B^2 - 3C (= 3 \times \frac{1}{9} - 3 \times \frac{1}{9}) = 0$, and $3D - 6BC + B^3 (= 3 \times \frac{5}{81} - 6 \times \frac{1}{3} \times \frac{1}{9} + \frac{1}{27} = \frac{5}{27} - \frac{6}{27} + \frac{1}{27} = \frac{6}{27} - \frac{6}{27}) = 0$.

7. And, when n is $= 4$, the third of the foregoing compound serieses (which is equal to the fourth power of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$), to wit, the compound series

$$1 + 4Bx - 4Cx^2 + 4Dx^3 \&c \\ + 6B^2x^2 - 12BCx^3 \&c \\ + 4B^3x^3 \&c,$$

must be equal to $1 + x$; and consequently $4B$ must be $= 1$, and $6B^2 - 4C$ must be $= 0$, and $4D - 12BC + 4B^3$ must likewise be $= 0$.

And so we shall find these quantities to be. For, if n is $= 4$, we shall have $B (= \frac{1}{n}) = \frac{1}{4}$, and $C (= \frac{n-1}{2n} B = \frac{4-1}{2 \times 4} \times \frac{1}{4} = \frac{3}{8} \times \frac{1}{4}) = \frac{3}{32}$, and $D (= \frac{2n-1}{3n} C = \frac{2 \times 4-1}{3 \times 4} \times \frac{3}{32} = \frac{8-1}{3 \times 4} \times \frac{3}{32} = \frac{7}{4} \times \frac{1}{32}) = \frac{7}{128}$; and consequently $4B (= 4 \times \frac{1}{4}) = 1$, and $6B^2 - 4C (= 6 \times \frac{1}{16} - 4 \times \frac{3}{32} = \frac{3}{8} - \frac{3}{8}) = 0$, and $4D - 12BC + 4B^3 (= 4 \times \frac{7}{128} - 12 \times \frac{1}{4} \times \frac{3}{32} + 4 \times \frac{1}{64} = \frac{7}{32} - \frac{9}{32} + \frac{2}{32} = \frac{9}{32} - \frac{9}{32}) = 0$.

8. And, when n is $= 5$, the last of the foregoing compound serieses (which is equal to the fifth power of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$), to wit, the compound series

$$1 + 5Bx - 5Cx^2 + 5Dx^3 \&c \\ + 10B^2x^2 - 20BCx^3 \&c \\ + 10B^3x^3 \&c,$$

must be equal to $1 + x$, and consequently $5B$ must be $= 1$, and $10B^2 - 5C$ must be $= 0$, and $5D - 20BC + 10B^3$ must likewise be $= 0$.

And

And so we shall find these quantities to be. For, if n is $= 5$, we shall have $B (= \frac{1}{n} A = \frac{1}{5} \times A = \frac{1}{5} \times 1) = \frac{1}{5}$, and $C (= \frac{n-1}{2n} B = \frac{5-1}{2 \times 5} \times \frac{1}{5} = \frac{4}{10} \times \frac{1}{5} = \frac{2}{5} \times \frac{1}{5}) = \frac{2}{25}$, and $D (= \frac{2n-1}{3n} \times C = \frac{2 \times 5-1}{3 \times 5} \times \frac{2}{25} = \frac{10-1}{15} \times \frac{2}{25} = \frac{9}{15} \times \frac{2}{25} = \frac{3}{5} \times \frac{2}{25}) = \frac{6}{125}$; and consequently $5 B (= 5 \times \frac{1}{5}) = 1$, and $10 B^2 - 5 C (= 10 \times \frac{1}{5} \times \frac{1}{5} - 5 \times \frac{2}{25} = \frac{10}{25} - \frac{10}{25}) = 0$, and $5 D - 20 BC + 10 B^3 (= 5 \times \frac{6}{125} - 20 \times \frac{1}{5} \times \frac{2}{25} + 10 \times \frac{1}{125} = \frac{30}{125} - \frac{40}{125} + \frac{10}{125} = \frac{40}{125} - \frac{40}{125}) = 0$.

9. Thus we see in all these instances, that, when the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (which is equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$) is raised to the n th power, or multiplied into itself $n-1$ times, the co-efficient of x in the compound series obtained by such multiplication is always equal to 1, and the co-efficients of x^2 and x^3 (which are compound quantities, or quantities consisting of more than 1 term) are, each of them, equal to 0, the term, or terms, which are marked with the sign $-$, or subtracted from the other terms, being equal to the term, or terms, which are marked with the sign $+$, and from which they are to be subtracted. And the same thing must take place in the co-efficients of all the following terms of the compound series obtained by such multiplication. For otherwise the said series could not be equal to $1+x$. We may therefore lay it down universally, as an undoubted truth, resulting from the nature of powers and roots, “that, if the series which is equal to $\sqrt[n]{1+x}$, or “ $\sqrt[n]{1+x}$, to wit, the series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, or the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (in which latter series the co-efficients $B, C, D, E, F, \&c$, are respectively equal to the former co-efficients $\frac{1}{n} A$, or $\frac{1}{n}$, and $\frac{n-1}{2n} B$, “ $\frac{2n-1}{3n} C$, $\frac{3n-1}{4n} D$, $\frac{4n-1}{5n} E$, &c), be raised to the n th power, or multiplied into “itself $n-1$ times, the co-efficient of x in the compound series produced by such “multiplication will be $= 1$, and the co-efficients of x^2, x^3, x^4, x^5 , and all the “following powers of x in the said series, will be, each of them, equal to 0; the “said co-efficients being compound quantities consisting of simple terms, or mem- “bers, connected with each other, by the signs $+$ and $-$, and the sum of the “members of each of the said co-efficients that are marked with the sign $-$, or “subtracted from the other members of them, which are marked with the sign $+$, “being always equal to the sum of the said other members from which they are “subtracted.” This fundamental proposition being well understood, we may derive from it a proof of the proposition asserted above in art. 2, concerning the quantity

quantity $\sqrt[n]{1+x}$, or $\sqrt[n]{1-x}$, or the n th root of the residual quantity $1-x$, to wit, "that it will be equal to the series $1 - \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 - \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 - \frac{4n-1}{5n} Ex^5 - \&c$, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$; which consists of the very same terms as the former series which is equal to the n th root of the binomial quantity $1+x$, but with the sign $-$ prefixed to every term after the first term 1 , instead of every other term."

10. To render the proof of this proposition as easy as possible, it will be convenient to multiply each of the two serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, into itself, and then to compare together the squares, or products thence arising. These multiplications will be as follow.

$1 +$	$Bx -$	$Cx^2 +$	$Dx^3 -$	$Ex^4 +$	$Fx^5 -$	$\&c$
$1 +$	$Bx -$	$Cx^2 +$	$Dx^3 -$	$Ex^4 +$	$Fx^5 -$	$\&c$
$1 +$	$Bx -$	$Cx^2 +$	$Dx^3 -$	$Ex^4 +$	$Fx^5 -$	$\&c$
	$+ Bx +$	$B^2x^2 -$	$BCx^3 +$	$BDx^4 -$	$BEx^5 +$	$\&c$
		$- Cx^2 -$	$BCx^3 +$	$C^2x^4 -$	$CDx^5 +$	$\&c$
			$+ Dx^3 +$	$BDx^4 -$	$CDx^5 +$	$\&c$
				$- Ex^4 -$	$BEx^5 +$	$\&c$
					$+ Fx^5 -$	$\&c$
$1 +$	$2 Bx -$	$2 Cx^2 +$	$2 Dx^3 -$	$2 Ex^4 +$	$2 Fx^5 -$	$\&c$
		$+ B^2x^2 -$	$2 BCx^3 +$	$2 BDx^4 -$	$2 BEx^5 -$	$\&c$
			$+ C^2x^4 -$	$2 CDx^5$	$\&c$	$\&c$
$1 -$	$Bx -$	$Cx^2 -$	$Dx^3 -$	$Ex^4 -$	$Fx^5 -$	$\&c$
$1 -$	$Bx -$	$Cx^2 -$	$Dx^3 -$	$Ex^4 -$	$Fx^5 -$	$\&c$
$1 -$	$Bx -$	$Cx^2 -$	$Dx^3 -$	$Ex^4 -$	$Fx^5 -$	$\&c$
	$- Bx +$	$B^2x^2 +$	$BCx^3 +$	$BDx^4 +$	$BEx^5 +$	$\&c$
		$- Cx^2 +$	$BCx^3 +$	$C^2x^4 +$	$CDx^5 +$	$\&c$
			$- Dx^3 +$	$BDx^4 +$	$CDx^5 +$	$\&c$
				$- Ex^4 +$	$BEx^5 +$	$\&c$
					$- Fx^5 +$	$\&c$
$1 -$	$2 Bx -$	$2 Cx^2 -$	$2 Dx^3 -$	$2 Ex^4 -$	$2 Fx^5 -$	$\&c$
		$+ B^2x^2 +$	$2 BCx^3 +$	$2 BDx^4 +$	$2 BEx^5$	$\&c$
			$+ C^2x^4 +$	$2 CDx^5$	$\&c$	$\&c$

11. Now, if we compare this last compound series (which is equal to the square of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$) with the former compound series (which is equal to the square of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$), we shall find the following observations to be true concerning them.

In

In the first place, all the terms of the second of these compound serieses are the very same with the corresponding terms of the former of them, but are differently connected with each other by the signs $+$ and $-$. And this, it is evident, must be the case with all the following terms of both these compound serieses (to whatever number of terms these serieses may be continued) as well as with the few terms here computed. For, since the terms of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, are the same with the terms of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, though differently connected with each other by the signs $+$ and $-$, it follows that the products of the multiplication of the terms of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, into each other must be equal to the products of the multiplication of the correspondent terms of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, into each other, that is, the several members of the square of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, must be equal to the corresponding members of the square of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, though differently connected with each other by the signs $+$ and $-$.

And secondly, we may observe that the first term in both these compound serieses is 1.

And, thirdly, we may observe that the third and fifth terms in the second of these products, or compound serieses, which is equal to the square of the second simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ (in which all the terms after the first term 1 are marked with the sign $-$) to wit, the terms $- 2Cx^2 + B^2x^2$ and $- 2Ex^4 + 2BDx^4 + C^2x^4$, have the same signs $+$ and $-$ prefixed to their several members $2Cx^2$, B^2x^2 , $2Ex^4$, $2BDx^4$, and C^2x^4 , respectively, as are prefixed to the same members of the third and fifth terms of the former of these products, or compound serieses, which is equal to the square of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (in which the second and other following terms are marked with the signs $+$ and $-$ alternately), to wit, the terms $- 2Cx^2 + B^2x^2$ and $- 2Ex^4 + 2BDx^4 + C^2x^4$. And the same observation is true likewise of the seventh and ninth, and eleventh, and all the following odd terms of these two compound serieses (to whatever number of terms the said serieses may be continued), to wit, that the signs $+$ and $-$ that are to be prefixed to the several members of any of the said odd terms in the latter of these compound serieses will be the same that are to be prefixed to the same members of the same odd terms in the former of these compound serieses; as will appear from an attentive consideration of the two foregoing operations of multiplication set down in the foregoing article.

And, 4thly, we may observe that the second, fourth, and sixth terms of the second of these products, or compound serieses (which is equal to the square of the second simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$), to wit, the terms $- 2Bx$, $- 2Dx^3 + 2BCx^3$, and $- 2Fx^5 + 2BEx^5 + 2CDx^5$, have their several members $2Bx$, $2Dx^3$, $2BCx^3$, $2Fx^5$, $2BEx^5$, and $2CDx^5$, marked with contrary signs to those which are prefixed to the same members of the second, fourth, and sixth, terms of the former

mer of those compound serieses (which is equal to the square of the first simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$), to wit, the terms $+ 2 Bx$, $+ 2 Dx^3 - 2 BCx^3$, and $+ 2 Fx^5 - 2 BEx^5 - 2 CDx^5$. And the same observation is true likewise of the eighth, and tenth, and twelfth, and other following even terms of these two compound serieses (to whatever number of terms they may be continued), to wit, that the signs $+$ and $-$ that are to be prefixed to the several members of any of the said even terms in the latter of these compound serieses will be, respectively, contrary to those which are to be prefixed to the same members of the same even terms in the former of these compound serieses; as will appear from an attentive consideration of the two foregoing operations of multiplication set down in the foregoing article.

12. And if we were to repeat the foregoing multiplications of the two serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ into themselves any number of times whatsoever, so as to obtain the cubes, and the fourth powers, and the fifth powers, or any higher powers of the said serieses, the foregoing observations would be true of all the compound serieses, or of all the powers of the said two simple serieses, which would be thereby obtained; to wit, 1st, That the terms of every power of the one series would be the very same with the corresponding terms of the same power of the other series; and, 2dly, That the first term of every power of both serieses would be 1; and, 3dly, That the signs $+$ and $-$, which would be prefixed to the several members of the third, fifth, seventh, ninth, eleventh, and other following odd terms of any power of one of these serieses, would be the very same that were to be prefixed to the same members of the same odd terms, respectively, of the same power of the other series; and, 4thly, That the signs $+$ and $-$, which would be prefixed to the second term, and to the several members of the fourth, sixth, eighth, tenth, twelfth, and other following even terms of any power of one of these two serieses would be, respectively, contrary to the signs that would be prefixed to the second term, and to the same members of the same following even terms of the same power of the other series.

13. Of the truth of these observations in the case of higher powers of the two foregoing simple serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ than the square, I shall here give one example, by raising the said two serieses to the cube, or third power.

We have already seen in art. 10 that the square of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ is the compound series

$$\begin{aligned} 1 + 2 Bx - 2 Cx^2 + 2 Dx^3 - 2 Ex^4 + 2 Fx^5 - \&c \\ + B^2x^2 - 2 BCx^3 + 2 BDx^4 - 2 BEx^5 + \&c \\ + C^2x^4 - 2 CDx^5 + \&c. \end{aligned}$$

Therefore to find its cube, we must multiply this compound series into the original series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$; which may be done as follows.

$$\begin{array}{r}
 1 + 2Bx - 2Cx^2 + 2Dx^3 - 2Ex^4 + 2Fx^5 - \&c \\
 + B^2x^2 - 2BCx^3 + 2BDx^4 - 2BEx^5 + \&c \\
 + C^2x^4 - 2CDx^5 + \&c
 \end{array}$$

$$\begin{array}{r}
 1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c \\
 \hline
 1 + 2Bx - 2Cx^2 + 2Dx^3 - 2Ex^4 + 2Fx^5 - \&c \\
 + B^2x^2 - 2BCx^3 + 2BDx^4 - 2BEx^5 + \&c \\
 + C^2x^4 - 2CDx^5 + \&c \\
 + Bx + 2B^2x^2 - 2BCx^3 + 2BDx^4 - 2BEx^5 + \&c \\
 + B^3x^3 - 2B^2Cx^4 + 2B^2Dx^5 - \&c \\
 + BC^2x^5 - \&c \\
 - Cx^2 - 2BCx^3 + 2C^2x^4 - 2CDx^5 + \&c \\
 - B^2Cx^4 + 2BC^2x^5 - \&c \\
 + Dx^3 + 2BDx^4 - 2CDx^5 + \&c \\
 + B^2Dx^5 - \&c \\
 - Ex^4 - 2BEx^5 + \&c \\
 + Fx^5 - \&c
 \end{array}$$

$$\begin{array}{r}
 1 + 3Bx - 3Cx^2 + 3Dx^3 - 3Ex^4 + 3Fx^5 - \&c \\
 + 3B^2x^2 - 6BCx^3 + 6BDx^4 - 6BEx^5 + \&c \\
 + B^3x^3 + 3C^2x^4 - 6CDx^5 + \&c \\
 - 3B^2Cx^4 + 3B^2Dx^5 - \&c \\
 + 3BC^2x^5 - \&c.
 \end{array}$$

Therefore the cube of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ is the compound series

$$\begin{array}{r}
 1 + 3Bx - 3Cx^2 + 3Dx^3 - 3Ex^4 + 3Fx^5 - \&c \\
 + 3B^2x^2 - 6BCx^3 + 6BDx^4 - 6BEx^5 + \&c \\
 + B^3x^3 + 3C^2x^4 - 6CDx^5 + \&c \\
 - 3B^2Cx^4 + 3B^2Dx^5 - \&c \\
 + 3BC^2x^5 - \&c.
 \end{array}$$

And we have already seen in art. 10 that the square of the second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ is the compound series

$$\begin{array}{r}
 1 - 2Bx - 2Cx^2 - 2Dx^3 - 2Ex^4 - 2Fx^5 - \&c \\
 + B^2x^2 + 2BCx^3 + 2BDx^4 + 2BEx^5 + \&c \\
 + C^2x^4 + 2CDx^5 + \&c.
 \end{array}$$

Therefore, to find the cube of the said second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, we must multiply this last compound series into the said second series itself; which may be done as follows.

$$1 - 2Bx$$

$$\begin{array}{r}
 1 - 2Bx - 2Cx^2 - 2Dx^3 - 2Ex^4 - 2Fx^5 - \&c \\
 + B^2x^2 + 2BCx^3 + 2BDx^4 + 2BEx^5 + \&c \\
 + C^2x^4 + 2CDx^5 + \&c \\
 \hline
 1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c \\
 1 - 2Bx - 2Cx^2 - 2Dx^3 - 2Ex^4 - 2Fx^5 - \&c \\
 + B^2x^2 + 2BCx^3 + 2BDx^4 + 2BEx^5 + \&c \\
 + C^2x^4 + 2CDx^5 + \&c \\
 - Bx + 2B^2x^2 + 2BCx^3 + 2BDx^4 + 2BEx^5 + \&c \\
 - B^3x^3 - 2B^2Cx^4 - 2B^2Dx^5 - \&c \\
 - BC^2x^5 - \&c \\
 - Cx^2 + 2BCx^3 + 2C^2x^4 + 2CDx^5 + \&c \\
 - B^2Cx^4 - 2BC^2x^5 - \&c \\
 - Dx^3 + 2BDx^4 + 2CDx^5 + \&c \\
 - B^2Dx^5 + \&c \\
 - Ex^4 + 2BEx^5 + \&c \\
 - Fx^5 + \&c \\
 \hline
 1 - 3Bx - 3Cx^2 - 3Dx^3 - 3Ex^4 - 3Fx^5 - \&c \\
 + 3B^2x^2 + 6BCx^3 + 6BDx^4 + 6BEx^5 + \&c \\
 - B^3x^3 + 3C^2x^4 + 6CDx^5 + \&c \\
 - 3B^2Cx^4 - 3B^2Dx^5 - \&c \\
 - 3BC^2x^5 - \&c
 \end{array}$$

Therefore the cube of the said second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ is the compound series

$$\begin{array}{r}
 1 - 3Bx - 3Cx^2 - 3Dx^3 - 3Ex^4 - 3Fx^5 - \&c \\
 + 3B^2x^2 + 6BCx^3 + 6BDx^4 + 6BEx^5 + \&c \\
 - B^3x^3 + 3C^2x^4 + 6CDx^5 + \&c \\
 - 3B^2Cx^4 - 3B^2Dx^5 - \&c \\
 - 3BC^2x^5 - \&c.
 \end{array}$$

14. The four observations made in art. 11 and 12 are evidently true of the two compound series obtained in the foregoing article 13, and which are equal to the cubes of the two simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$. For, in the 1st place, the several terms of the latter compound series are exactly the same with the corresponding terms of the former compound series; and 2dly, the first term in both these compound series is 1; and, 3dly, the signs + and - that are prefixed to the several members of the third and fifth terms, $-3Cx^2 + 3B^2x^2$ and $-3Ex^4 + 6BDx^4 + 3C^2x^4 - 3B^2Cx^4$ of the latter compound series (which is equal to the cube of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$) are the same with those which are prefixed to the same members of the third and fifth terms, $-3Cx^2 + 3B^2x^2$ and $-3Ex^4 + 6BDx^4 + 3C^2x^4 - 3B^2Cx^4$, of the former compound series, which is equal to the cube of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$; and, 4thly, the signs + and - which are prefixed to the second term, $-3Bx$, and to the several members of the fourth and sixth terms, $-3Dx^3 + 6BCx^3 - B^3x^3$ and $-3Fx^5 + 6BEx^5 + 6CDx^5 - 3B^2Dx^5 - 3BC^2x^5$, are the same with those which are prefixed to the corresponding terms of the former compound series.

$3BC^2x^5$, of the latter compound series (which is equal to the cube of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$) are, respectively, contrary to those which are prefixed to the second term, $+ 3Bx$, and to the several members of the fourth and sixth terms, $+ 3Dx^3 - 6BCx^3 + B^3x^3$ and $+ 3Fx^5 - 6BEx^5 - 6CDx^5 + 3B^2Dx^5 + 3BC^2x^5$, of the former compound series, which is equal to the cube of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$.

And the same observations will be found to be true of the several members of all the following terms of the said two compound series after the sixth terms (to whatever number of terms the said series may be continued) and of the signs $+$ and $-$ that are to be prefixed to the said members. And they will be true also of the terms of the several compound series that are equal to the fourth powers, and to the fifth powers, and to all higher powers, of the said two simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$.

15. This relation between the signs that are to be prefixed to the same members of the several terms of the said compound series which are equal to the squares and the cubes and other higher powers of the two simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ is the consequence of the known rules of Algebraick multiplication; according to which the product of the multiplication of two quantities which are both marked with the sign $-$ is to be marked with the sign $+$, as well as the product of the multiplication of two quantities which are both marked with the sign $+$; and the product of the multiplication of two quantities, of which one is marked with the sign $+$, and the other with the sign $-$, is to be marked with the sign $-$. For it will follow from hence that in the compound series which is equal to the square, or cube, or fourth power, or fifth power, or other higher power, of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, all the terms that will involve the odd powers of x , to wit, $x, x^3, x^5, x^7, x^9, x^{11}, \&c$, will be marked with contrary signs, respectively, to those which are to be prefixed to the same terms in the compound series which is equal to the square, or cube, or fourth power, or fifth power, or other corresponding higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$; and all the terms that will involve the even powers of x , to wit, $x^2, x^4, x^6, x^8, x^{10}, x^{12}, \&c$, in the former compound series, which is equal to the square, or cube, or other higher power, of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, will be marked with the same signs $+$ and $-$, respectively, as are prefixed to the same terms in the latter compound series, which is equal to the square, or cube, or other corresponding higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$.

16. To make this more evident, let the capital letter P be put $= Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} + \&c$, or the sum of all the terms in the first series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - Ix^8 + Kx^9 - Lx^{10} + Mx^{11} - \&c$, which involve the odd powers of x , and which in this series have, all of them, the sign $+$ prefixed to them; and let

let the capital letter Q be put $= Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c$, or the sum of all the terms in the same series which involve the even powers of x , and which in this series, as well as in the second series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - Gx^6 - Hx^7 - Ix^8 - Kx^9 - Lx^{10} - Mx^{11} - \&c$, are all marked with the sign $-$.

Then will the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ be $= 1 + P - Q$, and the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ will be $= 1 - P - Q$; and consequently the square of the former series will be $(= 1 + P - Q)^2 = 1 + [P - Q]^2 = 1 + 2 \times 1 \times [P - Q] + [P - Q]^2 = 1 + 2P - 2Q + P^2 - 2PQ + Q^2$, and the square of the latter series will be $(= 1 - P - Q)^2 = 1 - [P + Q]^2 = 1 - 2 \times 1 \times [P + Q] + [P + Q]^2 = 1 - 2P - 2Q + P^2 + 2PQ + Q^2$.

17. The terms of this latter quantity (which is equal to the square of the latter series) are exactly the same with the terms of the former quantity $1 + 2P - 2Q + P^2 - 2PQ + Q^2$, (which is equal to the square of the former series) but are not connected with each other by the signs $+$ and $-$ in the same manner as the terms of the said former quantity. And the difference between the terms of these two quantities in this respect is as follows. The third term $2Q$, and the fourth term P^2 , and the sixth, or last, term Q^2 , in both the quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$ are marked with the same signs $+$ and $-$, being $- 2Q + P^2 + Q^2$ in both quantities; but the second term $2P$, and the fifth term $2PQ$, have different signs in the two quantities, being $+ 2P$ and $- 2PQ$ in the former quantity, and $- 2P$ and $+ 2PQ$ in the latter quantity.

18. Now the three terms $2Q$, P^2 , and Q^2 , which are marked with the same signs in both these quantities, will be equal to serieses that will involve only the even powers of x .

For, in the first place, the second term $2Q$ is $= 2 \times$ the series $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c =$ the series $2Cx^2 + 2Ex^4 + 2Gx^6 + 2Ix^8 + 2Lx^{10} + \&c$, which contains only the even powers of x .

And, in the second place, the sixth term Q^2 (being equal to the square of the series $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c$) will evidently be equal to a series that will contain only x^4, x^6, x^8, x^{10} , and the following even powers of x .

And, lastly, the fourth term P^2 (being equal to the square of the series $Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} + \&c$, which contains only the odd powers of x) will also be equal to a series that will contain only the even powers of x ; because its terms will be the products of the multiplication of the terms of the series $Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} + \&c$ (which involve only the odd powers of x) into each other: it being evident that the multiplication of two odd powers of x into each other must always produce an even power of it.

Therefore the three terms $2Q$, P^2 , and Q^2 , which are marked with the same signs $+$ and $-$ in both the quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and

and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$, will be equal to serieses which will involve only the even powers of x . Q. E. D.

19. And the second term $2P$, and the fifth term $2PQ$, of the said quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$, which are marked with contrary signs in the latter of these two quantities to those with which they are marked in the former quantity, will be equal to serieses which will contain only the odd powers of x .

For, in the 1st place, $2P$ is $= 2 \times$ the series $Bx + Dx^3 + Fx^5 + Hx^7 + Kx^9 + Mx^{11} + \&c =$ the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c$, which contains only the odd powers of x .

And, 2dly, $2PQ$ is $= Q \times 2P = Q \times$ the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c =$ the series $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c$ (which involves the even powers of x) $\times$ the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c$ (which involves the odd powers of x) $=$ a series consisting of terms which will involve only $x^3, x^5, x^7, x^9, x^{11}$, and the following odd powers of x ; because all the terms of it will be the products of the multiplication of some of the terms of the series $Cx^2 + Ex^4 + Gx^6 + Ix^8 + Lx^{10} + \&c$ (which involve only the even powers of x) into some of the terms of the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + \&c$, which involve only the odd powers of x ; it being evident that the product of the multiplication of an even power of x into an odd power of it must always be an odd power of it.

Therefore the two terms $2P$ and $2PQ$ of the two quantities $1 + 2P - 2Q + P^2 - 2PQ + Q^2$ and $1 - 2P - 2Q + P^2 + 2PQ + Q^2$, which are marked with contrary signs $+$ and $-$ in those two quantities, will be equal to serieses which will contain only the odd powers of x . Q. E. D.

20. It appears, therefore, from the two preceeding articles, that, to whatever number of terms the two serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, and the squares of these two serieses, may be continued, the terms involving the even powers of x in the squares of both serieses, will be marked with the same signs $+$ and $-$, and the terms involving the odd powers of x in the square of the latter series will be marked with the contrary signs to those with which the same terms are marked in the square of the former series.

21. The reasonings used in the fix preceeding articles to prove, "that, "in the two compound serieses which are equal to the squares of the two simple "serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, the several terms, and members of "terms, which involve the even powers of x will be marked with the same "signs $+$ and $-$ respectively, and that the several terms, and members of "terms, which involve the odd powers of x in the latter of the said compound serieses will be marked with the contrary signs to those which are to be prefixed "to the same terms, and members of terms, in the former of the said compound

“pound serieses,” may be extended to the more complicated compound serieses which are equal to the cubes of the said simple serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, and to the fourth powers of the said serieses, and to the fifth powers of the said serieses, and to all higher powers of the said serieses, so as to prove, “that in all these compound serieses the several quantities which involve the even powers of x will be marked with the same signs in every two corresponding serieses, and that in the compound series which is equal to the cube, or the fourth power, or the fifth power, or any higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, the quantities which involve the odd powers of x will be marked with the contrary signs to those which are prefixed to the same quantities in the compound series which is equal to the cube, or the fourth power, or the fifth power, or other corresponding higher power of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$.” And therefore I think, we may now consider the four observations set forth above in art. 11 and 12 as sufficiently established.

22. Now, if these observations are admitted to be true, it will follow that, whenever the co-efficients of the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to the square, or the cube, or the fourth power, or the fifth power, or any higher power, of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ are all equal to 0, or those of the members of the said co-efficients which are marked with the sign $-$, taken together, are equal to the other members of the said co-efficients which are marked with the sign $+$, and from which the former members marked with the sign $-$, are to be subtracted, the same thing will also take place in the compound series which is equal to the square, or the cube, or the fourth power, or the fifth power, or the other corresponding higher power, of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$; to wit, that the co-efficients of the third, fourth, fifth, sixth, and other following terms of the said compound series will likewise all be equal to 0, or that those members of the said co-efficients which are marked with the sign $-$, or are subtracted from the other members of them which are marked with the sign $+$, will be equal to the said other members.

For in the said correspondent compound serieses the members of the several corresponding terms, or terms involving the same powers of x , will be the very same quantities, by the 1st observation in art. 11 and 12. And in the third, fifth, seventh, and other following odd terms of the said compound serieses (which will involve the even powers of x) the signs $+$ and $-$, to be prefixed to the several members of the said terms, will be the same in both serieses; and in the second terms and the several members of the fourth, sixth, eighth, and other following even terms of the said compound serieses (which will involve the odd powers of x) the signs $+$ and $-$, to be prefixed to the said second terms, and to the several members of the fourth, sixth, eighth, and other following even terms of the said compound serieses, will be, respectively, contrary in one of those compound serieses to what they are in the other; as is evident from the third and fourth observations in art. 11 and 12. Therefore, if in the third, fourth,

fourth, fifth, sixth, and other following terms of the compound series which is equal to any power of the first simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, the sum of the members of each of the said terms that are marked with the sign $-$ is equal to the sum of the members of the same term which are marked with the sign $+$, the same thing will also take place in the compound series which is equal to the same power of the second simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or the sum of the members which are marked with the sign $-$ in the third, and the fourth, and the fifth, and the sixth, and every other following term of the said second compound series, will be equal to the sum of the members of the same term which are marked with the sign $+$; that is, in other words, when the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to any power of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, become, all, equal to 0, and the whole series consists of only the two first terms, the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to the same power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ will also be, all, equal to 0, and the whole series will consist of only the two first terms.

Q. E. D.

23. And hence we may derive a proof of the proposition asserted above, in art. 2, concerning the quantity $\sqrt[n]{1-x}$, or $\sqrt[n]{1-x}$, or the n th root of the residual quantity $1-x$; to wit, that, if A be $= 1$, and B be $= \frac{1}{n} A$, and C be $= \frac{n-1}{2n} B$, and D be $= \frac{2n-1}{3n} C$, and $E, F, G, H, \&c$ be equal to $\frac{3n-1}{4n} D$, $\frac{4n-1}{5n} E$, $\frac{5n-1}{6n} F$, $\frac{6n-1}{7n} G$, $\&c$, respectively, the said quantity $\sqrt[n]{1-x}$, or $\sqrt[n]{1-x}$, will be equal to the infinite series $1 - \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 - \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 - \frac{4n-1}{5n} Ex^5 - \&c$, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, in which all the terms after the first term 1 are marked with the sign $-$, or are subtracted from the said first term.

For, since the series $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, has been shewn in art. 47 of the preceeding discourse concerning the Bino-

mial Theorem, page 237, to be equal to $\sqrt[n]{1+x}$, or $\sqrt[n]{1+x}$, it follows that, if the said series were to be raised to the n th power (n being any whole number whatsoever) or to be multiplied $n-1$ times into itself, the compound series thence arising would be equal to $1+x$, which would be the two first terms of it, and consequently the co-efficients of $x^2, x^3, x^4, x^5, \&c$ *ad infinitum* in the following terms of the said compound series would be, each of them, equal to 0, or those members of the said co-efficients which would be marked with

with the sign $-$, or would be to be subtracted from the others which would be marked with the sign $+$, would be equal to the said other members, from which they would be to be subtracted; of which we have given some examples above in art. 5, 6, 7, and 8. But, by the foregoing article 22, whenever the co-efficients of the third, fourth, fifth, sixth, and other following terms of the compound series which is equal to any power of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, are all equal to 0, or the subtracted members of each of the said co-efficients are equal, taken together, to the members from which they are subtracted, the co-efficients of the third, fourth, fifth, sixth, and other following terms of the compound series, which is equal to the same power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, will also be, all of them, equal to 0, or the subtracted members of each of the said co-efficients will be equal, taken together, to the members from which they are subtracted. Therefore, if the series $1 - \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 - \sqrt{\frac{2n-1}{3n}} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 - \sqrt{\frac{4n-1}{5n}} Ex^5 - \&c$, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ (in which B is $= \frac{1}{n} A$, or $\frac{1}{n} \times 1$, or $\frac{1}{n}$, and C is $= \frac{n-1}{2n} B$, and D is $= \frac{2n-1}{3n} C$, and E is $= \frac{3n-1}{4n} D$, and F , G , H , I , $\&c$, are equal to $\frac{4n-1}{5n} E$, $\frac{5n-1}{6n} F$, $\frac{6n-1}{7n} G$, and $\frac{7n-1}{8n} H$, $\&c$, respectively), be raised to the n th power, or multiplied $n - 1$ times into itself, the co-efficients of the third, fourth, fifth, sixth, and other following terms of the compound series which will be produced by such multiplication, and which will be equal to the said n th power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, will, each of them, be equal to 0, or the subtracted members of each of the said co-efficients will be, all taken together, equal to the other members of them, from which they are to be subtracted; and consequently the whole of the said compound series will be equal to its two first terms; which will be $1 - x$, because the two first terms of the compound series which is equal to the n th power of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 + \sqrt{\frac{2n-1}{3n}} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 + \sqrt{\frac{4n-1}{5n}} Ex^5 - \&c$ (to which the compound series that is equal to the n th power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, has been shewn to be analogous in the manner above-described), are $1 + x$.

And, since the n th power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or $1 - \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 - \sqrt{\frac{2n-1}{3n}} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 - \sqrt{\frac{4n-1}{5n}} Ex^5 - \&c$, will be equal to $1 - x$, it follows that the said series itself will be equal to $\sqrt[n]{1 - x}$, or $\sqrt[n]{1 - x}$, or the n th root of the residual quantity $1 - x$.

24. We must now proceed to consider the quantity $\sqrt[n]{1-x}$, or the m th power of the n th root of the residual quantity $1-x$.

Of the series which is equal to the quantity $\sqrt[n]{1-x}$, or the m th power of the n th root of the residual quantity $1-x$, when m and n are any whole numbers whatsoever.

25. It has been shewn in the foregoing discourse concerning the binomial theorem, art. 77, p. 269, that, if m be any whole number whatsoever, and n

any other whole number greater than m , the quantity $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n} Ax - \frac{\frac{n-m}{2n}}{2n} Bx^2 + \frac{\frac{2n-m}{3n}}{3n} Cx^3 - \frac{\frac{3n-m}{4n}}{4n} Dx^4 + \frac{\frac{4n-m}{5n}}{5n} Ex^5 - \&c$ *ad infinitum*; in which series the second, and third, and fourth, and other following terms, are alternately marked with the sign $+$ and the sign $-$. And it has been shewn in art. 104 of the said discourse, page 289, that, when m is greater than

n , but less than $2n$, the quantity $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 - \frac{\frac{2n-m}{3n}}{3n} Cx^3 + \frac{\frac{3n-m}{4n}}{4n} Dx^4 - \frac{\frac{4n-m}{5n}}{5n} Ex^5 + \&c$; in which the three first terms of the series are added together, and the fourth and fifth, and other following terms of it are marked with the sign $-$ and the sign $+$ alternately, or are alternately subtracted from, and added to, the said three first terms. And it has been shewn in art. 115 of the said discourse, page 299, that, if m be of any magnitude greater than $2n$, and pn is the greatest multiple of n that is less than m , so that m is greater than pn , but less than $(p+1)n$, or $pn+n$, the quan-

ty $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1+x$, will be equal to the series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 + \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \frac{m-5n}{6n} Fx^6 + \&c$, $-\frac{\frac{pn+n-m}{pn+2n}}{pn+2n} C'x^{p+2} + \frac{\frac{pn+2n-m}{pn+3n}}{pn+3n} D'x^{p+3} - \frac{\frac{pn+3n-m}{pn+4n}}{pn+4n} E'x^{p+4} + \frac{\frac{pn+4n-m}{pn+5n}}{pn+5n} F'x^{p+5} - \frac{\frac{pn+5n-m}{pn+6n}}{pn+6n} G'x^{p+6} + \&c$ *ad infinitum*; in which series all the terms after the first term 1 are to be added to the said first term, till we come to the term $\frac{pn+n-m}{pn+2n} C'x^{p+2}$, which

is to be subtracted from it; and all the terms after the said term $\frac{pn+n-m}{pn+2n}$ $C'x^{p+2}$ are to be added to, and subtracted from, the said first term alternately.

Now the quantity $\sqrt[n]{1-x}$, or the m th power of the n th root of the residual quantity $1-x$, will always be equal to a series consisting of the very same terms as the series that is equal to the quantity $\sqrt[n]{1+x}$, or the same power of the same root of the binomial quantity $1+x$, but with the signs $+$ and $-$ changed into their contraries in all the terms that involve x , x^3 , x^5 , x^7 , x^9 , x^{11} , and the other following odd powers of x , or in the second, and fourth, and sixth, and eighth, and tenth, and other following even terms of the series. This I shall now endeavour to demonstrate.

26. It has been shewn above in art. 23, that $\sqrt[n]{1-x}$, or the n th root of the residual quantity $1-x$, is equal to the infinite series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 - \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5 - \&c$, which consists of the very same terms, only differently connected together by the signs $+$ and $-$, as the series $1 + \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 + \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 + \frac{4n-1}{5n}Ex^5 - \&c$, which is equal to $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$. And the difference between these two serieses with respect to the signs $+$ and $-$, that are to be prefixed to their second and other following terms, is that in the series which is equal to $\sqrt[n]{1-x}$ the sign $-$ is to be prefixed to all the terms after the first term, whereas in the series which is equal to $\sqrt[n]{1+x}$ the sign $-$ is to be prefixed only to its third, and fifth, and seventh, and other following odd terms, and the sign $+$ is to be prefixed to its second, and fourth, and sixth, and other following even terms.

Now, since $\sqrt[n]{1-x}$ is equal to the infinite series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 - \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5 - \&c$, it follows that the quantity $\sqrt[n]{1-x}$, or the m th power of $\sqrt[n]{1-x}$ will be equal to the m th power of the said infinite series $1 - \frac{1}{n}Ax - \frac{n-1}{2n}Bx^2 - \frac{2n-1}{3n}Cx^3 - \frac{3n-1}{4n}Dx^4 - \frac{4n-1}{5n}Ex^5 - \&c$, or (as it will be convenient to denote it for the sake of brevity), of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ *ad infinitum*. We must therefore inquire what will be the terms of the series that is equal

equal to the m th power of the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, and what will be the signs $+$ and $-$ that will be to be prefixed to them, or what will be the terms of the series that will be produced by multiplying the said series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$ $m - 1$ times into itself, and which of the signs $+$ and $-$ will be to be prefixed to each of them: and we must compare the series so produced with the series which is produced by the multiplication of the series $1 + \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$ (which is equal to $\sqrt[n]{1+x}$) $m - 1$ times into itself, and which is therefore equal to the quantity $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1 + x$.

27. Now, from what has been shewn above in art. 11, 12, ———— 21, it is evident, in the first place, that if we raise both the serieses $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, and $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, to the m th power (m being any whole number whatsoever), or multiply each of them $m - 1$ times into itself, the products of the said multiplications will be two compound serieses consisting of exactly the same terms, of which the first term will be 1; and, 2dly, that the signs $+$ and $-$, that will be prefixed to the members of the several terms of the said compound serieses, will be the same in all the terms which will involve the even powers of x in both serieses, but will be different in those terms of the said two serieses which will involve the odd powers of x . Therefore the compound series which is equal to the m th power of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or $1 - \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 - \sqrt{\frac{2n-1}{3n}} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 - \sqrt{\frac{4n-1}{5n}} Ex^5 - \&c$, may be derived from the compound series which is equal to the m th power of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, by changing the signs of all those terms of the said latter compound series which involve the odd powers of x , into their contraries. But the compound series which is equal to the m th power of the said simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3n} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c$, is equal to the m th power of $\sqrt[n]{1+x}$, or to the quantity $\sqrt[n]{1+x}$, and consequently (by what is shewn in art. 115 of the foregoing discourse, page 299) is also equal to the simple series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 + \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \&c - \sqrt{\frac{pn+n-m}{pn+2n}} C'x^{p+2}$

$\left[\frac{pn+n-m}{pn+2n} C'x^{p+2} + \frac{pn+2n-m}{pn+3n} D'x^{p+3} - \&c. \right]$ Therefore, if the signs + and - of those terms in this last simple series which involve the odd powers of x , that is, of the second, fourth, sixth, and other following even terms of it, be changed into their contraries, the simple series thereby produced will be equal to the compound series produced by changing the signs of those terms of the compound series that is equal to the m th power of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n} Ax - \left[\frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \left[\frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c \right] \right]$, which involve the odd powers of x , or the signs of its second, fourth, sixth, and other following even terms, into their contraries. But it has been just now shewn, that the compound series produced by changing the signs of those terms of the compound series that is equal to the m th power of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or $1 + \frac{1}{n} Ax - \left[\frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \left[\frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \&c \right] \right]$, which involve the odd powers of x , into their contraries, is equal to the compound series, or rather is the compound series, which is equal to the m th power of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or $1 - \frac{1}{n} Ax - \left[\frac{n-1}{2n} Bx^2 - \left[\frac{2n-1}{3n} Cx^3 - \left[\frac{3n-1}{4n} Dx^4 - \left[\frac{4n-1}{5n} Ex^5 - \&c \right] \right] \right] \right]$. Therefore the simple series that is produced by changing the signs of those terms of the simple series $1 + \frac{m}{n} Ax + \frac{m-n}{2n} Bx^2 + \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \&c$, - $\left[\frac{pn+n-m}{pn+2n} C'x^{p+2} + \frac{pn+2n-m}{pn+3n} D'x^{p+3} - \left[\frac{pn+3n-m}{pn+4n} E'x^{p+4} + \&c \right] \right]$, which involve the odd powers of x , or the signs of the second, fourth, sixth, and other following even terms of it, into their contraries, will be equal to the compound series which is equal to the m th power of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or $1 - \frac{1}{n} Ax - \left[\frac{n-1}{2n} Bx^2 - \left[\frac{2n-1}{3n} Cx^3 - \left[\frac{3n-1}{4n} Dx^4 - \left[\frac{4n-1}{5n} Ex^5 - \&c \right] \right] \right] \right]$, and consequently will be equal to the m th power of $1 - x$ (which is equal to the said last-mentioned simple series), or to the m th power of the n th root of the residual quantity $1 - x$.

Q. E. D.

28. The reasonings in the foregoing article appear somewhat perplexed in consequence of the multitude of words which have been made use of in describing the several infinite serieses mentioned in them. I will therefore, now repeat them in a conciser manner, which will, I hope, remove all obscurity from them; and for this purpose I shall denote all the serieses that we shall have occasion to consider, by single letters.

29. Let

29. Let the Greek capital letter Γ be put for the series $1 + \frac{1}{n} Ax - \sqrt[n]{\frac{n-1}{2n}} Bx^2 + \frac{2n-1}{3^n} Cx^3 - \sqrt[n]{\frac{3n-1}{4^n}} Dx^4 + \frac{4n-1}{5^n} Ex^5 - \&c, ad infinitum$, or $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c ad infinitum$, which is equal to $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$; and let the Greek capital letter Δ be put for the series $1 - \frac{1}{n} Ax - \sqrt[n]{\frac{n-1}{2n}} Bx^2 - \sqrt[n]{\frac{2n-1}{3^n}} Cx^3 - \sqrt[n]{\frac{3n-1}{4^n}} Dx^4 - \sqrt[n]{\frac{4n-1}{5^n}} Ex^5 - \&c ad infinitum$, or $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c ad infinitum$, which is equal to $\sqrt[n]{1-x}$, or the n th root of the residual quantity $1-x$. And let Γ^m stand for the compound series which is equal to the m th power (m being any whole number whatsoever) of the simple series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - \&c$, or Γ , or the product which arises by multiplying the said simple series $m-1$ times into itself; and let Δ^m stand for the compound series which is equal to the same, or the m th, power of the simple series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - \&c$, or Δ , or the product which arises by multiplying it $m-1$ times into itself. And let the Greek capital letter Λ denote the simple series $1 + \frac{m}{n} Ax + \frac{m-2n}{2n} Bx^2 + \frac{m-2n}{3n} Cx^3 + \frac{m-3n}{4n} Dx^4 + \frac{m-4n}{5n} Ex^5 + \&c - \sqrt[n]{\frac{pn+n-m}{pn+2n}} C'x^{p+2} + \frac{pn+2n-m}{pn+3n} D'x^{p+3} - \frac{pn+3n-m}{pn+4n} E'x^{p+4} + \frac{pn+4n-m}{pn+5n} F'x^{p+5} - \&c ad infinitum$, which is equal to the quantity $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1+x$. And, lastly, let the Greek capital letter Π denote the simple series which is derived from the simple series Λ by changing the signs of those terms in it which involve the odd powers of x , that is, of the second, fourth, sixth, eighth, and other following even terms of it, into their contraries. With this notation the reasonings contained in the foregoing article 27 will be as follows.

30. From what has been shewn above, in art. 11, 12, &c, — — — 21, it is evident, in the 1st place, that the compound serieses Γ^m and Δ^m will consist of exactly the same terms, of which the first term will be 1; and, 2dly, that the signs $+$ and $-$ that will be prefixed to the members of the several terms of the said two compound serieses will be the same in all the quantities, or members of the terms of the said serieses, which will involve the even powers of x , that is, in the third, and fifth, and seventh, and other following odd terms of the said serieses; but will be different in those terms of the said two serieses which will involve the odd powers of x , that is, in the second, and fourth, and sixth, and other following even terms of the said serieses. Therefore the compound series Δ^m may be derived from the compound series Γ^m by changing the signs of all the members of those terms of the series Γ^m which involve in them the odd powers of

of x , that is, the signs of all the members of the second, fourth, sixth, and other following even terms of it, into their contraries. But, because the simple series

Γ is equal to $\sqrt[n]{1+x}$, the compound series Γ^m will be equal to the m th power of $\sqrt[n]{1+x}$, or to the quantity $\sqrt[n]{1+x}^m$, and consequently (by what is shewn in art. 115 of the foregoing discourse, page 299) will be also equal to the simple series Λ . Therefore, if the signs $+$ and $-$ in those terms of this simple series Λ which involve the odd powers of x , that is, in the second, fourth, sixth, and other following even terms of it, be changed into their contraries, the simple series thereby produced will be equal to the compound series that is produced by changing the signs $+$ and $-$ of all the members of those terms of the compound series Γ^m , which also involve the odd powers of x , or the signs of all the members of the second, fourth, sixth, and other following even terms of the said compound series Γ^m , into their contraries. But it has been just now shewn, that the compound series which is produced by changing the signs $+$ and $-$ of all the members of those terms of the compound series Γ^m which involve the odd powers of x , or of all the members of the second, fourth, sixth, and other following even terms of it, is the compound series Δ^m . Therefore, if the signs $+$ and $-$ in those terms of the simple series Λ which involve the odd powers of x , or in the second, fourth, sixth, and other following even terms of it, are changed into their contraries, the simple series thereby produced will be equal to the compound series Δ^m , and consequently

will be equal to the m th power of $\sqrt[n]{1-x}$, or of the n th root of the residual quantity $1-x$, which is equal to the simple series Δ ; that is, the simple series

Π will be equal to the m th power of $\sqrt[n]{1-x}$, or of the n th root of the residual quantity $1-x$, or to the quantity $\sqrt[n]{1-x}^m$. Q. E. D.

31. And, if still greater brevity be desir'd, this demonstration of the equality between the simple series Π and the quantity $\sqrt[n]{1-x}^m$ may be expressed in the manner following.

Since the simple series Γ is $= \sqrt[n]{1+x}$, it follows that the compound series Γ^m will be $=$ the m th power of $\sqrt[n]{1+x}$, or $= \sqrt[n]{1+x}^m$.

But (by the foregoing discourse, art 115, page 299) $\sqrt[n]{1+x}^m$ is equal the simple series Λ .

Therefore the compound series Γ^m is equal the simple series Λ .

Therefore, if we change the signs $+$ and $-$ of all the even terms, or terms involving the odd powers of x , in both these serieses Γ^m and Λ , into their contraries, the serieses thereby produced will be equal to each other.

But

But the series produced by this change in the signs of the terms of the compound series Γ^m will be the compound series Δ^m , by what has been shewn above in art. 11, 12, 13, &c — — — 21. And the series produced by this change in the signs of the terms of the simple series Λ is the simple series Π .

Therefore the simple series Π will be $=$ the compound series Δ^m .

But because the simple series Δ is $= \sqrt[n]{1-x}$, it follows that the compound series Δ^m must be $=$ the m th power of $\sqrt[n]{1-x}$, or $=$ the quantity $\sqrt[n]{1-x}^m$.

Therefore the simple series Π will be equal the quantity $\sqrt[n]{1-x}^m$. Q. E. D.

32. We have now demonstrated in a manner that, I hope, will be thought satisfactory, 1st, that the quantity $\sqrt[n]{1-x}$, or the n th root of the residual quantity $1-x$, is equal to the infinite series $1 - \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 - \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 - \frac{4n-1}{5n} Ex^5 - \&c \text{ ad infinitum}$, in which all the terms that come after the first term 1, are marked with the sign $-$, or subtracted from

the said first term; and secondly that the quantity $\sqrt[n]{1-x}^m$, or the m th power of the n th root of the said residual quantity, is equal to a simple series (which we have called Π) consisting of the very same terms as the series (which we have

called Λ) that is equal to the quantity $\sqrt[n]{1+x}^m$, or to the same m th power of the same n th root of the binomial quantity $1+x$, and derived from the said series Λ by changing the signs of those terms of it which involve the odd powers of x , or the signs of its second, fourth, sixth, and other following even terms, into their contraries. Nothing therefore seems now to remain to be done with respect to this subject, but to illustrate these two serieses by applying them to a few particular cases, or examples, in the same manner as we illustrated the se-

rieses which are equal to the quantities $\sqrt[n]{1+x}$ and $\sqrt[n]{1-x}$ in the beginning of the foregoing discourse concerning the binomial theorem in art. 8, 9, 10, 11, 12, &c. — — — 19 of the said discourse, pages 201, 202, 203, &c. — — — 209. These examples may be as follows.

Examples of the extraction of some particular roots of the residual quantity $1-x$ by means of the series given above in art. 2.

33. In the first place we will extract the square root of the residual quantity $1-x$ by means of the series given in art. 2, to wit, the series $1 - \frac{1}{n} Ax - \frac{n-1}{2n}$

$$\sqrt{\frac{n-1}{2n}} Bx^2 - \sqrt{\frac{2n-1}{3n}} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 - \sqrt{\frac{4n-1}{5n}} Ex^5 - \&c \text{ ad infinitum.}$$

Now in this case $\sqrt{1-x}^{\frac{1}{n}}$ is $= \sqrt{1-x}^{\frac{1}{2}}$, or n is $= 2$. Therefore $2n$ is $(= 2 \times 2) = 4$, and $3n$ is $(= 3 \times 2) = 6$, and $4n$ is $(= 4 \times 2) = 8$, and $5n$ is $(= 5 \times 2) = 10$, and consequently $n-1$ is $(= 2-1) = 1$, and $2n-1$ is $(= 4-1) = 3$, and $3n-1$ is $(= 6-1) = 5$, and $4n-1$ is $(= 8-1) = 7$. We shall therefore have $1 - \frac{1}{n} Ax - \sqrt{\frac{n-1}{2n}} Bx^2 - \sqrt{\frac{2n-1}{3n}} Cx^3 - \sqrt{\frac{3n-1}{4n}} Dx^4 - \sqrt{\frac{4n-1}{5n}} Ex^5 - \&c = 1 - \frac{1}{2} Ax - \frac{1}{4} Bx^2 - \frac{3}{6} Cx^3 - \frac{5}{8} Dx^4 - \frac{7}{10} Ex^5 - \&c = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \&c.$

Therefore, if the series given above in art. 2 is really equal to $\sqrt{1-x}^{\frac{1}{n}}$, the quantity $\sqrt{1-x}^{\frac{1}{2}}$, or the square-root of the residual quantity $1-x$, will be equal to the series $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \&c.$ Q. E. I.

34. Now "that this series is really equal to the square root of $1-x$," will appear by multiplying it into itself. For we shall find that the product of the said multiplication will be $1-x$. This multiplication may be performed in the manner following.

$$\begin{array}{r}
 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \&c \\
 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \&c \\
 \hline
 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \&c \\
 \quad - \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{16} + \frac{x^4}{32} + \frac{5x^5}{256} + \&c \\
 \quad \quad - \frac{x^2}{8} + \frac{x^3}{16} + \frac{x^4}{64} + \frac{x^5}{128} + \&c \\
 \quad \quad \quad - \frac{x^3}{16} + \frac{x^4}{32} + \frac{x^5}{128} + \&c \\
 \quad \quad \quad \quad - \frac{5x^4}{128} + \frac{5x^5}{256} + \&c \\
 \quad \quad \quad \quad \quad - \frac{7x^5}{256} + \&c \\
 \hline
 1 - x \quad * \quad * \quad * \quad * \quad \&c.
 \end{array}$$

It appears therefore that the series $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \&c$, is really and truly, as far as relates to the six first terms of it, the square-root of the residual quantity $1-x$; and consequently that the series given above in

art. 2 for the value of $1 - x^{\frac{1}{n}}$, or $\sqrt[n]{1 - x}$, is true in the case of square-roots.

35. In the next place we will investigate the value of $\sqrt[3]{1 - x}$, or the cube-root of the residual quantity $1 - x$, by means of the same series $1 - \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 - \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 - \frac{4n-1}{5n} Ex^5 - \&c$ *ad infinitum*.

Now in this case n is $= 3$; and consequently we shall have $2n = 6$, and $3n = 9$, and $4n = 12$, and $5n = 15$, and $n - 1 = 2$, $2n - 1 = 5$, $3n - 1 = 8$, and $4n - 1 = 11$. And therefore the series $1 - \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 - \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 - \frac{4n-1}{5n} Ex^5 - \&c$, will in this case be $= 1 - \frac{1}{3} Ax - \frac{2}{6} Bx^2 - \frac{5}{9} Cx^3 - \frac{8}{12} Dx^4 - \frac{11}{15} Ex^5 - \&c = 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \&c$. Q. E. I.

36. Now "that this series is really equal to the cube root of $1 - x$," will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be equal to $1 - x$. These multiplications will be as follow.

$$\begin{array}{r}
 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \&c \\
 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \&c \\
 \hline
 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \&c \\
 - \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \frac{5x^4}{243} + \frac{10x^5}{729} + \&c \\
 - \frac{x^2}{9} + \frac{x^3}{27} + \frac{x^4}{81} + \frac{5x^5}{729} + \&c \\
 - \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{5x^5}{729} + \&c \\
 - \frac{10x^4}{243} + \frac{10x^5}{729} + \&c \\
 - \frac{22x^5}{729} + \&c \\
 \hline
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c.
 \end{array}$$

$$1 - \frac{2x}{3}$$

$$\begin{array}{r}
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c \\
 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \&c \\
 \hline
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c \\
 - \frac{x}{3} + \frac{2x^2}{9} + \frac{x^3}{27} + \frac{4x^4}{243} + \frac{7x^5}{729} + \&c \\
 - \frac{x^2}{9} + \frac{2x^3}{27} + \frac{x^4}{81} + \frac{4x^5}{729} + \&c \\
 - \frac{5x^3}{81} + \frac{10x^4}{243} + \frac{5x^5}{729} + \&c \\
 - \frac{10x^4}{243} + \frac{20x^5}{729} + \&c \\
 - \frac{22x^5}{729} + \&c \\
 \hline
 1 - x \quad * \quad * \quad * \quad * \quad \&c.
 \end{array}$$

It appears therefore that the series $1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \&c$, is really and truly, as far as relates to the first six terms of it, the cube-root of the residual quantity $1 - x$, and consequently that the series given above in art. 2 for the value of $\sqrt[n]{1 - x}$, or $\sqrt[n]{1 - x}$, is true in the case of cube-roots as well as in that of square-roots.

Examples of the extraction of the roots of some particular powers of the residual quantity $1 - x$ by means of the series given above in art. 25, and which in art. 29, 30, and 31 is denoted by the Greek capital letter Π .

37. In the next place we will investigate the value of the quantity $\sqrt[3]{1 - x^2}$, or the cube-root of the square of the residual quantity $1 - x$, or (which comes to the same thing) the square of its cube-root, by means of the series given above in art. 25, and denoted by the Greek capital letter Π in art. 29, 30, and 31.

Now it is shewn in the foregoing discourse, art. 12, page 203, that $\sqrt[3]{1 + x^2}$, or the cube-root of the square of the binomial quantity $1 + x$, is equal to the series $1 + \frac{2x}{3} - \frac{x^2}{9} + \frac{4x^3}{81} - \frac{7x^4}{243} + \frac{14x^5}{729} - \&c$, *ad infinitum*. Therefore the quantity $\sqrt[3]{1 - x^2}$ must, according to art. 25, be equal to a series derived from this

3 B 2

this series by changing the signs of its second, fourth, sixth, and other following even terms into their contraries, that is, to the series $1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c$, in which all the terms after the first term 1 are marked with the sign $-$, or are subtracted from the said first term. Therefore, if the proposition contained in art. 25, is true, and the series just now set down, and which, in the foregoing articles, is denoted by Π , is really equal to $\sqrt[n]{1 - x^m}$, the quantity $\sqrt[2]{1 - x}$, or the cube-root of the square of the residual quantity $1 - x$, will be equal to the series $1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c$.

38. Now "that this series is really equal to $\sqrt[2]{1 - x}$, or to the cube root of "the square of the residual quantity $1 - x$, or to the cube-root of the trinomial "quantity $1 - 2x + xx$," (for both the quantities $1 + 2x + xx$ and $1 - 2x + xx$ are equally called *trinomial* quantities, though in the latter quantity the middle term $2x$ is marked with the sign $-$, or subtracted from the sum of the two other terms) will appear by multiplying the said series twice into itself. These multiplications will be as follow.

$$\begin{array}{r}
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c \\
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c \\
 \hline
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c \\
 - \frac{2x}{3} + \frac{4x^2}{9} + \frac{2x^3}{27} + \frac{8x^4}{243} + \frac{14x^5}{729} + \&c \\
 - \frac{x^2}{9} + \frac{2x^3}{27} + \frac{x^4}{81} + \frac{4x^5}{729} + \&c \\
 - \frac{4x^3}{81} + \frac{8x^4}{243} + \frac{4x^5}{729} + \&c \\
 - \frac{7x^4}{243} + \frac{14x^5}{729} + \&c \\
 - \frac{14x^5}{729} + \&c \\
 \hline
 1 - \frac{4x}{3} + \frac{2x^2}{9} + \frac{4x^3}{81} + \frac{5x^4}{243} + \frac{8x^5}{729} + \&c.
 \end{array}$$

$$1 - \frac{4x}{3}$$

$$\begin{array}{r}
 1 - \frac{4x}{3} + \frac{2x^2}{9} + \frac{4x^3}{81} + \frac{5x^4}{243} + \frac{8x^5}{729} \quad \&c \\
 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} \quad \&c \\
 \hline
 1 - \frac{4x}{3} + \frac{2x^2}{9} + \frac{4x^3}{81} + \frac{5x^4}{243} + \frac{8x^5}{729} \quad \&c \\
 - \frac{2x}{3} + \frac{8x^2}{9} - \frac{4x^3}{27} - \frac{243}{81x^4} - \frac{10x^5}{729} \quad \&c \\
 \hline
 - \frac{x^2}{9} + \frac{4x^3}{27} - \frac{2x^4}{81} - \frac{4x^5}{729} \quad \&c \\
 - \frac{4x^3}{81} + \frac{16x^4}{243} - \frac{8x^5}{729} \quad \&c \\
 - \frac{7x^4}{243} + \frac{28x^5}{729} \quad \&c \\
 - \frac{14x^5}{729} + \&c \\
 \hline
 1 - 2x + xx \quad * \quad * \quad * \quad \&c.
 \end{array}$$

It appears therefore that the series $1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} - \&c$ is really and truly the cube-root of the trinomial quantity $1 - 2x + xx$, or of the square of the residual quantity $1 - x$, and consequently that the proposition contained in art. 25 is true, or that the series Π is really equal to the quantity

$\sqrt[n]{1 - x}^m$, in the case of the cube-root of the square of a residual quantity, or when m , the numerator of the fraction $\frac{m}{n}$ (which is the index of the power of

$1 - x$ in the quantity $\sqrt[n]{1 - x}^m$) is $= 2$, and n , the denominator of the said fraction, is $= 3$.

39. As another example of the investigation of the value of $\sqrt[n]{1 - x}^m$ by means of the series Π , we will suppose m to be $= 3$, and n to be $= 5$, or

$\sqrt[n]{1 - x}^m$ to be equal to $\sqrt[5]{1 - x}^3$, or to the fifth root of the cube of the residual quantity $1 - x$, or to the fifth root of the quadrinomial quantity $1 - 3x + 3x^2 - x^3$.

Now it appears by art. 14 of the foregoing discourse concerning the binomial theorem, page 205, that the quantity $\sqrt[5]{1 + x}^3$ is $=$ the series $1 + \frac{3x}{5} - \frac{3x^2}{25}$

$+ \frac{7x^3}{125} - \frac{21x^4}{625} + \frac{357x^5}{15625} - \&c$. Therefore the series Π will be $1 - \frac{3x}{5} - \frac{3x^2}{25}$

$- \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c$, and consequently, by art. 25, the quantity $\sqrt[5]{1 - x}^3$ will be equal to the series $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c$.

40. Now

40. Now "that this series is really equal to the quantity $\sqrt[3]{1-x}$, or to the "fifth root of the cube of the residual quantity $1-x$, or to the fifth root of the "quadrinomial quantity $1-3x+3x^2-x^3$," will appear by raising the said series to the fifth power by multiplying it, first, into itself, whereby we shall obtain its square, and afterwards multiplying the said square into itself, whereby we shall obtain its fourth power, and, lastly, multiplying the said fourth power of it into the said series itself, whereby we shall obtain its fifth power. For we shall find that the product of these three multiplications will be the said quadrinomial quantity $1-3x+3x^2-x^3$. These multiplications will be as follow.

$$\begin{array}{r}
 1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c \\
 1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c \\
 \hline
 1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c \\
 \quad - \frac{3x}{5} + \frac{9x^2}{25} + \frac{9x^3}{125} + \frac{21x^4}{625} + \frac{63x^5}{3125} + \&c \\
 \quad \quad - \frac{3x^2}{25} + \frac{9x^3}{125} + \frac{9x^4}{625} + \frac{21x^5}{3125} + \&c \\
 \quad \quad \quad - \frac{7x^3}{125} + \frac{21x^4}{625} + \frac{21x^5}{3125} + \&c \\
 \quad \quad \quad \quad - \frac{21x^4}{625} + \frac{63x^5}{3125} + \&c \\
 \quad \quad \quad \quad \quad - \frac{357x^5}{15625} + \&c \\
 \hline
 1 - \frac{6x}{5} + \frac{3x^2}{25} + \frac{4x^3}{125} + \frac{9x^4}{625} + \frac{126x^5}{15625} \&c. \\
 \\
 1 - \frac{6x}{5} + \frac{3x^2}{25} + \frac{4x^3}{125} + \frac{9x^4}{625} + \frac{126x^5}{15625} \&c \\
 1 - \frac{6x}{5} + \frac{3x^2}{25} + \frac{4x^3}{125} + \frac{9x^4}{625} + \frac{126x^5}{15625} \&c \\
 \hline
 1 - \frac{6x}{5} + \frac{3x^2}{25} + \frac{4x^3}{125} + \frac{9x^4}{625} + \frac{126x^5}{15625} \&c \\
 \quad - \frac{6x}{5} + \frac{36x^2}{25} - \frac{18x^3}{125} - \frac{24x^4}{625} - \frac{54x^5}{3125} - \&c \\
 \quad \quad + \frac{3x^2}{25} - \frac{18x^3}{125} + \frac{9x^4}{625} + \frac{12x^5}{3125} + \&c \\
 \quad \quad \quad + \frac{4x^3}{125} - \frac{24x^4}{625} + \frac{12x^5}{3125} + \&c \\
 \quad \quad \quad \quad + \frac{9x^4}{625} - \frac{54x^5}{3125} + \&c \\
 \quad \quad \quad \quad \quad + \frac{126x^5}{15625} - \&c \\
 \hline
 1 - \frac{12x}{5} + \frac{42x^2}{25} - \frac{28x^3}{125} - \frac{21x^4}{625} - \frac{168x^5}{15625} \&c.
 \end{array}$$

This

This is the fourth power of the series $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c.$

$$\begin{array}{r}
 1 - \frac{12x}{5} + \frac{42x^2}{25} - \frac{28x^3}{125} - \frac{21x^4}{625} - \frac{168x^5}{15625} - \&c \\
 1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c \\
 \hline
 1 - \frac{12x}{5} + \frac{42x^2}{25} - \frac{28x^3}{125} - \frac{21x^4}{625} - \frac{168x^5}{15625} - \&c \\
 - \frac{3x}{5} + \frac{36x^2}{25} - \frac{126x^3}{125} + \frac{84x^4}{625} + \frac{63x^5}{3125} + \&c \\
 - \frac{3x^2}{25} + \frac{36x^3}{125} - \frac{126x^4}{625} + \frac{84x^5}{3125} + \&c \\
 - \frac{7x^3}{125} + \frac{84x^4}{625} - \frac{294x^5}{3125} + \&c \\
 - \frac{21x^4}{625} + \frac{252x^5}{3125} - \&c \\
 - \frac{357x^5}{15625} + \&c \\
 \hline
 1 - 3x + 3x^2 - x^3 \quad * \quad *
 \end{array}$$

It appears therefore that the series $1 - \frac{3x}{5} - \frac{3x^2}{25} - \frac{7x^3}{125} - \frac{21x^4}{625} - \frac{357x^5}{15625} - \&c$, is really and truly the fifth root of the quadrinomial quantity $1 - 3x + 3x^2 - x^3$, or of the cube of the residual quantity $1 - x$, and consequently that the

residual theorem laid down in art. 25, to wit, that the quantity $\sqrt[n]{1 - x}$ is equal to the series Π , is true in the case of the fifth root of the cube of a residual quantity, or when m , the numerator of the fraction $\frac{m}{n}$ (which is the index of

the power of $1 - x$ in the quantity $\sqrt[n]{1 - x}$) is $= 3$, and n , the denominator of the said fraction, is $= 5$.

41. In the foregoing examples the denominator n of the fraction $\frac{m}{n}$ (which is the index of the power of the residual quantity $1 - x$ in the quantity $\sqrt[n]{1 - x}$) has been greater than m , the numerator of the said fraction. We will now give an example, or two, of the expression of the value of the quantity $\sqrt[n]{1 - x}$ by means of the series Π , when the denominator n of the fraction, or index, $\frac{m}{n}$, is less than its numerator m .

And, first, we will investigate the value of $\sqrt[3]{1 - x}$, or of the square-root of the cube of the residual quantity $1 - x$ by means of the said series Π .

Now it appears by art. 16 of the foregoing discourse concerning the binomial theorem,

theorem, page 207, that the quantity $\sqrt[3]{1+x}$ is equal to the series $1 + \frac{3x}{2} + \frac{3x^2}{8} - \frac{x^3}{16} + \frac{3x^4}{128} - \frac{3x^5}{256} + \&c.$ Therefore the series Π will be $1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + \&c.$ and consequently, if the proposition contained in art. 25 is true, $\sqrt[3]{1-x}$, or the square-root of the cube of the residual quantity $1-x$, or the square-root of the quadrinomial quantity $1-3x+3x^2-x^3$, will be equal to the series $1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + \&c.$

42. Now "that this series is really equal to the square-root of the cube of $1-x$, or to the square-root of the quadrinomial quantity $1-3x+3x^2-x^3$," will appear by multiplying the said series into itself. For we shall find that the product of the said multiplication will be the said quadrinomial quantity. This multiplication will be as follows.

$$\begin{array}{r}
 1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + \&c \\
 1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + \&c \\
 \hline
 1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + \&c \\
 - \frac{3x}{2} + \frac{9x^2}{4} - \frac{9x^3}{16} - \frac{3x^4}{32} - \frac{9x^5}{256} - \&c \\
 + \frac{3x^2}{8} - \frac{9x^3}{16} + \frac{9x^4}{64} + \frac{3x^5}{128} + \&c \\
 + \frac{x^3}{16} - \frac{3x^4}{32} + \frac{3x^5}{128} + \&c \\
 + \frac{3x^4}{128} - \frac{9x^5}{256} + \&c \\
 + \frac{3x^5}{256} - \&c \\
 \hline
 1 - 3x + 3x^2 - x^3 \quad * \quad *
 \end{array}$$

It appears therefore that the series $1 - \frac{3x}{2} + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{3x^4}{128} + \frac{3x^5}{256} + \&c$ is really and truly the square-root of the quadrinomial quantity $1-3x+3x^2-x^3$, or of the cube of the residual quantity $1-x$, and consequently that the

residual theorem laid down in art. 25, to wit, that the quantity $\sqrt[3]{1-x}$ is equal to the series Π , is true in the case of the square-root of the cube of the residual quantity $1-x$, or when m , the numerator of the fraction $\frac{m}{n}$ (which is

the index of the power of $1-x$ in the quantity $\sqrt[3]{1-x}$) is $= 3$, and n , the denominator of the said fraction, is $= 2$.

43. I shall add one more example of the series Π , in which the numerator m of the index $\frac{m}{n}$ shall be greater than its denominator n .

Let it be required to find, by means of the said series Π , the value of $\sqrt[5]{1-x}$, or of the cube-root of the fifth power of the residual quantity $1-x$, or of the cube-root of the sextinomial quantity $1-5x+10x^2-10x^3+5x^4-x^5$.

It appears by art. 18 of the foregoing discourse concerning the binomial theorem, page 208, that the quantity $\sqrt[5]{1+x}$ is equal to the series $1 + \frac{5x}{3} + \frac{5x^2}{9} - \frac{5x^3}{81} + \frac{5x^4}{243} - \frac{7x^5}{729} + \&c.$

Therefore the series Π will be $1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c.$

and consequently $\sqrt[5]{1-x}$, or the cube-root of the fifth power of the residual quantity $1-x$, or the cube-root of the sextinomial quantity $1-5x+10x^2-10x^3+5x^4-x^5$, will be equal to the series $1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c.$

44. Now "that this series is really equal to the cube-root of the fifth power of $1-x$, or to the cube-root of the sextinomial quantity $1-5x+10x^2-10x^3+5x^4-x^5$," will appear by multiplying the said series twice into itself. For we shall find that the product of the said multiplications will be the said sextinomial quantity. These multiplications will be as follow.

$$\begin{array}{r}
 1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c \\
 1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c \\
 \hline
 1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c \\
 - \frac{5x}{3} + \frac{25x^2}{9} - \frac{25x^3}{27} - \frac{25x^4}{243} - \frac{25x^5}{729} - \&c \\
 + \frac{5x^2}{9} - \frac{25x^3}{27} + \frac{25x^4}{81} + \frac{25x^5}{729} + \&c \\
 + \frac{5x^3}{81} - \frac{25x^4}{243} + \frac{25x^5}{729} + \&c \\
 + \frac{5x^4}{243} - \frac{25x^5}{729} + \&c \\
 + \frac{7x^5}{729} - \&c \\
 \hline
 1 - \frac{10x}{3} + \frac{35x^2}{9} - \frac{140x^3}{81} + \frac{35x^4}{243} + \frac{14x^5}{729} \&c.
 \end{array}$$

$$\begin{array}{r}
 1 - \frac{10x}{3} + \frac{35x^2}{9} - \frac{140x^3}{81} + \frac{35x^4}{243} + \frac{14x^5}{729} \quad \&c \\
 1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c \\
 \hline
 1 - \frac{10x}{3} + \frac{35x^2}{9} - \frac{140x^3}{81} + \frac{35x^4}{243} + \frac{14x^5}{729} \quad \&c \\
 - \frac{5x}{3} + \frac{50x^2}{9} - \frac{175x^3}{27} + \frac{700x^4}{243} - \frac{175x^5}{729} - \&c \\
 + \frac{5x^2}{9} - \frac{50x^3}{27} + \frac{175x^4}{81} - \frac{700x^5}{729} + \&c \\
 + \frac{5x^3}{81} - \frac{50x^4}{243} + \frac{175x^5}{729} - \&c \\
 + \frac{5x^4}{243} - \frac{50x^5}{729} + \&c \\
 + \frac{7x^5}{729} - \&c \\
 \hline
 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5 \quad \&c.
 \end{array}$$

It appears therefore that the series $1 - \frac{5x}{3} + \frac{5x^2}{9} + \frac{5x^3}{81} + \frac{5x^4}{243} + \frac{7x^5}{729} + \&c$ is really and truly the cube-root of the fifth power of the residual quantity $1 - x$, or the cube-root of the sextinomial quantity $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$, and consequently that the residual theorem laid down in art. 25, to wit, that

the quantity $\sqrt[n]{1 - x^{\frac{m}{n}}}$ is equal to the series Π , is true in the case of the cube-root of the fifth power of the residual quantity $1 - x$, or when m , the numerator of the fraction $\frac{m}{n}$ (which is the index of the power of $1 - x$ in the quantity

$\sqrt[n]{1 - x^{\frac{m}{n}}}$) is $= 5$, and n , the denominator of the said fraction, is $= 3$.

45. These examples will, I apprehend, be sufficient to illustrate the two theorems delivered above in art. 2 and 25 of this discourse concerning the

quantities $\sqrt[n]{1 - x^{\frac{1}{n}}}$ and $\sqrt[n]{1 - x^{\frac{m}{n}}}$, or the n th root of the residual quantity $1 - x$ and the m th power of the said n th root. And therefore, having already given demonstrations of these theorems in art. 11, 12, 13, &c. . . . and 26, 27, 30, 31, 32, &c, I shall here put an end to this discourse.

End of the discourse concerning the residual theorem in the case of fractional powers.

A
M E T H O D
OF EXTENDING
C A R D A N ' s R U L E

FOR RESOLVING THE CUBICK EQUATION

$$y^3 + qy = r, \text{ or } qy + y^3 = r,$$

TO THE RESOLUTION OF THE CUBICK EQUATION

$$qy - y^3 = r, \text{ when } \frac{rr}{4} \text{ is of any magnitude less than } \frac{q^3}{54},$$

$$\text{or } \frac{1}{2} \times \frac{q^3}{27}, \text{ or when } r \text{ is less than } \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}};$$

*By the Help of Sir Isaac Newton's binomial and residual Theorems in the Case of
Roots, which have been demonstrated in the two preceeding Discourses.*

BY FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

M. E. T. H. O. D.

OF THE

C. A. R. D. A. N. S. R. U. L. E.

BY FRANCIS MARRIAGE, Esq. T. R. S.

CHURCH OF HIS MAJESTY'S COURT OF RECORDS.

M E T H O D

OF EXTENDING

C A R D A N's R U L E, &c.

ART. I. **T**HE binomial and residual theorems, which have been demonstrated in the preceeding tracts both with respect to integral and to fractional powers of the quantities $1 + x$ and $1 - x$, are of very extensive use in many other branches of the mathematicks as well as in the construction of the logarithms, or measures, of ratios. And, amongst other subjects, they may be applied to the resolution of cubick equations in which the square of the unknown quantity is wanting, to wit, of the equations $y^3 + qy = r$, and $y^3 - qy = r$, and $qy - y^3 = r$, so as to enable us to find the values of y , in all the possible cases of these equations, or in all the different relative magnitudes of the co-efficient q , and the absolute term r , that can be supposed, not excepting that case of the second equation $y^3 - qy = r$ in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or rr is less than $\frac{4q^3}{27}$, or r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and which (from its not being capable of a direct and immediate resolution by the help of Cardan's rule for resolving the said equation $y^3 - qy = r$, in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or rr is greater than $\frac{4q^3}{27}$, or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) has obtained amongst algebraists the name of *the irreducible case*. Even in this case we may always, by the help of Sir Isaac Newton's binomial and residual theorems in the case of roots (which have been investigated in the two preceeding discourses) derive from one or other, of Cardan's rules an expression of the true value of y in the said equation $y^3 - qy = r$. The method of doing this, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, has been explained at considerable length in a paper of mine

mine published in the Philosophical Transactions for the year 1778, which proceeds upon clear and intelligible principles, without any mention of impossible roots or impossible quantities of any kind, or even of negative quantities. But when $\frac{rr}{4}$ is less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, or rr is less than $(\frac{4q^3}{2 \times 27}, \text{ or } \frac{2q^3}{27})$, or r is less than $\sqrt[3]{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, the method explained in that paper will not enable us to find the value of y in that equation; because the series obtained in that paper for the said value will not in that case be a converging series. Nor had I, at the time of publishing that paper, discovered any method of deriving from Cardan's rules an expression of the value of y in the said equation $y^3 - qy = r$ in this second branch of the irreducible case of it, or when $\frac{rr}{4}$ was less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, or r was less than $\sqrt[3]{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. But I have since found out a way of doing it by the help of Cardan's first rule, which gives us the value of y in the equation $y^3 + qy = r$, or $qy + y^3 = r$. For from the expression of this value of y in the said equation $qy + y^3 = r$ we may, in the case supposed, or when $\frac{rr}{4}$ is less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, or r is less than $\sqrt[3]{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, derive an expression of the value of the lesser of the two roots of the equation $qy - y^3 = r$, by the help of the two serieses for expressing the cube-roots of the binomial quantity $1 + x$ and the residual quantity $1 - x$ obtained by means of the binomial and residual theorems: and from the value of the said lesser root of the equation $qy - y^3 = r$, we may afterwards derive the value of y in the equation $y^3 - qy = r$ by the resolution of a quadratick equation. And this extension of Cardan's first rule (which gives us the value of y in the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$), to the discovery of the lesser root of the cubick equation $qy - y^3 = r$, when r is less than $\sqrt[3]{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, may be made in a clear and intelligible manner, without any mention of impossible roots, or other impossible quantities, or even of negative quantities, as well as the former extension of Cardan's second rule (which gives us the value of y in the equation $y^3 - qy = r$ in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) to the discovery of the value of y in the first branch of the second case of that equation, or when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$. The manner of making this extension of Cardan's first rule to the discovery of the lesser root of the equation $qy - y^3 = r$, I shall now endeavour to explain.

The values of the quantities $\sqrt[3]{1+x}$ and $\sqrt[3]{1-x}$, or $\sqrt[3]{1+x}$ and $\sqrt[3]{1-x}$, expressed in infinite serieses by means of the binomial and residual theorems.

2. It has been shewn above in the discourse concerning the binomial theorem in the case of fractional powers, art. 51, page 242, that $\sqrt[3]{1+x}$, or $\sqrt[3]{1+x}$, or the cube-root of the binomial quantity $1+x$, is equal to the infinite series

$$1 + \frac{1}{3}Ax - \frac{2}{6}Bx^2 + \frac{5}{9}Cx^3 - \frac{8}{12}Dx^4 + \frac{11}{15}Ex^5 - \frac{14}{18}Fx^6 + \frac{17}{21}Gx^7 - \frac{20}{24}Hx^8 + \frac{23}{27}Ix^9 - \frac{26}{30}Kx^{10} + \frac{29}{33}Lx^{11} - \frac{32}{36}Mx^{12} + \frac{35}{39}Nx^{13} - \frac{38}{42}Ox^{14} + \frac{41}{45}Px^{15} - \frac{44}{48}Qx^{16} + \frac{47}{51}Rx^{17} - \frac{50}{54}Sx^{18} + \&c \text{ ad infinitum, or}$$

$$1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \frac{22x^5}{729} - \frac{154x^6}{6561} + \frac{374x^7}{19,683} - \frac{935x^8}{59,049} + \frac{21,505x^9}{1,594,323} - \frac{55,913x^{10}}{4,782,969} + \frac{147,407x^{11}}{14,348,907} - \frac{1,179,256x^{12}}{129,140,163} + \frac{3,174,920x^{13}}{387,420,489} - \frac{8,617,640x^{14}}{1,162,261,467} + \frac{70,664,648x^{15}}{10,460,353,203} - \frac{194,327,782x^{16}}{31,381,059,609} + \frac{537,259,162x^{17}}{94,143,178,827} - \frac{13,431,479,050x^{18}}{2,541,865,828,329} + \&c \text{ ad infinitum.}$$

Therefore (by what is shewn in the foregoing discourse concerning the residual theorem in the case of fractional powers, art. 23, page 360) the quantity

$\sqrt[3]{1-x}$, or $\sqrt[3]{1-x}$, or the cube-root of the residual quantity $1-x$, will be equal to a series consisting of the very same terms as the foregoing series, but with the sign $-$ prefixed to every term after the first term 1 , instead of every other term, that is, to the series

$$1 - \frac{1}{3}Ax - \frac{2}{6}Bx^2 - \frac{5}{9}Cx^3 - \frac{8}{12}Dx^4 - \frac{11}{15}Ex^5 - \frac{14}{18}Fx^6 - \frac{17}{21}Gx^7 - \frac{20}{24}Hx^8 - \frac{23}{27}Ix^9 - \frac{26}{30}Kx^{10} - \frac{29}{33}Lx^{11} - \frac{32}{36}Mx^{12} - \frac{35}{39}Nx^{13} - \frac{38}{42}Ox^{14} - \frac{41}{45}Px^{15} - \frac{44}{48}Qx^{16} - \frac{47}{51}Rx^{17} - \frac{50}{54}Sx^{18} - \&c \text{ ad infinitum, or}$$

$$1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \frac{22x^5}{729} - \frac{154x^6}{6561} - \frac{374x^7}{19,683} - \frac{935x^8}{59,049} - \frac{21,505x^9}{1,594,323} - \frac{55,913x^{10}}{4,782,969} - \frac{147,407x^{11}}{14,348,907} - \frac{1,179,256x^{12}}{129,140,163} - \frac{3,174,920x^{13}}{387,420,489} - \frac{8,617,640x^{14}}{1,162,261,467} - \frac{70,664,648x^{15}}{10,460,353,203} - \frac{194,327,782x^{16}}{31,381,059,609} - \frac{537,259,162x^{17}}{94,143,178,827} - \frac{13,431,479,050x^{18}}{2,541,865,828,329} - \&c.$$

The value of $\sqrt[3]{1+x} - \sqrt[3]{1-x}$ expressed in an infinite series.

3. Therefore the difference of the two quantities $\sqrt[3]{1+x}$ and $\sqrt[3]{1-x}$, or the excess of $\sqrt[3]{1+x}$, (or the cube-root of the binomial quantity $1+x$), above $\sqrt[3]{1-x}$, (or the cube root of the residual quantity $1-x$, will be) equal to the excess of the former of these two series above the latter, that is (if, for the sake of brevity, we denote the co-efficients of the second, and third, and fourth, and other following terms of these two series by the single letters B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, T, &c), to the excess of the series $1 + Bx - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - Ix^8 + Kx^9 - Lx^{10} + Mx^{11} - Nx^{12} + Ox^{13} - Px^{14} + Qx^{15} - Rx^{16} + Sx^{17} - Tx^{18} + \&c$ above the series $1 - Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - Gx^6 - Hx^7 - Ix^8 - Kx^9 - Lx^{10} - Mx^{11} - Nx^{12} - Ox^{13} - Px^{14} - Qx^{15} - Rx^{16} - Sx^{17} - Tx^{18} - \&c$, and consequently to the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + 2Ox^{13} + 2Qx^{15} + 2Sx^{17} + \&c$.

Of the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, according to Cardan's first rule.

4. The first of the two rules for the resolution of certain cubick equations usually known by the name of *Cardan's rules* was not invented by Cardan himself, but, about 30 years before the publication of his book, by one *Scipio Ferreus*, of *Bononia*, or *Bologna*, in Italy; as Cardan himself informs us. It relates to the resolution of the cubick equation $y^3 + qy = r$, and is true in all the possible cases of that equation, or in all the different relative magnitudes of q (the co-efficient of the unknown quantity y), and the absolute term r , and consequently when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, which is the case that we are about to consider. And it gives us two different expressions of the value of y in the said equation, to wit,

1st, $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}} + \frac{-q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}}}$, and, 2dly, $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}} - \sqrt[3]{-\frac{r}{2} + \sqrt{\frac{q^3}{27} + \frac{rr}{4}}}$, or (if, for the sake of brevity, we put $ee = \frac{rr}{4}$, and $ss = \frac{q^3}{27} + \frac{rr}{4}$) 1st, $\sqrt[3]{e+s} + \frac{-q}{3\sqrt[3]{e+s}}$, and, 2dly,

2dly, $\sqrt[3]{e+s} - \sqrt[3]{-e+s}$, or, rather, (because ss is greater than $\frac{rr}{4}$, or ee , and consequently s is greater than $\frac{r}{2}$, or e , and therefore ought to be

placed before it) 1st, $\sqrt[3]{s+e} - \frac{q}{3\sqrt[3]{s+e}}$, and, 2dly, $\sqrt[3]{s+e} - \sqrt[3]{s-e}$. See my dissertation on the use of the negative sign in Algebra, art. 208, 209, 210, 211, 212, pages 178, 179, 180.

It is the second of these two expressions, to wit, the expression $\sqrt[3]{s+e} - \sqrt[3]{s-e}$, that we shall have occasion to consider in the course of the following articles.

The value of the root of the said cubick equation expressed by an infinite series by means of the binomial and residual theorems.

5. Now $s+e$ is $= s \times \sqrt[3]{1+\frac{e}{s}}$, and $s-e$ is $= s \times \sqrt[3]{1-\frac{e}{s}}$. Therefore $\sqrt[3]{s+e}$ will be $= \sqrt[3]{s} \times \sqrt[3]{1+\frac{e}{s}}$, and $\sqrt[3]{s-e}$ will be $= \sqrt[3]{s} \times \sqrt[3]{1-\frac{e}{s}}$; and consequently $\sqrt[3]{s+e} - \sqrt[3]{s-e}$ will be $= \sqrt[3]{s} \times \sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{s} \times \sqrt[3]{1-\frac{e}{s}} = \sqrt[3]{s} \times \left(\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}} \right)$. Therefore the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, will be $= \sqrt[3]{s} \times \left(\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}} \right)$.

But, since $\sqrt[3]{1+x} - \sqrt[3]{1-x}$ is equal to the series $2Bx + 2Dx^3 + 2Fx^5 + 2Hx^7 + 2Kx^9 + 2Mx^{11} + 2Ox^{13} + 2Qx^{15} + 2Sx^{17} + \&c$ *ad infinitum* (as is shewn in art. 3) it follows (by substituting $\frac{e}{s}$ instead of x in the terms of the said equation) that $\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}}$ will be equal to the series $\frac{2Be}{s} + \frac{2De^3}{s^3} + \frac{2Fe^5}{s^5} + \frac{2He^7}{s^7} + \frac{2Ke^9}{s^9} + \frac{2Me^{11}}{s^{11}} + \frac{2Oe^{13}}{s^{13}} + \frac{2Qe^{15}}{s^{15}} + \frac{2Se^{17}}{s^{17}} + \&c$ *ad infinitum*. Therefore $\sqrt[3]{s} \times \left(\sqrt[3]{1+\frac{e}{s}} - \sqrt[3]{1-\frac{e}{s}} \right)$ will be equal to $\sqrt[3]{s} \times$ the series $\frac{2Be}{s} + \frac{2De^3}{s^3} + \frac{2Fe^5}{s^5} + \frac{2He^7}{s^7} + \frac{2Ke^9}{s^9} + \frac{2Me^{11}}{s^{11}} + \frac{2Oe^{13}}{s^{13}} + \frac{2Qe^{15}}{s^{15}} + \frac{2Se^{17}}{s^{17}} + \&c$ *ad infinitum*; and consequently the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, will be equal to $\sqrt[3]{s} \times$ the said series $\frac{2Be}{s} + \frac{2De^3}{s^3} + \frac{2Fe^5}{s^5} + \frac{2He^7}{s^7} + \frac{2Ke^9}{s^9} + \frac{2Me^{11}}{s^{11}} + \frac{2Oe^{13}}{s^{13}} + \frac{2Qe^{15}}{s^{15}} + \frac{2Se^{17}}{s^{17}} + \&c$

ad infinitum, or to $2\sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \frac{Q e^{15}}{s^{15}} + \frac{S e^{17}}{s^{17}} + \&c \text{ ad infinitum}$, or (restoring the values of the co-efficients B, C, D, E, F, G, H, &c) $2\sqrt[3]{s} \times$ the series $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19,683s^7} + \frac{21,505e^9}{1,594,323s^9} + \frac{147,407e^{11}}{14,348,907s^{11}} + \frac{3,174,920e^{13}}{387,420,489s^{13}} + \frac{70,664,648e^{15}}{10,460,353,203s^{15}} + \frac{537,259,162e^{17}}{94,143,178,827s^{17}} + \&c \text{ ad infinitum}$.

We will now proceed to give an example of the resolution of a cubick equation of the said form $y^3 + qy = r$, or $qy + y^3 = r$, by means of this expression.

An example of the resolution of a cubick equation of the foregoing form $y^3 + qy = r$, or $qy + y^3 = r$, by means of the expression $2\sqrt[3]{s} \times$ the series $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19,683s^7} + \frac{21,505e^9}{1,594,323s^9} + \frac{147,407e^{11}}{14,348,907s^{11}} + \frac{3,174,920e^{13}}{387,420,489s^{13}} + \frac{70,664,648e^{15}}{10,460,353,203s^{15}} + \frac{537,259,162e^{17}}{94,143,178,827s^{17}} + \&c \text{ ad infinitum}$.

6. Let the equation that is to be resolved by means of this expression, be $y^3 + 15y = 4$, or $15y + y^3 = 4$.

Here q is $= 15$, and r is $= 4$; and consequently $\frac{q}{3}$ is $(= \frac{15}{3}) = 5$, and $\frac{r}{2}$ is $(= \frac{4}{2}) = 2$, and $\frac{q^3}{27}$ is $(= \frac{15^3}{27} = 5^3) = 125$, and $\frac{rr}{4}$ is $(= \frac{4 \times 4}{4} = 2^2) = 4$.

We shall therefore have $ss (= \frac{q^3}{27} + \frac{rr}{4} = 125 + 4) = 129$, and $s (= \sqrt[3]{129}) = 11.357,816,691$, and $\sqrt[3]{s} (= \sqrt[3]{11.357,816,691}) = 2.247,835$, and consequently $2\sqrt[3]{s} (= 2 \times 2.247,835) = 4.495,670$.

And we shall also have $e (= \frac{r}{2}) = 2$, and consequently $\frac{e}{s} (= \frac{2.000,000,000}{11.357,816,691}) = 0.176,090,181$, and $\frac{ee}{ss} (= \overline{0.176,090,181}^2) = 0.031,006,752$, and $\frac{e^3}{s^3} (= \frac{e}{s} \times \frac{e^2}{s^2} = 0.176,090,181 \times 0.031,006,752) = 0.005,459,984$, and $\frac{e^5}{s^5} (= \frac{e^3}{s^3} \times \frac{e^2}{s^2} = 0.005,459,984 \times 0.031,006,752) = 0.000,169,296$, and $\frac{e^7}{s^7} (= \frac{e^5}{s^5} \times \frac{e^2}{s^2} = 0.000,169,296 \times 0.031,006,752) = 0.000,005,249$, and $\frac{e^9}{s^9} (= \frac{e^7}{s^7} \times \frac{e^2}{s^2} = 0.000,005,249 \times 0.031,006,752) = 0.000,000,162$, and

and $\frac{e^{11}}{s^{11}} (= \frac{e^9}{s^9} \times \frac{e^2}{s^2} = 0.000,000,162 \times 0.031,006,752) = 0.000,000,005$,
 and $\frac{e^{13}}{s^{13}} (= \frac{e^{11}}{s^{11}} \times \frac{e^2}{s^2} = 0.000,000,005 \times 0.031,006,752) = 0.000,000,000$,
 and $\frac{e}{3s} (= \frac{1}{3} \times \frac{e}{s} = \frac{1}{3} \times 0.176,090,181) = 0.058,696,727$,
 and $\frac{5e^3}{81s^3} (= \frac{5}{81} \times \frac{e^3}{s^3} = \frac{5}{81} \times 0.005,459,984 = \frac{5 \times 0.005,459,984}{81} =$
 $\frac{0.027,299,920}{81}) = 0.000,337,036$, and $\frac{22e^5}{729s^5} (= \frac{22}{729} \times \frac{e^5}{s^5} = \frac{22}{729} \times 0.000,$
 $169,296 = \frac{22 \times 0.000,169,296}{729} = \frac{0.003,724,512}{729}) = 0.000,005,109$, and $\frac{374e^7}{19,683s^7}$
 $(= \frac{374}{19,683} \times \frac{e^7}{s^7} = \frac{374}{19,683} \times 0.000,005,249 = \frac{374 \times 0.000,005,249}{19,683} =$
 $\frac{0.001,963,126}{19,683}) = 0.000,000,099$, and $\frac{21,505e^9}{1,594,323s^9} (= \frac{21,505}{1,594,323} \times \frac{e^9}{s^9} = \frac{21,505}{1,594,323}$
 $\times 0.000,000,162 = \frac{21,505 \times 0.000,000,162}{1,594,323} = \frac{0.003,483,810}{1,594,323}) = 0.000,000,002$,
 and $\frac{147,407e^{11}}{14,348,907s^{11}} (= \frac{147,407}{14,348,907} \times \frac{e^{11}}{s^{11}} = \frac{147,407}{14,348,907} \times 0.000,000,005 =$
 $\frac{147,407 \times 0.000,000,005}{14,348,907} = \frac{0.000,737,035}{14,348,907}) = 0.000,000,000$. Therefore the series
 $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19,683s^7} + \frac{21,505e^9}{1,594,323s^9} + \frac{147,407e^{11}}{14,348,907s^{11}} + \&c$ will be
 $(= 0.058,696,727 + 0.000,337,036 + 0.000,005,109 + 0.000,000,099$
 $+ 0.000,000,002, + 0.000,000,000, + \&c) = 0.059,038,973$; and conse-
 quently $2\sqrt[3]{s}$ multiplied into the said series will be $= 2\sqrt[3]{s}$ multiplied into
 $0.059,038,973 = 4.495,670 \times 0.059,038,973 = 0.265,419,739$. There-
 fore y , or the root of the cubick equation $y^3 + 15y = 4$, or $15y + y^3 = 4$,
 will be $= 0.265,419,739$. Q. E. I.

7. This value of y approaches very nearly to the truth. For, if we suppose y to be equal to $0.265,419,739$, we shall have $yy = 0.070,447,637,850$, and $y^3 = 0.018,698,193$, and $15y (= 15 \times 0.265,419,739) = 3.981,296,085$, and consequently $y^3 + 15y (= 0.018,698,193 + 3.981,296,085) = 3.999,994,278$; which differs from the absolute term, 4, of the proposed equation $y^3 + 15y = 4$ by only $0.000,005,722$, or less than $0.000,006$, or $\frac{6}{1,000,000}$, or 6 millionth parts of an unit, or 6 four-millionth parts, or 3 two-millionth parts, of the said absolute term 4 itself.

8. If we were to make the value of y already found, to wit, $0.265,419,739$, the basis of a further approximation to its true value according to Mr. Raphson's method of resolving equations, by supposing y to be equal to $0.265,419,739 + z$, and substituting this compound quantity instead of y in the proposed equation $y^3 + 15y = 4$, and then resolving the equation resulting from such substitution in the same manner as a simple equation by omitting all the terms that involve either the square or cube of z , we should find that z was equal to

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0.000,005,722,000,000,000

15.211,342,913

$\frac{0.000,005,722,000,000,000}{15.211,342,913} = 0.000,000,376$, and consequently that y , or $0.265,419,739 + z$, would be $(= 0.265,419,739 + 0.000,000,376) = 0.265,420,115$. We may therefore consider this last number, $0.265,420,115$, as being the true value of y as far as the said true value can be expressed in nine places of figures.

End of the resolution of the equation $y^3 + 15y = 4$, or $15y + y^3 = 4$.

9. It has been shewn in the foregoing articles, that, if e be put $= \frac{r}{2}$, and ss be $= \frac{q^3}{27} + \frac{rr}{4}$, or $\frac{q^3}{27} + ee$, the root y of the cubick equation $y^3 + qy = r$ will be equal to $2\sqrt[3]{s} \times$ the series $\frac{e}{3s} + \frac{5e^3}{81s^3} + \frac{22e^5}{729s^5} + \frac{374e^7}{19,683s^7} + \frac{21,505e^9}{1,594,323s^9} + \frac{147,407e^{11}}{14,348,907s^{11}} + \&c \text{ ad infinitum}$, or $2\sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \frac{Q e^{15}}{s^{15}} + \frac{S e^{17}}{s^{17}} + \&c \text{ ad infinitum}$, the terms of which are the second, fourth, sixth, eighth, tenth, and other following terms of the series which is equal to $1 + \frac{e}{s} \Big| \frac{1}{3}$, or $\sqrt[3]{1 + \frac{e}{s}}$, or the cube-root of the binomial quantity $1 + \frac{e}{s}$. Now from this expression, which is equal to the value of y in the equation $y^3 + qy = r$, or $qy + y^3 = r$, we may, by a peculiar train of reasoning, derive another expression, very much resembling the former, which shall be equal to the lesser of the two roots of the equation $qx - x^3 = r$ in which the letters q and r denote the same known quantities as in the foregoing equation $y^3 + qy = r$, or $qy + y^3 = r$. The method of doing this I shall now endeavour to explain.

Of the cubick equation $qx - x^3 = r$.

10. The equation $qx - x^3 = r$ is not always possible, whatever be the magnitudes of q and r , but only when r is equal to, or less than, the quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or that value of the compound quantity $qx - x^3$ which results from the substitution of $\frac{\sqrt{q}}{\sqrt{3}}$, or $\sqrt{\frac{q}{3}}$, in its terms instead of x . If r is equal to this quantity, the equation $qx - x^3 = r$ will have only one root, to wit, $\frac{\sqrt{q}}{\sqrt{3}}$, or $\sqrt{\frac{q}{3}}$; and, when the absolute term r is less than the said quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qx - x^3 = r$ will have two roots, of which the lesser will be less than $\frac{\sqrt{q}}{\sqrt{3}}$.

$\frac{\sqrt{q}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{q}}{\sqrt{3}}$, but less than $\sqrt{q}$. See my dissertation on the use of the negative sign in Algebra, art. 114, page 92. It is the lesser of these two roots of the equation $qx - x^3 = r$ that I now propose to find in those cases of the said equation in which the absolute term is less not only than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ (which is its greatest possible magnitude), but than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, by means of an expression to be derived from the foregoing expression $2\sqrt[3]{s} \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \frac{Qe^{15}}{s^{15}} + \frac{Se^{17}}{s^{17}} + \&c$ *ad infinitum*, which is equal to the root y of the equation $y^3 + qy = r$, or $qy + y^3 = r$, the letters q and r being supposed to stand for the same quantities in both equations.

The value of the lesser root of the said equation expressed in an infinite series.

11. Now, when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or rr is less than $\frac{2q^3}{27}$, or $\frac{rr}{4}$ is less than $(\frac{1}{4} \times \frac{2q^3}{27}, \text{ or } \frac{q^3}{2 \times 27}, \text{ or } \frac{q^3}{54})$, let zz be taken $= \frac{q^3}{27} - \frac{rr}{4}$, and let z be substituted every where, instead of s , in the foregoing expression $2\sqrt[3]{s} \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \&c$ *ad infinitum*, whereby the said expression will be changed into the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{De^3}{z^3} + \frac{Fe^5}{z^5} + \frac{He^7}{z^7} + \frac{Ke^9}{z^9} + \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} + \&c$ *ad infinitum*; and, lastly, let the sign $-$ be prefixed to the second, fourth, sixth, eighth, tenth, and every following even term of this last series, instead of the sign $+$. The new expression thereby obtained, to wit, the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ will be equal to the lesser root of the equation $qx - x^3 = r$. This proposition we will now endeavour to demonstrate.

A proof that the infinite series set forth in the foregoing article is a converging series.

12. In the first place it will be proper to shew that this series will be a converging series. Now this may be shewn in the manner following.

Since $\frac{rr}{4}$, or ee , is less than $\frac{q^3}{54}$, it follows that $\frac{rr}{2}$, or $2ee$, will be less than $\frac{2q^3}{54}$, or $\frac{q^3}{27}$. Therefore (subtracting ee , or $\frac{rr}{4}$, from both sides) $2ee - ee$ will be less than $\frac{q^3}{27} - \frac{rr}{4}$, that is ee will be less than zz . Consequently e will be less than

than z , and the fraction $\frac{e}{z}$ will be less than 1. Therefore the fractions $\frac{e}{z}$, $\frac{e^3}{z^3}$, $\frac{e^5}{z^5}$, $\frac{e^7}{z^7}$, $\frac{e^9}{z^9}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{13}}{z^{13}}$, &c will form a decreasing progression, and consequently *à fortiori*, the several terms of the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ (of which the numeral co-efficients B, D, F, H, K, M, O, &c, also form a decreasing progression) will also form a decreasing progression, or a converging series. Q. E. D.

Therefore the expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ will converge to a certain finite magnitude, and consequently may be equal to the lesser root of the cubick equation $qx - x^3 = r$. It remains that we shew that it is so.

Preparations for demonstrating that the foregoing infinite series is equal to the lesser root of the equation $qx - x^3 = r$.

13. In order to demonstrate that the expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ is equal to the lesser root of the equation $qx - x^3 = r$, it will be necessary to resume the consideration of the expression $2 \sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ *ad infinitum*, from which it is derived.

Now, since the expression $2 \sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ *ad infinitum* is equal to the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$, it follows that, if we were, first, to multiply the said expression $2 \sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ *ad infinitum* into the co-efficient q , and then to raise the said expression $2 \sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ *ad infinitum* to its cube, or third power, and lastly, to add the said cube to the said product, the sum thence arising would be equal to r , to whatever number of terms the said series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ may be continued. For, if this sum were not equal to r , it would not be true that the expression $2 \sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ *ad infinitum*, was equal to the root of the equation $qy + y^3 = r$.

These operations may be performed in the following manner.

14. Since

14. Since $\frac{q^3}{27} + \frac{rr}{4}$ is $= ss$, or (because $\frac{rr}{4}$ is $= ee$), $\frac{q^3}{27} + ee$ is $= ss$, we shall have $\frac{q^3}{27} = ss - ee$, and $q^3 = 27 \times \sqrt[3]{ss - ee} = 27 \times ss \times \sqrt[3]{1 - \frac{ee}{ss}}$, and consequently $q = 3 \times \sqrt[3]{ss} \times \sqrt[3]{1 - \frac{ee}{ss}} = 3 \times s^{\frac{2}{3}} \times \sqrt[3]{1 - \frac{ee}{ss}}$.
 $\sqrt[3]{1 - \frac{ee}{ss}} = 1 - \frac{ee}{3ss} - \frac{ee^2}{3s^4} - \frac{ee^3}{s^6} - \frac{ee^4}{s^8} - \frac{ee^5}{s^{10}} - \frac{ee^6}{s^{12}} -$
 &c *ad infinitum*. Therefore the product qy will be $= 3s^{\frac{2}{3}} \times$ the series $1 - \frac{Bee}{ss} - \frac{Cee^2}{s^4} - \frac{Dee^3}{s^6} - \frac{Eee^4}{s^8} - \frac{Fee^5}{s^{10}} - \frac{Gee^6}{s^{12}} -$
 &c *ad infinitum* $\times y = 3s^{\frac{2}{3}} \times$ the series $1 - \frac{Bee}{ss} - \frac{Cee^2}{s^4} - \frac{Dee^3}{s^6} - \frac{Eee^4}{s^8} - \frac{Fee^5}{s^{10}} - \frac{Gee^6}{s^{12}} -$
 &c *ad infinitum* $\times$ the expression $2s^{\frac{1}{3}} \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$
 &c *ad infinitum* $= 6s \times$ the series $1 - \frac{Bee}{ss} - \frac{Cee^2}{s^4} - \frac{Dee^3}{s^6} - \frac{Eee^4}{s^8} - \frac{Fee^5}{s^{10}} - \frac{Gee^6}{s^{12}} -$
 &c *ad infinitum* $\times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$
 &c *ad infinitum* $= 6s$ into the following compound series, to wit,

$$\begin{aligned} & \frac{Be}{s} - \frac{B^2e^3}{s^3} - \frac{BCe^5}{s^5} - \frac{BDe^7}{s^7} - \frac{BEe^9}{s^9} - \frac{BFe^{11}}{s^{11}} - \frac{BGe^{13}}{s^{13}} - \&c \\ & + \frac{De^3}{s^3} - \frac{BD^2e^5}{s^5} - \frac{CDe^7}{s^7} - \frac{D^2e^9}{s^9} - \frac{DEe^{11}}{s^{11}} - \frac{DFe^{13}}{s^{13}} - \&c \\ & + \frac{Fe^5}{s^5} - \frac{BF^2e^7}{s^7} - \frac{CF^2e^9}{s^9} - \frac{DFe^{11}}{s^{11}} - \frac{EF^2e^{13}}{s^{13}} - \&c \\ & + \frac{He^7}{s^7} - \frac{BH^2e^9}{s^9} - \frac{CH^2e^{11}}{s^{11}} - \frac{DH^2e^{13}}{s^{13}} - \&c \\ & + \frac{Ke^9}{s^9} - \frac{BK^2e^{11}}{s^{11}} - \frac{CK^2e^{13}}{s^{13}} - \&c \\ & + \frac{Me^{11}}{s^{11}} - \frac{BM^2e^{13}}{s^{13}} - \&c \\ & + \frac{Oe^{13}}{s^{13}} - \&c; \end{aligned}$$

or, if, for the sake of brevity, we denote this compound series by the Greek capital letter Γ , we shall have the product $qy = 6s \times$ the compound series Γ .

15. In the foregoing compound series Γ , which is the product of the multiplication of the series $1 - \frac{Bee}{ss} - \frac{Cee^2}{s^4} - \frac{Dee^3}{s^6} - \frac{Eee^4}{s^8} - \frac{Fee^5}{s^{10}} - \frac{Gee^6}{s^{12}} -$ &c (of which all the terms after the first term 1 are marked with the sign $-$, or subtracted from the said first term) into the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} +$ &c (of which all the terms after the first term $\frac{Be}{s}$ are marked with

with the sign +, or added to the said first term), it is evident that the first term of every horizontal row of terms will be marked with the sign +, and that every following term of the same row will be marked with the sign -, to whatever number of terms the said horizontal rows of terms may be continued, and likewise that the first term of every new horizontal row of terms is placed immediately under the second term of the next preceeding horizontal row, because they both involve the same power of the fraction $\frac{e}{s}$; whence it follows that all the terms in every vertical column of terms, except the lowest term, will be marked with the sign -, and that the said lowest term will be marked with the sign +. This will appear most evidently upon performing the multiplication of the former of the said serieses into the latter.

16. If we multiply the foregoing compound series called Γ into $6s$, the product will be the following compound series, to wit,

$$\begin{array}{r}
 6Be - \frac{6B^2e^3}{s^2} - \frac{6BCE^5}{s^4} - \frac{6BDE^7}{s^6} - \frac{6BEe^9}{s^8} - \frac{6BFe^{11}}{s^{10}} - \frac{6BGe^{13}}{s^{12}} - \&c \\
 + \frac{6De^3}{s^2} - \frac{6BDe^5}{s^4} - \frac{6CDe^7}{s^6} - \frac{6D^2e^9}{s^8} - \frac{6DEe^{11}}{s^{10}} - \frac{6DFe^{13}}{s^{12}} - \&c \\
 + \frac{6Fe^5}{s^4} - \frac{6BFe^7}{s^6} - \frac{6CFe^9}{s^8} - \frac{6DFe^{11}}{s^{10}} - \frac{6EF e^{13}}{s^{12}} - \&c \\
 + \frac{6He^7}{s^6} - \frac{6BHe^9}{s^8} - \frac{6CHE^{11}}{s^{10}} - \frac{6DHe^{13}}{s^{12}} - \&c \\
 + \frac{6Ke^9}{s^8} - \frac{6BKe^{11}}{s^{10}} - \frac{6CKe^{13}}{s^{12}} - \&c \\
 + \frac{6Me^{11}}{s^{10}} - \frac{6BMe^{13}}{s^{12}} - \&c \\
 + \frac{6Oe^{13}}{s^{12}} - \&c.
 \end{array}$$

Let this compound series, for the sake of brevity, be denoted by the Greek capital letter Δ . Then, since the product qy is equal to $6s \times$ the compound series Γ , it will also be equal to the compound series Δ .

17. In the next place we must find the cube of the expression $2\sqrt[3]{s} \times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \&c$ *ad infinitum*, which is equal to y .

Now the cube of this expression is $= 8s \times$ the cube of the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \&c$; which cube may be found by multiplying the said series twice into itself in the manner following.

The multiplication of the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ twice into itself, in order to obtain its cube.

$$\begin{array}{cccccccc} \frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c \\ \frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c \end{array}$$

$$\frac{B^2 e^2}{s^2} + \frac{BD e^4}{s^4} + \frac{BF e^6}{s^6} + \frac{BH e^8}{s^8} + \frac{BK e^{10}}{s^{10}} + \frac{BM e^{12}}{s^{12}} + \&c$$

$$+ \frac{BD e^4}{s^4} + \frac{D^2 e^6}{s^6} + \frac{DF e^8}{s^8} + \frac{DH e^{10}}{s^{10}} + \frac{DK e^{12}}{s^{12}} + \&c$$

$$+ \frac{BF e^6}{s^6} + \frac{DF e^8}{s^8} + \frac{F^2 e^{10}}{s^{10}} + \frac{FH e^{12}}{s^{12}} + \&c$$

$$+ \frac{BH e^8}{s^8} + \frac{DH e^{10}}{s^{10}} + \frac{FH e^{12}}{s^{12}} + \&c$$

$$+ \frac{BK e^{10}}{s^{10}} + \frac{DK e^{12}}{s^{12}} + \&c$$

$$+ \frac{BM e^{12}}{s^{12}} + \&c$$

$$\frac{B^2 e^2}{s^2} + \frac{2 BD e^4}{s^4} + \frac{2 BF e^6}{s^6} + \frac{2 BH e^8}{s^8} + \frac{2 BK e^{10}}{s^{10}} + \frac{2 BM e^{12}}{s^{12}} + \&c$$

$$+ \frac{D^2 e^6}{s^6} + \frac{2 DF e^8}{s^8} + \frac{2 DH e^{10}}{s^{10}} + \frac{2 DK e^{12}}{s^{12}} + \&c$$

$$+ \frac{F^2 e^{10}}{s^{10}} + \frac{2 FH e^{12}}{s^{12}} + \&c$$

$$\begin{array}{cccccccc} \frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c \\ \frac{B^3 e^3}{s^3} + \frac{2 B^2 D e^5}{s^5} + \frac{2 B^2 F e^7}{s^7} + \frac{2 B^2 H e^9}{s^9} + \frac{2 B^2 K e^{11}}{s^{11}} + \frac{2 B^2 M e^{13}}{s^{13}} + \&c \\ + \frac{BD^2 e^7}{s^7} + \frac{2 BDF e^9}{s^9} + \frac{2 BDH e^{11}}{s^{11}} + \frac{2 BDK e^{13}}{s^{13}} + \&c \\ + \frac{BF^2 e^{11}}{s^{11}} + \frac{2 BFH e^{13}}{s^{13}} + \&c \end{array}$$

$$+ \frac{B^2 D e^5}{s^5} + \frac{2 BD^2 e^7}{s^7} + \frac{2 BDF e^9}{s^9} + \frac{2 BDH e^{11}}{s^{11}} + \frac{2 BDK e^{13}}{s^{13}} + \&c$$

$$+ \frac{D^3 e^9}{s^9} + \frac{2 D^2 F e^{11}}{s^{11}} + \frac{2 D^2 H e^{13}}{s^{13}} + \&c$$

$$+ \frac{DF^2 e^{13}}{s^{13}} + \&c$$

$$+ \frac{B^2 F e^7}{s^7} + \frac{2 BDF e^9}{s^9} + \frac{2 BF^2 e^{11}}{s^{11}} + \frac{2 BFH e^{13}}{s^{13}} + \&c$$

$$+ \frac{D^2 F e^{11}}{s^{11}} + \frac{2 DF^2 e^{13}}{s^{13}} + \&c$$

$$\begin{aligned}
& + \frac{B^2 H e^9}{s^9} + \frac{2 BDH e^{11}}{s^{11}} + \frac{2 BFH e^{13}}{s^{13}} + \&c \\
& + \frac{D^2 H e^{13}}{s^{13}} + \&c \\
& + \frac{B^2 K e^{11}}{s^{11}} + \frac{2 BDK e^{13}}{s^{13}} + \&c \\
& + \frac{B^2 M e^{13}}{s^{13}} + \&c \\
\hline
& \frac{B^3 e^3}{s^3} + \frac{3 B^2 D e^5}{s^5} + \frac{3 B^2 F e^7}{s^7} + \frac{3 B^2 H e^9}{s^9} + \frac{3 B^2 K e^{11}}{s^{11}} + \frac{3 B^2 M e^{13}}{s^{13}} + \&c \\
& + \frac{3 BD^2 e^7}{s^7} + \frac{6 BDF e^9}{s^9} + \frac{6 BDH e^{11}}{s^{11}} + \frac{6 BDK e^{13}}{s^{13}} + \&c \\
& + \frac{D^3 e^9}{s^9} + \frac{3 BF^2 e^{11}}{s^{11}} + \frac{6 BFH e^{13}}{s^{13}} + \&c \\
& + \frac{3 D^2 F e^{11}}{s^{11}} + \frac{3 D^2 H e^{13}}{s^{13}} + \&c \\
& + \frac{3 DF^2 e^{13}}{s^{13}} + \&c.
\end{aligned}$$

This last compound series is the cube of the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ *ad infinitum*.

Therefore $8s$ into the cube of the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ will be $= 8s \times$ the foregoing compound series; or, if, for the sake of brevity, we denote the said compound series by the Greek capital letter Λ , $8s \times$ the cube of the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$ will be $= 8s \times$ the compound series Λ ; that is, the cube of the expression $2 \sqrt[3]{s} \times$ the series $\frac{B e}{s} + \frac{D e^3}{s^3} + \frac{F e^5}{s^5} + \frac{H e^7}{s^7} + \frac{K e^9}{s^9} + \frac{M e^{11}}{s^{11}} + \frac{O e^{13}}{s^{13}} + \&c$, or the cube of y , will be $= 8s \times$ the compound series Λ .

18. If the foregoing compound series Λ be multiplied into $8s$, the product will be the following compound series, to wit,

$$\begin{aligned}
& \frac{8 B^3 e^3}{s^2} + \frac{24 B^2 D e^5}{s^4} + \frac{24 B^2 F e^7}{s^6} + \frac{24 B^2 H e^9}{s^8} + \frac{24 B^2 K e^{11}}{s^{10}} + \frac{24 B^2 M e^{13}}{s^{12}} + \&c \\
& + \frac{24 BD^2 e^7}{s^6} + \frac{48 BDF e^9}{s^8} + \frac{48 BDH e^{11}}{s^{10}} + \frac{48 BDK e^{13}}{s^{12}} + \&c \\
& + \frac{8 D^3 e^9}{s^8} + \frac{24 BF^2 e^{11}}{s^{10}} + \frac{48 BFH e^{13}}{s^{12}} + \&c \\
& + \frac{24 D^2 F e^{11}}{s^{10}} + \frac{24 D^2 H e^{13}}{s^{12}} + \&c \\
& + \frac{24 DF^2 e^{13}}{s^{12}} + \&c,
\end{aligned}$$

which

which we will denote by the Greek capital letter Π . Then will y^3 be equal the compound series Π .

19. Since the compound series Δ , obtained in art. 16, is equal to the product qy , and the compound series Π , obtained in art. 18, is equal to y^3 , it follows that the sum of the two compound series Δ and Π will be equal to $qy + y^3$, and consequently to its equal, the absolute term r ; that is, the series $\Delta + \Pi$ will be $= r$. Therefore the series Π will be $= r -$ the series Δ . But r is equal to the first term, $6Be$, of the series Δ . For B is $\frac{1}{3}$, and e is $= \frac{r}{2}$, and consequently $6Be$ is $(= 6 \times \frac{1}{3} \times \frac{r}{2} = \frac{6r}{6}) = r$. Therefore the series Π is $= 6Be -$ the series Δ ; that is, the compound series Π is equal to the excess of the first term, $6Be$, of the compound series Δ above the whole of the said series. But $6Be -$ the series Δ , or the excess of the first term, $6Be$, of the series Δ above the whole of the said series, will be a compound series consisting of all the terms of the compound series Δ , except the first term $6Be$, with their signs $+$ and $-$ every where changed. Therefore the compound series Π will be equal to a compound series consisting of all the terms of the compound series Δ , except the first term $6Be$, with their signs $+$ and $-$ every where changed; that is, the compound series

$$\begin{aligned} & \frac{8B^3e^3}{s^2} + \frac{24B^2De^5}{s^4} + \frac{24B^2Fe^7}{s^6} + \frac{24B^2He^9}{s^8} + \frac{24B^2Ke^{11}}{s^{10}} + \frac{24B^2Me^{13}}{s^{12}} + \&c \\ & + \frac{24BD^2e^7}{s^6} + \frac{48BDFe^9}{s^8} + \frac{48BDHe^{11}}{s^{10}} + \frac{48BDKe^{13}}{s^{12}} + \&c \\ & + \frac{8D^3e^9}{s^8} + \frac{24BF^2e^{11}}{s^{10}} + \frac{48BFHe^{13}}{s^{12}} + \&c \\ & + \frac{24D^2Fe^{11}}{s^{10}} + \frac{24D^2He^{13}}{s^{12}} + \&c \\ & + \frac{24DF^2e^{13}}{s^{12}} + \&c \end{aligned}$$

will be equal to the compound series

$$\begin{aligned} & \frac{6B^2e^3}{s^2} + \frac{6BCe^5}{s^4} + \frac{6BD e^7}{s^6} + \frac{6BE e^9}{s^8} + \frac{6BF e^{11}}{s^{10}} + \frac{6BG e^{13}}{s^{12}} + \&c \\ & - \frac{6De^3}{s^2} + \frac{6BD e^5}{s^4} + \frac{6CD e^7}{s^6} + \frac{6D^2e^9}{s^8} + \frac{6DE e^{11}}{s^{10}} + \frac{6DF e^{13}}{s^{12}} + \&c \\ & - \frac{6Fe^5}{s^4} + \frac{6BF e^7}{s^6} + \frac{6CF e^9}{s^8} + \frac{6DF e^{11}}{s^{10}} + \frac{6EF e^{13}}{s^{12}} + \&c \\ & - \frac{6He^7}{s^6} + \frac{6BH e^9}{s^8} + \frac{6CH e^{11}}{s^{10}} + \frac{6DH e^{13}}{s^{12}} + \&c \\ & - \frac{6Ke^9}{s^8} + \frac{6BK e^{11}}{s^{10}} + \frac{6CK e^{13}}{s^{12}} + \&c \\ & - \frac{6Me^{11}}{s^{10}} + \frac{6BM e^{13}}{s^{12}} + \&c \\ & - \frac{6Oe^{13}}{s^{12}} + \&c; \end{aligned}$$

which, for the sake of brevity, we will denote by the Greek capital letter Σ . And then we shall have the compound series $\Pi =$ the compound series Σ .

Of the signs + and — that are to be prefixed to the several terms of the foregoing compound series Σ .

20. In this compound series Σ all the terms of the first horizontal row of terms are to be added together, and consequently all the terms after the first term, $\frac{6 B^2 e^3}{s^2}$, are to be marked with the sign +, to whatever number of terms the said horizontal row of terms may be continued; and the first terms of the second, third, fourth, fifth, and other following horizontal rows of terms (to whatever number of horizontal rows the said compound series may be continued) will be marked with the sign —, and all the following terms of the said horizontal rows after the first terms will be marked with the sign +; and consequently all the terms of every vertical column of terms in this compound series, except the lowest term, will be marked with the sign +, and the said lowest term of each vertical column will be marked with the sign —.

This follows necessarily from art. 15 and 16; because the signs + and — that are prefixed to the terms of the compound series Δ , set down in art. 16, are the same with the signs of the corresponding terms of the preceeding compound series Γ , set down in art. 14, from which the compound series Δ is derived by only multiplying its terms into $6s$; and the compound series Σ is derived from the compound series Δ by omitting its first term $6 B e$, and changing the signs of all its following terms. Consequently the signs to be prefixed to the terms of the compound series Σ must be contrary to those which are to be prefixed to the corresponding terms of the compound series Γ , and therefore must be contrary to those which are described in art. 15, or must be such as they are described to be in the present article.

Of the equality between the co-efficients of the terms of the compound series II and the co-efficients of the corresponding terms of the compound series Σ .

21. Since ee is $= \frac{rr}{4}$, and ss is $= \frac{q^3}{27} + \frac{rr}{4}$, we may, by lessening the value of $\frac{rr}{4}$, or of r , without altering that of q , lessen the value of the fraction $\frac{\frac{rr}{4}}{\frac{q^3}{27} + \frac{rr}{4}}$; or $\frac{ee}{ss}$, and consequently that of $\frac{e}{s}$, as far as we please. Yet in all

these values of the fraction $\frac{e}{s}$ it will always be true that the compound series II, which involves in its terms the fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, which will be equal to the compound series Σ , which involves in its terms the same fractions. It therefore follows from this constant equality between these two serieses in all possible

possible magnitudes (how small soever) of the fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}},$ &c, that the term, or terms, that involve any given powers of e and s in one of the two serieses must be equal to the terms that involve the same powers of them in the other series. And consequently the several co-efficients of the terms which involve any given powers of e and s in one of the two serieses must be equal to the several co-efficients of the terms that involve the same powers of e and s in the other series. Thus, for example, the co-efficient of the fraction $\frac{e^3}{s^2}$ in the series Π , to wit, $8 B^3$, must be equal to the two co-efficients of the same fraction $\frac{e^3}{s^2}$ in the series Σ , to wit, $6 B^2 - 6 D$; and in like manner the co-efficient of the fraction $\frac{e^5}{s^4}$ in the series Π , to wit, $24 B^2 D$, must be equal to the three co-efficients of the same fraction $\frac{e^5}{s^4}$ in the series Σ , to wit, $6 BC + 6 BD - 6 F$; and the two co-efficients of the fraction $\frac{e^7}{s^6}$ in the series Π , to wit, $24 B^2 F + 24 BD^2$, must be equal to the four co-efficients of the same fraction $\frac{e^7}{s^6}$ in the series Σ , to wit, $6 BD + 6 CD + 6 BF - 6 H$; and the same thing must take place with respect to the co-efficients of the following fractions $\frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}},$ &c *ad infinitum*, or to whatever number of terms the two serieses may be continued.

Examples of the said equality of the co-efficients of the terms of the said two compound serieses.

22. Of this equality between the co-efficients of the same fractions, consisting of the powers of e and s , in these two compound serieses Π and Σ , it may not be amiss to give a few instances by actually computing the values of the said co-efficients; which may be done in the manner following.

The capital letters $B, C, D, E, F,$ and $H,$ are equal to $\frac{1}{3}, \frac{1}{9}, \frac{5}{81}, \frac{10}{243}, \frac{22}{729},$ and $\frac{374}{19683},$ respectively. Therefore B^3 will be $= \frac{1}{27},$ and $8 B^3$ will be $= \frac{8}{27};$ and $6 B^2$ will be $(= 6 \times \frac{1}{9}) = \frac{2}{3},$ and $6 D$ will be $(= 6 \times \frac{5}{81} = 2 \times \frac{5}{27}) = \frac{10}{27},$ and consequently $6 B^2 - 6 D$ will be $(= \frac{2}{3} - \frac{10}{27} = \frac{18}{27} - \frac{10}{27}) = \frac{8}{27};$ that is, $8 B^3,$ the co-efficient of the fraction $\frac{e^3}{s^2}$ in the compound series $\Pi,$ and $6 B^2 - 6 D,$ the co-efficient of the same fraction $\frac{e^3}{s^2}$ in the compound series $\Sigma,$ will, each of them, be equal to the same quantity $\frac{8}{27}.$

In like manner $24 B^2 D$, or the co-efficient of the fraction $\frac{e^5}{s^4}$ in the compound series Π , is $(= 24 \times \frac{1}{3 \times 3} \times \frac{5}{81} = \frac{8}{3} \times \frac{5}{81}) = \frac{40}{243}$; and $6 BC + 6 BD - 6 F$, or the co-efficient of the same fraction $\frac{e^5}{s^4}$ in the compound series Σ , will be found to be equal to the same quantity. For $6 BC + 6 BD - 6 F$ is $= 6 \times \frac{1}{3} \times \frac{1}{9} + 6 \times \frac{1}{3} \times \frac{5}{81} - 6 \times \frac{22}{729} (= \frac{2}{9} + \frac{10}{81} - \frac{44}{243} = \frac{54}{243} + \frac{30}{243} - \frac{44}{243} = \frac{84}{243} - \frac{44}{243}) = \frac{40}{243}$. Q. E. D.

And $24 B^2 F + 24 BD^2$, or the co-efficient of the fraction $\frac{e^7}{s^6}$ in the compound series Π , is $(= 24 \times \frac{1}{9} \times \frac{22}{729} + 24 \times \frac{1}{3} \times \frac{5 \times 5}{81 \times 81} = \frac{8}{3} \times \frac{22}{729} + 8 \times \frac{25}{81 \times 81} = \frac{176}{2187} + \frac{200}{6561} = \frac{528}{6561} + \frac{200}{6561}) = \frac{728}{6561}$; and $6 BD + 6 CD + 6 BF - 6 H$, or the co-efficient of the same fraction $\frac{e^7}{s^6}$ in the other compound series Σ , is equal to the same quantity. For it is $= 6 \times \frac{1}{3} \times \frac{5}{81} + 6 \times \frac{1}{9} \times \frac{5}{81} + 6 \times \frac{1}{3} \times \frac{22}{729} - 6 \times \frac{374}{19,683} (= 2 \times \frac{5}{81} + \frac{2}{3} \times \frac{5}{81} + 2 \times \frac{22}{729} - \frac{2 \times 374}{6561} = \frac{10}{81} + \frac{10}{243} + \frac{44}{729} - \frac{748}{6561} = \frac{810}{6561} + \frac{270}{6561} + \frac{396}{6561} - \frac{748}{6561} = \frac{1476}{6561} - \frac{748}{6561}) = \frac{728}{6561}$. Q. E. D.

And the same equality will be found to take place between the co-efficients of the following fractions $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, *ad infinitum*, in the compound series Π and the co-efficients of the same fractions in the compound series Σ , respectively, to whatever number of terms the said serieses may be continued.

The reduction of the compound series Π to a simple series $\frac{P e^3}{s^2} +$

$$\frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \mathcal{E}c \text{ ad infinitum.}$$

23. In the compound series Π , obtained in art. 18, all the terms after the first term, $\frac{8 B^3 e^3}{s^2}$, are marked with the sign $+$ and added to the said first term. Therefore, if we reduce the said compound series to a simple series, with single letters for the co-efficients of its terms, by denoting the co-efficient, $8 B^3$, of the fraction $\frac{e^3}{s^2}$ in the first term $\frac{8 B^3 e^3}{s^2}$, by the single letter P , and the co-efficient,

$24 B^2 D$,

$24 B^2 D$, of the fraction $\frac{e^5}{s^4}$ in the second term, $\frac{24 B^2 D e^5}{s^4}$, by the single letter Q , and the compound co-efficient, $24 B^2 F + 24 B D^2$, of the fraction $\frac{e^7}{s^6}$ in the third term, by the single letter R , and the compound co-efficients of the following fractions $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the fourth, fifth, and sixth, and other following terms of the same series, by the single letters S , T , V , &c, the simple series that will be equal to the said compound series Π will be $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$, in which all the terms after the first term, $\frac{P e^3}{s^2}$, are marked with the sign $+$, or are added to the said first term.

The same simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$ ad infinitum will also be equal to the compound series Σ .

24. But by art. 19, the compound series Π is equal to the compound series Σ . Therefore the simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$ will be equal to the compound series Σ , which is set down in the latter part of art. 19.

And the several co-efficients P , Q , R , S , T , V , &c, of the terms of the said simple series will be respectively equal to the several compound co-efficients of the corresponding terms of the said compound series Σ .

25. And, further, it is shewn in art. 21, that the co-efficients of the several fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the compound series Π are respectively equal to the co-efficients of the same fractions in the other compound series Σ . Therefore the co-efficients P , Q , R , S , T , V , &c, of the several fractions $\frac{e^3}{s^2}$, $\frac{e^5}{s^4}$, $\frac{e^7}{s^6}$, $\frac{e^9}{s^8}$, $\frac{e^{11}}{s^{10}}$, $\frac{e^{13}}{s^{12}}$, &c, in the simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$ (which are respectively equal to the co-efficients of the same fractions in the compound series Π) will be also respectively equal to the co-efficients of the same fractions in the compound series Σ . And hence it follows that, if we were to reduce the said compound series Σ to a simple series by the addition and subtraction of the several members of each of its vertical columns

lums according to the signs $+$ and $-$ which are prefixed to them, we should find the sum of the co-efficients of the terms that are marked with the sign $+$ in each of the vertical columns of the said compound series (and which are all the terms of the column, except the last, or lowest) to be greater than the co-efficient of the last, or lowest, term in the column, which is marked with the sign $-$, and the resulting differences, or excesses, of the sums of the co-efficients of the terms marked with the sign $+$ in each column above the co-efficients of the lowest terms in them, which are marked with the sign $-$, to be equal to the co-efficients $P, Q, R, S, T, V, \&c$, of the fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}}, \&c$, in the simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$ respectively; that is, we should find the compound co-efficient of the fraction $\frac{e^3}{s^2}$ in the compound series Σ , to wit, $6 B^2 - 6 D$, to be $= P$; and the compound co-efficient of the fraction $\frac{e^5}{s^4}$ in the same series Σ , to wit, $6 BC + 6 BD - 6 F$, to be $= Q$; and the compound co-efficient of the fraction $\frac{e^7}{s^6}$ in the same series, to wit, $6 BD + 6 CD + 6 BF - 6 H$, to be $= R$; and the compound co-efficient of the fraction $\frac{e^9}{s^8}$, to wit, $6 BE + 6 D^2 + 6 CF + 6 BH - 6 K$, to be $= S$; and the compound co-efficient of the fraction $\frac{e^{11}}{s^{10}}$, to wit, $6 BF + 6 DE + 6 DF + 6 CH + 6 BK - 6 M$, to be $= T$; and the compound co-efficient of the fraction $\frac{e^{13}}{s^{12}}$, to wit, $6 BG + 6 DF + 6 EF + 6 DH + 6 CK + 6 BM - 6 O$, to be $= V$; and, in like manner, the compound co-efficients of the following fractions $\frac{e^{15}}{s^{14}}, \frac{e^{17}}{s^{16}}, \frac{e^{19}}{s^{18}}, \frac{e^{21}}{s^{20}}, \frac{e^{23}}{s^{22}}, \&c$, *ad infinitum* in the same compound series Σ to be respectively equal to the co-efficients of the same fractions in the following terms of the simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$ *ad infinitum*.

Conclusions contained in the three foregoing articles.

26. It appears from the three foregoing articles, that the simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$ will be equal to each of the two compound serieses, Π and Σ , and likewise that each of its terms will be equal to the correspondent term, or term that involves the same powers of e and s , in each of the said two compound serieses, and consequently that the co-efficient of each of its terms will be equal to the co-efficient of the correspondent term, or term involving the same powers of e and s , in each of the said two compound serieses.

Examination

Examination of the expression $2\sqrt[3]{z}$ $\times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ ad infinitum.

27. These things being thoroughly understood, we must now turn our attention to the expression $2\sqrt[3]{z}$ $\times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} + \&c$ ad infinitum, which we have asserted to be equal to the lesser root of the cubick equation $qx - x^3 = r$. This we must now proceed to demonstrate. And first we will endeavour to prove that this expression is equal to one of the two roots of the said equation; and afterwards we will shew that it cannot be equal to the greater of those two roots, and consequently that it must be equal to the lesser of them.

28. Now it will be evident that this expression $2\sqrt[3]{z}$ $\times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ ad infinitum is equal to one of the roots of the cubick equation $qx - x^3 = r$, if we can shew that, being substituted instead of x in the compound quantity $qx - x^3$, it will make that quantity be equal to the absolute term r , or that the product of the multiplication of the said expression into the co-efficient q will be greater than the cube of the said expression, and that its excess above the said cube will be equal to the absolute term r . This therefore is what I shall now endeavour to demonstrate.

The product of the multiplication of q into the said expression set forth in an infinite series.

29. The value of zz in the expression $2\sqrt[3]{z}$ $\times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ is $\frac{q^3}{27} - \frac{rr}{4}$, by art. 11, in which this series is derived from the expression $2\sqrt[3]{s}$ $\times$ the series $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \&c$, which is equal to the root of the equation $qy + y^3 = r$. But ee is $= \frac{rr}{4}$, and consequently $\frac{q^3}{27} - ee$ is $= \frac{q^3}{27} - \frac{rr}{4}$. Therefore zz will be $= \frac{q^3}{27} - ee$, and consequently $\frac{q^3}{27}$ will be $= zz + ee$, and q^3 will be

$$= 27 \times \overline{zz + ee} = 27 \times \overline{zz} \times \sqrt{1 + \frac{ee}{zz}}, \text{ and } q \text{ will be } = 3 \times \overline{zz}^{\frac{1}{3}} \times \overline{1 + \frac{ee}{zz}}^{\frac{1}{3}} = 3 \times \overline{z}^{\frac{2}{3}} \times \overline{1 + \frac{ee}{zz}}^{\frac{1}{3}} = (\text{by the binomial theorem in the case}$$

$$\text{of roots}) 3 \times \overline{z}^{\frac{2}{3}} \times \text{the series } 1 + \frac{1}{3} A \frac{ee}{zz} - \frac{2}{6} B \times \frac{e^4}{z^4} + \frac{5}{9} C \times \frac{e^6}{z^6} - \frac{8}{12} D \times \frac{e^8}{z^8} + \frac{11}{15} E \times \frac{e^{10}}{z^{10}} - \frac{14}{18} F \times \frac{e^{12}}{z^{12}} + \frac{17}{21} G \times \frac{e^{14}}{z^{14}} - \frac{20}{24} H \times \frac{e^{16}}{z^{16}} +$$

$$\&c = 3 \times \overline{z}^{\frac{2}{3}} \times \text{the series } 1 + \frac{B ee}{zz} - \frac{C e^4}{z^4} + \frac{D e^6}{z^6} - \frac{E e^8}{z^8} + \frac{F e^{10}}{z^{10}} - \frac{G e^{12}}{z^{12}} + \frac{H e^{14}}{z^{14}} - \frac{I e^{16}}{z^{16}} + \&c. \text{ Therefore the product of the multiplication of } q \text{ into}$$

$$\text{the expression } 2 \overline{z}^{\frac{1}{3}} \times \text{the series } \frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \text{ will be } = 3 \times \overline{z}^{\frac{2}{3}} \times \text{the series } 1 + \frac{B ee}{zz} - \frac{C e^4}{z^4} + \frac{D e^6}{z^6} - \frac{E e^8}{z^8}$$

$$+ \frac{F e^{10}}{z^{10}} - \frac{G e^{12}}{z^{12}} + \frac{H e^{14}}{z^{14}} - \frac{I e^{16}}{z^{16}} + \&c \times 2 \overline{z}^{\frac{1}{3}} \times \text{the series } \frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c = 6 \overline{z} \times \text{the series } 1 + \frac{B ee}{zz} - \frac{C e^4}{z^4} + \frac{D e^6}{z^6} - \frac{E e^8}{z^8} + \frac{F e^{10}}{z^{10}} - \frac{G e^{12}}{z^{12}} + \frac{H e^{14}}{z^{14}} - \frac{I e^{16}}{z^{16}} + \&c \times \text{the series } \frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c = 6 \overline{z} \times \text{the compound series}$$

$$\begin{aligned} & \frac{B e}{z} + \frac{B^2 e^3}{z^3} - \frac{B C e^5}{z^5} + \frac{B D e^7}{z^7} - \frac{B E e^9}{z^9} + \frac{B F e^{11}}{z^{11}} - \frac{B G e^{13}}{z^{13}} + \&c \\ & - \frac{D e^3}{z^3} - \frac{B D e^5}{z^5} + \frac{C D e^7}{z^7} - \frac{D^2 e^9}{z^9} + \frac{D E e^{11}}{z^{11}} - \frac{D F e^{13}}{z^{13}} + \&c \\ & + \frac{F e^5}{z^5} + \frac{B F e^7}{z^7} - \frac{C F e^9}{z^9} + \frac{D F e^{11}}{z^{11}} - \frac{E F e^{13}}{z^{13}} + \&c \\ & - \frac{H e^7}{z^7} - \frac{B H e^9}{z^9} + \frac{C H e^{11}}{z^{11}} - \frac{D H e^{13}}{z^{13}} + \&c \\ & + \frac{K e^9}{z^9} + \frac{B K e^{11}}{z^{11}} - \frac{C K e^{13}}{z^{13}} + \&c \\ & - \frac{M e^{11}}{z^{11}} - \frac{B M e^{13}}{z^{13}} + \&c \\ & + \frac{O e^{13}}{z^{13}} + \&c. \end{aligned}$$

Let us, for the sake of brevity, denote this compound series by the small Greek letter γ . And we shall then have the product of the multiplication of q into the expression $2 \overline{z}^{\frac{1}{3}} \times \text{the series } \frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ *ad infinitum* equal to $6 \overline{z} \times \text{the compound series } \gamma$.

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A comparison between the foregoing compound series γ and the compound series Γ obtained above in art. 14.

30. This compound series γ will consist of terms that will involve the several fractions $\frac{e}{z}$, $\frac{e^3}{z^3}$, $\frac{e^5}{z^5}$, $\frac{e^7}{z^7}$, $\frac{e^9}{z^9}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{13}}{z^{13}}$, &c, or the several odd powers of the fraction $\frac{e}{z}$, in the same manner as the terms of the compound series Γ , obtained above in art. 14, involve the corresponding fractions $\frac{e}{s}$, $\frac{e^3}{s^3}$, $\frac{e^5}{s^5}$, $\frac{e^7}{s^7}$, $\frac{e^9}{s^9}$, $\frac{e^{11}}{s^{11}}$, $\frac{e^{13}}{s^{13}}$, &c, or the several odd powers of the fraction $\frac{e}{s}$: and the co-efficients of the several powers of $\frac{e}{z}$ in the present series γ will be the same with the co-efficients of the same powers of $\frac{e}{s}$ in the former series Γ , though they will not be every where marked with the same signs $+$ and $-$. This equality, or, rather, identity, of the co-efficients of the corresponding terms in both these compound series arises from the equality, or identity, of the co-efficients of the terms of the two simple series, by the multiplication of which into each other the said compound series γ is produced, with the co-efficients of the corresponding terms of the two simple series, by the multiplication of which into each other the former compound series Γ is produced. For, since the co-efficients of the terms of the two serieses $1 + \frac{Bee}{zz} - \frac{Cee^4}{z^4} + \frac{De^6}{z^6} - \frac{Ee^8}{z^8} + \frac{Fe^{10}}{z^{10}} - \frac{Ge^{12}}{z^{12}} + \&c$ and $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ (by the multiplication of which into each other the compound series γ is produced) are the same with the co-efficients of the corresponding terms of the two serieses $1 - \frac{Bee}{ss} - \frac{Cee^4}{s^4} - \frac{De^6}{s^6} - \frac{Ee^8}{s^8} - \frac{Fe^{10}}{s^{10}} - \frac{Ge^{12}}{s^{12}} - \&c$ and $\frac{Be}{s} + \frac{De^3}{s^3} + \frac{Fe^5}{s^5} + \frac{He^7}{s^7} + \frac{Ke^9}{s^9} + \frac{Me^{11}}{s^{11}} + \frac{Oe^{13}}{s^{13}} + \&c$ (by the multiplication of which into each other the compound series Γ was produced) it is evident that the co-efficients of the several corresponding members of the two compound serieses Γ and γ produced by these multiplications must be the same combinations of the co-efficients B, C, D, E, F, G , &c and B, D, F, H, K, M, O , &c of the terms of the two original serieses, or in other words, must be the same quantities in both serieses.

Of the signs $+$ and $-$ that are to be prefixed to the terms of the compound series γ .

31. But the signs $+$ and $-$ that are to be prefixed to the several terms of the compound series γ will not be every where the same as those that are to be prefixed to the corresponding terms of the other compound series Γ , but only in the

terms of the third, fifth, seventh, and other following odd vertical columns of terms, which involve the fractions $\frac{e^5}{s^5}$, $\frac{e^9}{s^9}$, $\frac{e^{13}}{s^{13}}$, &c and $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, $\frac{e^{13}}{z^{13}}$ &c. In the second, fourth, sixth, and other following even vertical columns of terms in the said compound series γ , which involve the fractions $\frac{e^3}{z^3}$, $\frac{e^7}{z^7}$, $\frac{e^{11}}{z^{11}}$, &c the signs + and — to be prefixed to the several terms in the said vertical columns will be respectively contrary to those which are to be prefixed to the corresponding terms of the second, fourth, sixth, and other following even vertical columns of the compound series Γ , which involve the corresponding fractions $\frac{e^3}{s^3}$, $\frac{e^7}{s^7}$, $\frac{e^{11}}{s^{11}}$, &c. This will appear to be the case as far as the terms of the seventh vertical columns of the said two compound series, or the terms which involve the fractions $\frac{e^{13}}{s^{13}}$ and $\frac{e^{13}}{z^{13}}$, upon the inspection of the said two compound series, as herein before set down in art. 14 and 29. For in the former of these series the terms contained in the third, fifth, and seventh, vertical columns are $-\frac{BCe^5}{s^5}$ $-\frac{BD e^5}{s^5} + \frac{F e^5}{s^5}$, $-\frac{BE e^9}{s^9} - \frac{D^2 e^9}{s^9} - \frac{CF e^9}{s^9} - \frac{BH e^9}{s^9} + \frac{K e^9}{s^9}$, and $-\frac{BG e^{13}}{s^{13}} - \frac{DF e^{13}}{s^{13}} - \frac{EF e^{13}}{s^{13}} - \frac{DH e^{13}}{s^{13}} - \frac{CK e^{13}}{s^{13}} - \frac{BM e^{13}}{s^{13}} + \frac{O e^{13}}{s^{13}}$; in each of which columns, or sets, of terms, the sign — is prefixed to every term, except the last, and the sign + is prefixed to the last term; and in the latter of these series the terms contained in the third, fifth, and seventh vertical columns are $-\frac{BC e^5}{z^5} - \frac{BD e^5}{z^5} + \frac{F e^5}{z^5}$, $-\frac{BE e^9}{z^9} - \frac{D^2 e^9}{z^9} - \frac{CF e^9}{z^9} - \frac{BH e^9}{z^9} + \frac{K e^9}{z^9}$, and $-\frac{BG e^{13}}{z^{13}} - \frac{DF e^{13}}{z^{13}} - \frac{EF e^{13}}{z^{13}} - \frac{DH e^{13}}{z^{13}} - \frac{CK e^{13}}{z^{13}} - \frac{BM e^{13}}{z^{13}} + \frac{O e^{13}}{z^{13}}$; in each of which columns, or sets, of terms the sign — is, in like manner, prefixed to all the terms, except the last, and the sign + is prefixed to the last term. And in the former of these series, the terms contained in the second, fourth, and sixth, vertical columns are $-\frac{B^2 e^3}{s^3} + \frac{D e^3}{s^3}$, $-\frac{BD e^7}{s^7} - \frac{CD e^7}{s^7} - \frac{BF e^7}{s^7} + \frac{H e^7}{s^7}$, and $-\frac{BF e^{11}}{s^{11}} - \frac{DE e^{11}}{s^{11}} - \frac{DF e^{11}}{s^{11}} - \frac{CH e^{11}}{s^{11}} - \frac{BK e^{11}}{s^{11}} + \frac{M e^{11}}{s^{11}}$; in each of which columns, or sets of terms, the sign — is prefixed to all the terms, except the last, and the sign + is prefixed to the last term: and in the latter of these series the terms contained in the second, fourth, and sixth, vertical columns are $+\frac{B^2 e^3}{z^3} - \frac{D e^3}{z^3}$, $+\frac{BD e^7}{z^7} + \frac{CD e^7}{z^7} + \frac{BF e^7}{z^7} - \frac{H e^7}{z^7}$, and $+\frac{BF e^{11}}{z^{11}} + \frac{DE e^{11}}{z^{11}} + \frac{DF e^{11}}{z^{11}} + \frac{CH e^{11}}{z^{11}} + \frac{BK e^{11}}{z^{11}} - \frac{M e^{11}}{z^{11}}$; in each of which columns, or sets of terms the sign + is prefixed to all the terms, except the last term, and the sign — is prefixed to the last term.

And that the same thing will take place in the terms of all the following vertical columns of these two compound series, to whatever number of terms they may be continued, may be shewn in the manner following.

32. It is shewn above in art. 15, that, in the compound series Γ , all the terms in every vertical column of the series, except the last, or lowest, term, are marked with the sign —, and the lowest term is marked with the sign +. And we have seen that the same thing takes place in the terms of the third vertical column and of the fifth vertical column, and of the seventh vertical column, of the compound series γ , which involve in them the fractions $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, and $\frac{e^{13}}{z^{13}}$; to wit, that all the terms in each of these vertical columns, except the lowest, have the sign — prefixed to them, and that the lowest term is marked with the sign +; but that the contrary rule takes place in the terms of the second vertical column, and of the fourth vertical column, and of the sixth vertical column, of the said compound series γ , which involve in them the fractions $\frac{e^3}{z^3}$, $\frac{e^7}{z^7}$, and $\frac{e^{11}}{z^{11}}$; to wit, that all the terms in each of these vertical columns, except the lowest, are marked with the sign +, and the lowest term is marked with the sign —. We are now to prove that the same rules will hold as to the signs of the terms contained in the 9th, 11th, 13th, 15th, and other following odd vertical columns of terms, and the signs of the terms contained in the 8th, 10th, 12th, 14th, and other following even vertical columns of terms of the said compound series γ , to whatever number of terms, or columns of terms, the said compound series γ may be continued.

33. Now in the said compound series γ (which is set down in art. 29), it is evident, that in the 1st, and 3d, and 5th, and 7th, and other following odd horizontal rows of terms, the two first terms will be marked with the sign +, and all the following terms will be marked with the sign — and the sign + alternately; and in the 2d, and 4th, and 6th, and other following even horizontal rows of terms, the two first terms will be marked with the sign —, and all the following terms will be marked with the sign + and the sign — alternately. This is a necessary consequence of the order in which the signs + and — follow each other, in the two simple serieses, by the multiplication of which the compound series γ is produced. For the simple series which is the multiplicand of

his multiplication, is the series $1 + \frac{Bee}{zz} - \frac{Cee^4}{z^4} + \frac{Dee^6}{z^6} - \frac{Eee^8}{z^8} + \frac{Fee^{10}}{z^{10}} - \frac{Gee^{12}}{z^{12}} +$
 &c; in which the second term $\frac{Bee}{zz}$ is marked with the sign +, and all the following terms are marked with the signs — and + alternately: and consequently, whenever this series is multiplied by a quantity to which the sign + is prefixed, the product will be a series of terms, in which the signs + and — will follow each other in the same order as in the multiplicand, that is, the sign + will be prefixed to the two first terms of the said product, and the sign — and the sign + will be prefixed to the third, fourth, fifth, sixth, and other following terms of the said product alternately; and, whenever this simple series is multiplied by a quantity to which the sign — is prefixed, the product will be a series of terms in which the signs will be, respectively, contrary to the signs of the corresponding terms of the multiplicand, and therefore the sign — will be prefixed to the two first

first terms of the said product, and the signs $+$ and $-$ will be prefixed to the third, fourth, fifth, sixth, and other following terms of the said product alternately. But the simple series into which the said simple series $1 + \frac{B e e}{z z} - \frac{C e^4}{z^4} + \frac{D e^6}{z^6} - \frac{E e^8}{z^8} + \frac{F e^{10}}{z^{10}} - \frac{G e^{12}}{z^{12}} + \&c$ is multiplied, in order to produce the compound series γ , is the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$, in which the terms are marked with the sign $+$ and the sign $-$ alternately. Therefore in the 1st, and 3d, and 5th, and 7th, and other following odd horizontal rows of terms in the compound series γ (which is produced by the said multiplication) the sign $+$ will be prefixed to the two first terms of the said horizontal rows, and the sign $-$ and the sign $+$ will be prefixed to the third, and fourth, and fifth, and sixth, and other following terms of the said rows, alternately; and in the 2d, and 4th, and 6th, and 8th, and other following even horizontal rows of terms in the said compound series γ , the sign $-$ will be prefixed to the two first terms of the said horizontal rows, and the sign $+$ and the sign $-$ will be prefixed to the third, and fourth, and fifth, and sixth, and other following terms of the said horizontal rows, alternately.

34. Now from this order of succession of the signs $+$ and $-$ to each other in the horizontal rows of terms of the compound series γ , we may deduce the order of their succession in the terms contained in the several vertical columns of the said compound series. This may be done in the manner following.

Since the two first terms of the several odd horizontal rows of terms in this compound series γ are marked with the sign $+$, and the two first terms of the several even horizontal rows are marked with the sign $-$, and the first terms of the second, and third, and fourth, and other following horizontal rows of terms are the last, or lowest, terms of the second, and third, and fourth, and other following vertical columns in the said series, it follows that the last, or lowest terms of the second, and third, and fourth, and other vertical columns will be marked with the sign $-$ and the sign $+$ alternately; and it follows likewise that the sign prefixed to the second term of every horizontal row of terms (being the same as the sign prefixed to the first term of the same horizontal row) will be contrary to the sign of the term placed immediately under it, or of the first term of the next horizontal row, and consequently that the sign prefixed to the last term but one of every vertical column (which is always the second term of one of the horizontal rows of terms) will be contrary to the last, or lowest, term of the same vertical column, which is the first term of the next lower horizontal row of terms.

Therefore, since the first terms of the second, and fourth, and sixth, and other following even horizontal rows of terms, or the last or lowest terms of the second, fourth, sixth, and other following even vertical columns, are marked with the sign $-$, the last terms but one of the same columns, to wit, the second, fourth, sixth, and other following even columns, will be marked with the sign $+$; and, since the first terms of the third, and fifth, and seventh, and other following odd horizontal rows of terms, or the last, or lowest terms of the third, fifth,

fifth, seventh, and other following odd vertical columns, are marked with the sign $+$, the last terms but one of the same columns, to wit, the third, and fifth, and seventh, and other following odd columns, will be marked with the sign $-$.

35. In the foregoing article 34, we have shewn that the lowest terms of the second, fourth, sixth, and other following even columns of terms in the compound series γ will be marked with the sign $-$, and the next higher terms, or the lowest terms but one, of the same columns will be marked with the sign $+$; and that the lowest terms of the third, fifth, seventh, and other following odd columns of terms in the same series will be marked with the sign $+$, and the next higher terms, or the lowest terms but one, of the same columns will be marked with the sign $-$. We must now examine the signs that are to be prefixed to the other terms of the several columns that are higher than the two lowest terms.

Now the signs to be prefixed to all the terms of every column above the two lowest terms are the same with each other and with the sign of the lowest term but one of the same column; as may be shewn in the manner following.

The second term of every horizontal row of terms in the compound series γ is placed directly under the third term of the next horizontal row above it, and under the fourth term of the second horizontal row above it, and under the fifth term of the third horizontal row above it, and so on *ad infinitum*, the number of terms in every new horizontal row above the said second term that preceed the term that is placed immediately above such second term, increasing continually by an unit. But, because the signs $+$ and $-$ are prefixed to the second, and third, and fourth, and fifth, and other following terms of the first, the third, the fifth, the seventh, and other following odd horizontal rows of terms, alternately, and the signs $-$ and $+$ are prefixed to the second, and third, and fourth, and fifth, and other following terms of the second, the fourth, the sixth, the eighth, and other following even horizontal rows of terms, alternately, it is evident that the sign prefixed to the second term of every horizontal row of terms must be the same with the sign prefixed to the third term of the next horizontal row above it, and with the sign prefixed to the fourth term of the second horizontal row above it, and with the sign prefixed to the fifth term of the third horizontal row above it, and with the signs prefixed to the sixth, seventh, eighth, &c, terms of the fourth, fifth, sixth, &c, horizontal rows above it, *ad infinitum*. And therefore the sign $+$ or $-$, that is to be prefixed to the second term of every horizontal row of terms in the said compound series γ will be the same with the signs of all the terms placed immediately above the said second term, or of all the terms of the same vertical column that preceed it: or, the signs to be prefixed to all the terms of every vertical column of terms in the said compound series γ above the two lowest terms of the said column will be the same with each other, and with the sign of the lowest term but one of the said column.

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A recapitulation of what has been shewn in the six foregoing articles concerning the analogy between the compound series Γ and the compound series γ .

36. It follows from the six foregoing articles, 30, 31, 32, 33, 34, and 35, that, if we compare the compound series γ , obtained in art. 29, with the compound series Γ , obtained above in art. 14, the analogy and the differences between them will be as follows, to wit: In the 1st place, in the compound series γ the terms will involve the fractions $\frac{e}{z}$, $\frac{e^3}{z^3}$, $\frac{e^5}{z^5}$, $\frac{e^7}{z^7}$, $\frac{e^9}{z^9}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{13}}{z^{13}}$, &c, or the odd powers of the fraction $\frac{e}{z}$, instead of the fractions $\frac{e}{s}$, $\frac{e^3}{s^3}$, $\frac{e^5}{s^5}$, $\frac{e^7}{s^7}$, $\frac{e^9}{s^9}$, $\frac{e^{11}}{s^{11}}$, $\frac{e^{13}}{s^{13}}$, &c, or the odd powers of the fraction $\frac{e}{s}$, which are involved in the terms of the compound series Γ .

In the next place, the co-efficients of the several terms in the series γ that involve in them the several odd powers of the fraction $\frac{e}{z}$ in the compound series γ will be the very same with the co-efficients of the several corresponding terms, or terms involving the same odd powers of the fraction $\frac{e}{s}$, in the series Γ respectively.

And, lastly, the signs $-$ and $+$ to be prefixed to the several terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the series γ (which involve the fractions $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, $\frac{e^{13}}{z^{13}}$, &c) will be the same as are to be prefixed to the corresponding terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the series Γ , which involve the fractions $\frac{e^5}{s^5}$, $\frac{e^9}{s^9}$, $\frac{e^{13}}{s^{13}}$, &c; but the signs $+$ and $-$, which are to be prefixed to the several terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the series γ (which involve the fractions $\frac{e^3}{z^3}$, $\frac{e^7}{z^7}$, $\frac{e^{11}}{z^{11}}$, &c) will be respectively contrary to those which are to be prefixed to the corresponding terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the series Γ , which involve the fractions $\frac{e^3}{s^3}$, $\frac{e^7}{s^7}$, $\frac{e^{11}}{s^{11}}$, &c.

37. If we multiply the foregoing compound series called γ , (which was obtained in art. 29) by the quantity $6z$, the product will be the following compound series; to wit,

6 Be +

$$\begin{aligned}
& 6 B e + \frac{6 B^2 e^3}{z^2} - \frac{6 B C e^5}{z^4} + \frac{6 B D e^7}{z^6} - \frac{6 B E e^9}{z^8} + \frac{6 B F e^{11}}{z^{10}} - \frac{6 B G e^{13}}{z^{12}} + \&c \\
& - \frac{6 D e^3}{z^2} - \frac{6 B D e^5}{z^4} + \frac{6 C D e^7}{z^6} - \frac{6 D^2 e^9}{z^8} + \frac{6 D E e^{11}}{z^{10}} - \frac{6 D F e^{13}}{z^{12}} + \&c \\
& + \frac{6 F e^5}{z^4} + \frac{6 B F e^7}{z^6} - \frac{6 C F e^9}{z^8} + \frac{6 D F e^{11}}{z^{10}} - \frac{6 E F e^{13}}{z^{12}} + \&c \\
& - \frac{6 H e^7}{z^6} - \frac{6 B H e^9}{z^8} + \frac{6 C H e^{11}}{z^{10}} - \frac{6 D H e^{13}}{z^{12}} + \&c \\
& + \frac{6 K e^9}{z^8} + \frac{6 B K e^{11}}{z^{10}} - \frac{6 C K e^{13}}{z^{12}} + \&c \\
& - \frac{M e^{11}}{z^{10}} - \frac{B M e^{13}}{z^{12}} + \&c \\
& + \frac{O e^{13}}{z^{12}} + \&c.
\end{aligned}$$

Let this compound series, for the sake of brevity, be denoted by the small Greek letter δ . Then, since the product of the multiplication of q into the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ *ad infinitum* is equal to $6 z \times$ the compound series γ , it will also be equal to the compound series δ .

Of the analogy between the foregoing compound series δ and the compound series Δ obtained above in art. 16.

38. Now since the compound series Δ , obtained above in art. 16, is equal to $6 s \times$ the compound series Γ , and the compound series δ is equal to $6 z \times$ the compound series γ , it follows that there will be the same analogy and the same differences between the two compound serieses Δ and δ as between the two compound serieses Γ and γ . And consequently the analogy and the differences which are shewn in art. 36 to take place between the two compound serieses Γ and γ , will take place also between the two compound serieses Δ and δ , and therefore,

In the first place, the second and other following terms of the compound series δ will involve the fractions $\frac{e^3}{z^2}, \frac{e^5}{z^4}, \frac{e^7}{z^6}, \frac{e^9}{z^8}, \frac{e^{11}}{z^{10}}, \frac{e^{13}}{z^{12}}, \&c$, instead of the corresponding fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}}, \&c$, which are involved in the second and other following terms of the compound series Δ .

And 2dly, the co-efficients of the several fractions $\frac{e^3}{z^2}, \frac{e^5}{z^4}, \frac{e^7}{z^6}, \frac{e^9}{z^8}, \frac{e^{11}}{z^{10}}, \frac{e^{13}}{z^{12}}, \&c$, in the terms of the second and other following vertical columns of the compound series δ will be the same with the co-efficients of the corresponding fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}}, \&c$, respectively, in the terms of the second and other following vertical columns of the compound series Δ .

And, 3dly, the signs — and +, that are to be prefixed to the several terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the compound series δ (which involve in them the fractions $\frac{e^5}{z^4}, \frac{e^9}{z^8}, \frac{e^{13}}{z^{12}}, \&c$) will be the same as are to be prefixed to the corresponding terms of the 3d, 5th, 7th, and other following odd vertical columns of terms in the compound series Δ which involve in them the corresponding fractions $\frac{e^5}{s^4}, \frac{e^9}{s^8}, \frac{e^{13}}{s^{12}}, \&c$; and the signs + and — which are to be prefixed to the several terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the compound series δ (which involve in them the fractions $\frac{e^3}{z^2}, \frac{e^7}{z^6}, \frac{e^{11}}{z^{10}}, \frac{e^{15}}{z^{14}}, \&c$) will be, respectively, contrary to those which are to be prefixed to the corresponding terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of terms in the compound series Δ , which involve in them the fractions $\frac{e^3}{s^2}, \frac{e^7}{s^6}, \frac{e^{11}}{s^{10}}, \frac{e^{15}}{s^{14}}, \&c$.

The product of the multiplication of the co-efficient q into the expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ is equal to the simple series $6 B e + \frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c$.

39. The compound series mentioned above in art. 19, and denoted by the Greek capital letter Σ , is equal to $6 B e$ — the compound series Δ . And it is shewn in art. 25 and 26 that the compound series Σ , or $6 B e - \Delta$, is equal to the simple series $\frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$. Therefore (adding Δ to both sides) we shall have $6 B e = \Delta + \frac{P e^3}{s^2} + \frac{Q e^5}{s^4} + \frac{R e^7}{s^6} + \frac{S e^9}{s^8} + \frac{T e^{11}}{s^{10}} + \frac{V e^{13}}{s^{12}} + \&c$, and $\Delta =$ the simple series $6 B e - \frac{P e^3}{s^2} - \frac{Q e^5}{s^4} - \frac{R e^7}{s^6} - \frac{S e^9}{s^8} - \frac{T e^{11}}{s^{10}} - \frac{V e^{13}}{s^{12}} - \&c$, in which all the terms after the first term $6 B e$ are marked with the sign —, or subtracted from the said first term. But it has been shewn in the foregoing art. 38, that the co-efficients of the fractions $\frac{e^3}{z^2}, \frac{e^5}{z^4}, \frac{e^7}{z^6}, \frac{e^9}{z^8}, \frac{e^{11}}{z^{10}}, \frac{e^{13}}{z^{12}}, \&c$, in the compound series δ are equal to, or the same with, the co-efficients of the corresponding fractions $\frac{e^3}{s^2}, \frac{e^5}{s^4}, \frac{e^7}{s^6}, \frac{e^9}{s^8}, \frac{e^{11}}{s^{10}}, \frac{e^{13}}{s^{12}}, \&c$ in the compound series Δ , respectively; and that the signs — and +, that are to be prefixed to the several terms of the 3d, 5th, 7th, and other following odd vertical columns of the compound series δ , (which involve in them the fractions

$\frac{e^5}{z^4},$

$\frac{e^5}{z^4}, \frac{e^9}{z^8}, \frac{e^{13}}{z^{12}}, \&c$) are the same with those which are to be prefixed to the corresponding terms of the 3d, 5th, 7th, and other following odd vertical columns of the compound series Δ , which involve in them the corresponding fractions $\frac{e^5}{z^4}, \frac{e^9}{z^8}, \frac{e^{13}}{z^{12}}, \&c$; and that the signs which are to be prefixed to the several terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of the compound series δ (which involve in them the fractions $\frac{e^3}{z^2}, \frac{e^7}{z^6}, \frac{e^{11}}{z^{10}}, \frac{e^{15}}{z^{14}}, \&c$) are, respectively, contrary to those which are to be prefixed to the corresponding terms of the 2d, 4th, 6th, 8th, and other following even vertical columns of the compound series Δ , which involve in them the fractions $\frac{e^3}{z^2}, \frac{e^7}{z^6}, \frac{e^{11}}{z^{10}}, \frac{e^{15}}{z^{14}}, \&c$. And in both the said compound series Δ and δ the first term is $6 B e$. It follows therefore, that, since the compound series Δ is equal to the simple series $6 B e - \frac{P e^3}{z^2} - \frac{Q e^5}{z^4} - \frac{R e^7}{z^6} - \frac{S e^9}{z^8} - \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} - \&c$ *ad infinitum*, the compound series δ must be equal to the simple series $6 B e + \frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c$ *ad infinitum*. But, by art. 37, the product of the multiplication of the co-efficient q into the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ is equal to the compound series δ . Therefore the product of the multiplication of the co-efficient q into the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ will also be equal to the simple series $6 B e + \frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c$ *ad infinitum*.

Of the relation of the product of the multiplication of the co-efficient q into the expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ to the cube of the said expression.

40. We are now to shew that the product of the multiplication of the co-efficient q into the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ is greater than the cube of the said expression, and that the difference is equal to r , or to $6 B e$, which is $(= 6 \times \frac{1}{3} \times \frac{r}{2} = \frac{6}{6} \times r) = r$.

Now this will be evident, if we can shew that the simple series $6Be + \frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} + \&c$ *ad infinitum* (which we have shewn to be equal to the said product) is greater than the said cube, and that the difference is equal to r , or $6Be$; or that the said series, when its first term $6Be$ is taken from it, will be equal to the said cube; or that the simple series $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} + \&c$ *ad infinitum*, will be equal to the cube of the said expression $2z \frac{1}{3} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum*. This we will therefore now endeavour to prove in the manner following.

41. The cube of the expression $2z \frac{1}{3} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ is $= 8z \times$ the cube of the said series. We must therefore raise the said series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ to its cube, or third power, by multiplying it twice into itself. This may be done as follows.

The multiplication of the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ twice into itself, in order to obtain its cube.

$$\begin{array}{r}
 \frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c \\
 \frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c \\
 \hline
 \frac{B^2e^2}{z^2} - \frac{BD e^4}{z^4} + \frac{BF e^6}{z^6} - \frac{BH e^8}{z^8} + \frac{BK e^{10}}{z^{10}} - \frac{BM e^{12}}{z^{12}} + \&c \\
 - \frac{BD e^4}{z^4} + \frac{D^2e^6}{z^6} - \frac{DE e^8}{z^8} + \frac{DH e^{10}}{z^{10}} - \frac{DK e^{12}}{z^{12}} + \&c \\
 + \frac{BF e^6}{z^6} - \frac{DF e^8}{z^8} + \frac{F^2e^{10}}{z^{10}} - \frac{FH e^{12}}{z^{12}} + \&c \\
 - \frac{BH e^8}{z^8} + \frac{DH e^{10}}{z^{10}} - \frac{FH e^{12}}{z^{12}} + \&c \\
 + \frac{BK e^{10}}{z^{10}} - \frac{DK e^{12}}{z^{12}} + \&c \\
 - \frac{BM e^{12}}{z^{12}} + \&c
 \end{array}$$

$$\frac{B^3e^3}{z^3}$$

$$\begin{aligned}
& \frac{B^2 e^2}{z^2} - \frac{2 BD e^4}{z^4} + \frac{2 BF e^6}{z^6} - \frac{2 BH e^8}{z^8} + \frac{2 BK e^{10}}{z^{10}} - \frac{2 BM e^{12}}{z^{12}} + \&c \\
& + \frac{D^2 e^6}{z^6} - \frac{2 DF e^8}{z^8} + \frac{2 DH e^{10}}{z^{10}} - \frac{2 DK e^{12}}{z^{12}} + \&c \\
& + \frac{F^2 e^{10}}{z^{10}} - \frac{2 FH e^{12}}{z^{12}} + \&c \\
& \frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \\
& \frac{B^3 e^3}{z^3} - \frac{2 B^2 D e^5}{z^5} + \frac{2 B^2 F e^7}{z^7} - \frac{2 B^2 H e^9}{z^9} + \frac{2 B^2 K e^{11}}{z^{11}} - \frac{2 B^2 M e^{13}}{z^{13}} + \&c \\
& + \frac{BD^2 e^7}{z^7} - \frac{2 BDF e^9}{z^9} + \frac{2 BDH e^{11}}{z^{11}} - \frac{2 BDK e^{13}}{z^{13}} + \&c \\
& + \frac{BF^2 e^{11}}{z^{11}} - \frac{2 BFH e^{13}}{z^{13}} + \&c \\
& - \frac{B^2 D e^5}{z^5} + \frac{2 BD^2 e^7}{z^7} - \frac{2 BDF e^9}{z^9} + \frac{2 BDH e^{11}}{z^{11}} - \frac{2 BDK e^{13}}{z^{13}} + \&c \\
& - \frac{D^3 e^9}{z^9} + \frac{2 D^2 F e^{11}}{z^{11}} - \frac{2 D^2 H e^{13}}{z^{13}} + \&c \\
& - \frac{DF^2 e^{13}}{z^{13}} + \&c \\
& + \frac{B^2 F e^7}{z^7} - \frac{2 BDF e^9}{z^9} + \frac{2 BF^2 e^{11}}{z^{11}} - \frac{2 BFH e^{13}}{z^{13}} + \&c \\
& + \frac{D^2 F e^{11}}{z^{11}} - \frac{2 DF^2 e^{13}}{z^{13}} + \&c \\
& - \frac{B^2 H e^9}{z^9} + \frac{2 BDH e^{11}}{z^{11}} - \frac{2 BFH e^{13}}{z^{13}} + \&c \\
& - \frac{D^2 H e^{13}}{z^{13}} + \&c \\
& + \frac{B^2 K e^{11}}{z^{11}} - \frac{2 BDK e^{13}}{z^{13}} + \&c \\
& - \frac{B^2 M e^{13}}{z^{13}} + \&c \\
& \frac{B^3 e^3}{z^3} - \frac{3 B^2 D e^5}{z^5} + \frac{3 B^2 F e^7}{z^7} - \frac{3 B^2 H e^9}{z^9} + \frac{3 B^2 K e^{11}}{z^{11}} - \frac{3 B^2 M e^{13}}{z^{13}} + \&c \\
& + \frac{3 BD^2 e^7}{z^7} - \frac{6 BDF e^9}{z^9} + \frac{6 BDH e^{11}}{z^{11}} - \frac{6 BDK e^{13}}{z^{13}} + \&c \\
& - \frac{D^3 e^9}{z^9} + \frac{3 D^2 F e^{11}}{z^{11}} - \frac{6 D^2 H e^{13}}{z^{13}} + \&c \\
& + \frac{3 D^2 F e^{11}}{z^{11}} - \frac{3 D^2 H e^{13}}{z^{13}} + \&c \\
& - \frac{3 DF^2 e^{13}}{z^{13}} + \&c.
\end{aligned}$$

This last compound series is the cube of the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ *ad infinitum*.

Therefore

Therefore $8z \times$ the cube of the said series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ will be $= 8z \times$ the foregoing compound series; or, if, for the sake of brevity, we denote the said compound series by the small Greek letter λ , we shall have $8z \times$ the cube of the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ equal to $8z \times$ the compound series λ ; and consequently the cube of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum* will be $= 8z \times$ the compound series λ .

A comparison between the compound series λ , obtained in the foregoing article 41, and the compound series Λ , obtained above in art. 17.

42. We must now compare the compound series λ , which is equal to the cube of the simple series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$, with the compound series Λ , obtained above in art. 17, which is equal to the cube of the simple series $\frac{Be}{z} + \frac{De^3}{z^3} + \frac{Fe^5}{z^5} + \frac{He^7}{z^7} + \frac{Ke^9}{z^9} + \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$, in order to discover the analogies and the differences that subsist between them.

Now, upon comparing together these two compound serieses Λ and λ , we shall find that,

In the first place, wherever there is any power of the fraction $\frac{e}{z}$ in the compound series Λ , there will be the same power of the fraction $\frac{e}{z}$ in the corresponding term of the compound series λ .

And, in the second place, the co-efficients of the terms of the compound series λ are equal to, or the same with, the co-efficients of the corresponding terms of the compound series Λ .

And these two analogies between these two compound serieses must take place not only in the few first vertical columns of terms of the said serieses which are set down above in art. 17 and art. 41, but throughout all the terms of the said serieses, to whatever number of terms the said serieses may be continued: because the original series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ (by the multiplication of which twice into itself the compound series λ is produced) involves in its terms the same powers of the fraction $\frac{e}{z}$, and the same co-efficients B, D, F, H, K, M, O, &c, combined with those powers respectively,

tively, as the other original series $\frac{B e}{z} + \frac{D e^3}{z^3} + \frac{F e^5}{z^5} + \frac{H e^7}{z^7} + \frac{K e^9}{z^9} + \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} + \&c$ (by the multiplication of which twice into itself the compound series Λ was produced) involves of the fraction $\frac{e}{z}$.

And, in the 3d place, all the terms of the compound series Λ , after the first term $\frac{B^3 e^3}{z^3}$, are marked with the sign $+$, or added to the said first term: but in the compound series λ only the terms of the 3d, and 5th, and other following odd vertical columns of terms (which involve the fractions $\frac{e^7}{z^7}$, $\frac{e^{11}}{z^{11}}$, $\frac{e^{15}}{z^{15}}$, $\frac{e^{19}}{z^{19}}$, &c) are marked with the sign $+$, or added to the first term $\frac{B^3 e^3}{z^3}$; and the second term $\frac{3 B^2 D e^5}{z^5}$, and the terms in the 4th, 6th, 8th, and other following even vertical columns of terms in the said series (which involve the fractions $\frac{e^5}{z^5}$, $\frac{e^9}{z^9}$, $\frac{e^{13}}{z^{13}}$, $\frac{e^{17}}{z^{17}}$, &c) are marked with the sign $-$, or subtracted from the said first term.

43. That in the compound series λ the terms in the second, third, and other following vertical columns of terms are marked with the signs $-$ and $+$ alternately, is evident upon inspection as far as the multiplication is carried above in art. 41, that is, as far as the six first vertical columns of terms. And that the same alternate succession of those signs will take place in all the following columns of terms in the said series, to whatever number of terms the said series may be continued, will be evident from the following considerations.

Since in the multiplication of the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ into itself in art. 41, in order to obtain its square, the signs $-$ and $+$ follow each other alternately both in the multiplicand and the multiplier (which are both the same series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \&c$), it is evident that in the several products of the multiplicand by the first term $\frac{B e}{z}$ of the multiplier, and by its third term $\frac{F e^5}{z^5}$, and its fifth term $\frac{K e^9}{z^9}$, and its seventh term $\frac{O e^{13}}{z^{13}}$, and all its following odd terms (to which odd terms the sign $+$ is prefixed) the order of the signs $+$ and $-$ in the terms of the said products must be the same as in the multiplicand itself; that is, the terms of the said several products which constitute the 1st, 3d, 5th, 7th, and other following odd horizontal lines of the said general product (before the similar terms are collected together by addition at the bottom) will be marked with the signs $+$ and $-$ alternately. And it is likewise evident, that in the several products of the said multiplicand by the second term $\frac{D e^3}{z^3}$ of the multiplier (which is marked with

the

the sign —), and by the fourth term $\frac{H e^7}{z^7}$, and the sixth term $\frac{M e^{11}}{z^{11}}$, and the following even terms of the multiplicator (which are all marked with the sign —), the order of the signs + and — in the terms of the said products must be contrary to the order of them in the terms of the multiplicand itself; that is, the terms of the said several products which constitute the 2d, 4th, 6th, 8th, and other following even horizontal rows of terms in the said general product (before the similar terms are collected together by addition at the bottom) will be marked with the signs — and + alternately. Therefore the first term of every new horizontal row of terms will be marked with a contrary sign to that of the first term of the horizontal row next above it, and consequently with the same sign as the second term of the said horizontal row next above it, or as the term under which it is placed; the first term of every new horizontal row being placed immediately under the second term of the horizontal row next above it. And thus it appears that the last term of every vertical row and the last term but one will be marked with the same sign. And hence it follows (from the alternate succession of the signs + and — to each other in the terms of all the horizontal rows of terms in the said general product, before the terms are collected together at the bottom) that all the terms in every vertical column of terms will be marked with the same sign as the lowest term, and consequently that the terms in the second, third, fourth, fifth, sixth, and other following vertical columns of terms will be marked with the sign — and the sign + alternately. Q. E. D.

44. And, since all the terms in each of the vertical columns of the general product in art. 41 (before the similar terms are collected together at the bottom by addition) are marked with the same sign + or —, and the terms in the 2d, 3d, 4th, 5th, 6th, and other following vertical columns of terms in the said general product are marked with the sign — and the sign + alternately, it follows that in the compound series set down under the said general product (and which is equal to it and derived from it by adding together the similar terms in each vertical column so as to convert them into single terms) the several terms of the second, and third, and fourth, and fifth, and sixth, and other following vertical columns of terms will also be marked with the sign — and the sign + alternately. Q. E. D.

45. And, since in the compound series last mentioned, which is equal to the square of the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$, the second and other following vertical columns of terms are marked with the sign — and the sign + alternately, and the compound series Λ (which is the cube of the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$) is produced by multiplying the said compound series into the original series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$, in which the second and other following terms

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are

are also marked with the sign — and the sign + alternately, it will follow, by repeating the reasonings used in the two last articles 43 and 44, that in the product of this last multiplication, that is, in the compound series λ , the terms of the second, and third, and fourth, and fifth, and sixth, and other following vertical columns will also be marked with the sign — and the sign + alternately, agreeably to what we have seen by inspection of the said compound series λ (as set down in art. 41) to take place in the first six vertical columns of it, or as far as the terms that involve the fraction $\frac{e^{13}}{z^{13}}$, and, agreeably likewise, to what was asserted, in art. 42, concerning all the following vertical columns of the said compound series.

Q. E. D.

Of the compound series π , which is equal to $8z \times$ the compound series λ , and consequently to the cube of the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ ad infinitum.

46. If the foregoing compound series λ , obtained in art. 41, be multiplied into $8z$, the product will be the following compound series, to wit,

$$\begin{aligned} & \frac{8B^3e^3}{z^2} - \frac{24B^2De^5}{z^4} + \frac{24B^2Fe^7}{z^6} - \frac{24B^2He^9}{z^8} + \frac{24B^2Ke^{11}}{z^{10}} - \frac{24B^2Me^{13}}{z^{12}} + \&c \\ & + \frac{24BD^2e^7}{z^6} - \frac{48BDFe^9}{z^8} + \frac{48BDHe^{11}}{z^{10}} - \frac{48BDKe^{13}}{z^{12}} + \&c \\ & - \frac{8D^3e^9}{z^8} + \frac{24BF^2e^{11}}{z^{10}} - \frac{48BFHe^{13}}{z^{12}} + \&c \\ & + \frac{24D^2Fe^{11}}{z^{10}} - \frac{24D^2He^{13}}{z^{12}} + \&c \\ & + \frac{24DF^2e^{13}}{z^{12}} + \&c, \end{aligned}$$

which, for the sake of brevity, we will denote by the small Greek letter π .

Then will the cube of the expression $2z\sqrt[3]{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ (which is equal to $8z \times$ the compound series λ) be = the compound series π .

47. Since the compound series π is = $8z \times$ the compound series λ ; and the terms of the second, and third, and fourth, and other following vertical columns of the compound series λ are marked with the sign — and the sign + alternately; it is evident that the terms of the second, and third, and fourth, and other following vertical columns of the compound series π will also be marked with the sign — and the sign + alternately: because the multiplication into $8z$ can make no change in the signs of the quantities that are multiplied.

A comparison between the compound series π just now obtained in art. 46, and the compound series Π , obtained above in art. 18.

48. It has been shewn above in art. 42, that the co-efficients of the several terms of the compound series λ (set down in art. 41) are respectively equal to, or the same with, the co-efficients of the corresponding terms of the compound series Λ , set down in art. 17. And therefore the co-efficients of the several terms of the compound series π , or $8z \times$ the compound series λ , which are equal to 8 times the co-efficients of the corresponding terms of the compound series λ , must be equal to the co-efficients of the corresponding terms in the compound series Π , or $8s \times$ the compound series Λ (set down above in art. 18), which are equal to 8 times the co-efficients of the corresponding terms of the compound series Λ .

A proof that the compound series π is equal to the simple series $\frac{Pe^3}{z^2} - \frac{Qe^5}{z^4} + \frac{Re^7}{z^6} - \frac{Se^9}{z^8} + \frac{Te^{11}}{z^{10}} - \frac{Ve^{13}}{z^{12}} + \mathcal{E}c$ ad infinitum.

49. It appears from the last article 48 that the co-efficients of the several terms of the compound series π , which is set down in art. 46, are respectively equal to, or the same with, the co-efficients of the corresponding terms of the compound series Π , which is set down in art. 18; and it appears from art. 47 that all the terms in the second and other following vertical columns of the compound series π will be marked with the sign $-$ and the sign $+$ alternately, to wit, all the terms in the 2d, 4th, 6th, 8th, and other following even vertical columns of it with the sign $-$, and all the terms of the 3d, 5th, 7th, 9th, and other following odd vertical columns of it with the sign $+$; whereas in the compound series Π all the terms in the 2d, 3d, 4th, 5th, and all the following vertical columns, both odd and even, are marked with the sign $+$. It follows therefore that, if the compound series Π be converted into a simple series by adding together into one sum all the terms of each separate vertical column, and the compound series π be likewise converted into a simple series by adding together into one sum all the terms of each separate column, the co-efficients of the terms of the second simple series, which is equal to the compound series π , will be respectively equal to the co-efficients of the corresponding terms of the former simple series, which is equal to the compound series Π : but in the simple series that is equal to the compound series π , the second, and fourth, and sixth, and other following even terms will be marked with the sign $-$, and the third, and fifth, and seventh, and other following odd terms will be marked with the sign $+$; whereas in the simple series that is equal to the compound series Π all the terms after the first term will be marked with the sign $+$. Therefore,

fore, if we suppose (as was done above in art. 23) the co-efficients of the terms of the simple series that is equal to the compound series Π to be $P, Q, R, S, T, V, \&c$, and consequently the said simple series to be $\frac{P e^3}{z^2} + \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} + \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} + \frac{V e^{13}}{z^{12}} + \&c \text{ ad infinitum}$, the simple series that is equal to the compound series π will be $\frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c \text{ ad infinitum}$.

A proof derived from the foregoing articles, that the expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ is equal to one of the roots of the cubick equation $q x - x^3 = r$.

50. It is shewn in art. 46 that the compound series π is equal to the cube of the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$. Therefore the simple series $\frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c \text{ ad infinitum}$ (which is equal to the compound series π) will also be equal to the cube of the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \text{ ad infinitum}$.

51. It was shewn above in art. 39 that the product of the multiplication of the co-efficient q into the expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \text{ ad infinitum}$ is equal to the series $6 B e + \frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c \text{ ad infinitum}$. And now we have seen, in the last art. 50, that the cube of the said expression $2 z^{\frac{1}{3}} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c$ is equal to the series $\frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c \text{ ad infinitum}$, which consists of the very same terms as the series $6 B e + \frac{P e^3}{z^2} - \frac{Q e^5}{z^4} + \frac{R e^7}{z^6} - \frac{S e^9}{z^8} + \frac{T e^{11}}{z^{10}} - \frac{V e^{13}}{z^{12}} + \&c \text{ ad infinitum}$, except the first term $6 B e$, and therefore is less than the said last-mentioned

tioned series by the difference $6 B e$. It follows therefore, that the cube of the expression $2 z \frac{1}{3} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \text{ ad infinitum}$, is less than the product of the multiplication of the co-efficient q into the said expression, and that their difference is $6 B e$, or r . And consequently the said expression $2 z \frac{1}{3} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \text{ ad infinitum}$ must be equal to one of the roots of the cubick equation $q x - x^3 = r$. Q. E. D.

52. Having now shewn that the expression $2 z \frac{1}{3} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c \text{ ad infinitum}$ is equal to one of the roots of the cubick equation $q x - x^3 = r$, it remains that we shew that it cannot be equal to the greater of the two roots of that equation, being always less than the least possible magnitude of the said greater root; whence it will follow that it must be equal to the lesser root of the said equation. This may be shewn in the manner following.

Of the least possible magnitude, or the lower limit of the magnitude, of the greater root of the cubick equation $q x - x^3 = r$ upon the supposition that r is less than $\sqrt{2} \times \frac{q \sqrt{q}}{3 \sqrt{3}}$.

53. The greatest possible magnitude of the absolute term r of the cubick equation $q x - x^3 = r$ is $\frac{2q \sqrt{q}}{3 \sqrt{3}}$, or that value of the compound quantity $q x - x^3$ which results from a supposition that x is $= \frac{\sqrt{q}}{\sqrt{3}}$. If r is greater than this quantity $\frac{2q \sqrt{q}}{3 \sqrt{3}}$, the equation $q x - x^3 = r$ is impossible; and if r is exactly equal to $\frac{2q \sqrt{q}}{3 \sqrt{3}}$, the equation $q x - x^3 = r$ will have but one root, which will be $= \frac{\sqrt{q}}{\sqrt{3}}$; and, if r is less than $\frac{2q \sqrt{q}}{3 \sqrt{3}}$, the equation $q x - x^3 = r$ will have two roots, of which the lesser will be less than $\frac{\sqrt{q}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{q}}{\sqrt{3}}$, but less than $\sqrt{q}$. But in the foregoing articles it is supposed that the absolute term r is less, not only than $\frac{2q \sqrt{q}}{3 \sqrt{3}}$ (which is its greatest possible magnitude), but likewise than $\sqrt{2} \times \frac{q \sqrt{q}}{3 \sqrt{3}}$, which is less than $\frac{2q \sqrt{q}}{3 \sqrt{3}}$ in the proportion of ($\sqrt{2}$ to 2, or of) 1 to $\sqrt{2}$. Therefore the greater root of the equation $q x - x^3 = r$ must, on the

the present supposition, be greater, not only than $\frac{\sqrt{q}}{\sqrt{3}}$, but than the greater root of it when r is $= \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. Now, when r is $= \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, the greater root of the equation $qx - x^3 = r$, will be $\sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}}$. For, if we suppose x to be $= \sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}}$, we shall have $x^3 = 2\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, and $qx (= q \times \sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}} = \sqrt{2} \times \frac{q\sqrt{q}}{\sqrt{3}}) = 3\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, and consequently $qx - x^3 (= 3\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}} - 2\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}) = 1 \times \sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. Therefore, on the present supposition, to wit, that r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, the greater root of the equation $qx - x^3 = r$ must be greater than $\sqrt{2} \times \frac{\sqrt{q}}{\sqrt{3}}$, or than $\frac{\sqrt{2}}{\sqrt{3}} \times \sqrt{q}$, or than $\sqrt{\frac{2}{3}} \times \sqrt{q}$, or than $\sqrt{0.666,666}$, &c $\times \sqrt{q}$, or than $0.8165 \times \sqrt{q}$; or, in other words, $\sqrt{\frac{2}{3}} \times \sqrt{q}$, or $0.8165 \times \sqrt{q}$, will be the least possible magnitude, or the lower limit of the magnitude, of the greater root of the equation $qx - x^3 = r$ upon the present supposition.

A proof that the said lower limit of the magnitude of the greater root of the cubick equation $qx - x^3 = r$ is greater than the greatest possible magnitude of the expression $2\sqrt{\frac{2}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c.$

Now this quantity $\sqrt{\frac{2}{3}} \times \sqrt{q}$, or $0.8165 \times \sqrt{q}$, is greater than the greatest possible magnitude of the expression $2z \frac{1}{3} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum*; as may be shewn in the manner following.

54. This expression $2z \frac{1}{3} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ is equal to the series $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{10}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}} - \&c.$

Now, since zz is $= \frac{q^3}{27} - \frac{rr}{4}$, or $\frac{q^3}{27} - ee$, it is evident that, if ee , or $\frac{rr}{4}$, be supposed

supposed to increase, zz , or the excess of $\frac{q^3}{27}$ above ee or $\frac{rr}{4}$, will at the same time decrease. Therefore, if ee , or $\frac{rr}{4}$, is supposed to increase from 0 to its greatest possible magnitude on the present supposition, to wit, to $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, the numerators of the terms of the series $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}} - \&c$ (which involve in them the powers of e) will continually increase, and their denominators (which are powers of the decreasing quantity z) will at the same time continually decrease: and consequently the terms of the said series will, on account both of the increase of their numerators and the decrease of their denominators, continually increase at the same time. Therefore the said series $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}} - \&c$ will have attained its greatest magnitude when ee , or $\frac{rr}{4}$, has attained its greatest magnitude, or is $= \frac{q^3}{54}$; at which time zz , or $\frac{q^3}{27} - ee$, or $\frac{q^3}{27} - \frac{rr}{4}$, will be $= \frac{q^3}{27} - \frac{q^3}{54}$, or $\frac{2q^3}{54} - \frac{q^3}{54}$, or $\frac{q^3}{54}$, and consequently will be equal to ee . Therefore the

expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ (which is equal to the series $\frac{2Be}{z^{\frac{2}{3}}} - \frac{2De^3}{z^{\frac{8}{3}}} + \frac{2Fe^5}{z^{\frac{14}{3}}} - \frac{2He^7}{z^{\frac{20}{3}}} + \frac{2Ke^9}{z^{\frac{26}{3}}} - \frac{2Me^{11}}{z^{\frac{32}{3}}} + \frac{2Oe^{13}}{z^{\frac{38}{3}}} - \&c$) will also have attained its greatest magnitude when ee or $\frac{rr}{4}$, is equal to $\frac{q^3}{54}$, and zz is consequently equal to ee . But, when zz is equal to ee , and ee is equal to $\frac{q^3}{54}$, the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ becomes $= B - D + F - H + K - M + O \&c$, and z is ($= e = \frac{r}{2}$) $= \sqrt[6]{\frac{q^3}{54}}$, and consequently $z^{\frac{1}{3}} = \sqrt[6]{\frac{q^3}{54}}$, and $2z^{\frac{1}{3}} = 2\sqrt[6]{\frac{q^3}{54}} = \frac{2\sqrt[6]{q^3}}{\sqrt[6]{54}} = \frac{2\sqrt[2]{q}}{\sqrt[6]{54}} = \frac{\sqrt[6]{64} \times \sqrt[2]{q}}{\sqrt[6]{54}} = \frac{\sqrt[6]{64}}{\sqrt[6]{54}} \times \sqrt{q} = \sqrt[6]{\frac{64}{54}} \times \sqrt{q} = \sqrt[6]{\frac{32}{27}} \times \sqrt{q} = \sqrt[6]{1.185,185,185} \times \sqrt{q} = 1.028,7 \times \sqrt{q}$; and consequently the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ becomes in this case $= 1.0287 \times \sqrt{q} \times$ the series $B - D + F - H + K - M + O - \&c$; or the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ is $= 1.0287 \times \sqrt{q} \times$ the series $B - D + F - H + K - M + O - \&c$ *ad infinitum*.

55. Now

55. Now because the terms B, D, F, H, K, M, O, &c, are a decreasing progression, every term being less than that which immediately precedes it, the first term B alone of the said series $B - D + F - H + K - M + O - \&c$ must be greater than the whole series. Therefore $1.0287 \times \sqrt{q} \times B$ will be greater than $1.0287 \times \sqrt{q} \times$ the whole series $B - D + F - H + K - M + O - \&c$,

and consequently than the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum*.

But B is $= \frac{1}{3}$. Therefore $1.0287 \times \sqrt{q} \times \frac{1}{3}$, or $\frac{1.0287}{3} \times \sqrt{q}$, or $0.3429 \times \sqrt{q}$,

will be greater than the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$. But

$0.3429 \times \sqrt{q}$ is much less than $0.8165 \times \sqrt{q}$, which is the least possible magnitude of the greater root of the equation $qx - x^3 = r$ upon the present supposition that $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, or that r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$. Therefore, *a for-*

tiori, the greatest possible magnitude of the expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ is less than $0.8165 \times \sqrt{q}$, or the least possible magnitude of the greater root of the equation $qx - x^3 = r$.

Therefore the said expression $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum* cannot be equal to the greater root of the said equation. But it has been shewn above in art. 51 that the said expression is equal to one of the roots of the said equation $qx - x^3 = r$ in this case of the said equation, or when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$. Therefore it must be equal to the lesser root of the said equation. Q. E. D.

*End of the demonstration that the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum* is equal to the lesser root of the cubick equation $qx - x^3 = r$.*

56. I have now compleated the demonstration of the proposition laid down above in art. 11, to wit, that, if the absolute term r of the cubick equation $qx - x^3 = r$ be less, not only than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ (which is its greatest possible magnitude) but than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$; or if $\frac{rr}{4}$ be less, not only than $\frac{q^3}{27}$, but than $\frac{q^3}{2 \times 27}$,
or

or $\frac{q^3}{54}$; and e be taken $= \frac{r}{2}$, and zz be $= \frac{q^3}{27} - \frac{rr}{4}$, or $\frac{q^3}{27} - ee$; the leffer root

of the equation $qx - x^3 = r$ will be equal to the expreffion $2z^{\frac{1}{3}} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ *ad infinitum*. It is indeed a very long and complicated demonstration. But I know not how to make it shorter without taking from the perspicuity of the reasonings used in it, which are both various and abstruse. And "that they should be so" will appear the less surprizing, if we consider that they supply the place of those very obscure and intricate operations by which many writers of Algebra endeavour to find the roots of impossible quantities, such as $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$. See upon this subject *Monfieur Clairaut's Elémens d'Algèbre*, Part V, art. ix, pages 286, 287, 288, 289, and a paper of *Monfieur Nicole* in the *Memoirs of the Academy of Sciences at Paris* for the year 1738, pages 99 and 100, from which *Monfieur Clairaut* has extracted what he has delivered upon this subject in the pages of his Algebra just now cited. And see also *Dr. Wallis's Algebra*, chapter 48, pages 179, 180, of the folio edition at London in 1685, and *Professor Saunderson's Algebra*, pages 744, 745, 746, 747, and *Mac Laurin's Algebra*, Part I. the supplement to the 14th chapter, pages 127, 128, 129, 130, and the *Philosophical Transactions*, No 451.

57. We will now proceed to give an example of the resolution of a cubick equation of the foregoing form $qx - x^3 = r$, when r is less than $\sqrt[3]{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by means of the expreffion $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$, which we have shewn in the foregoing articles to be equal to its leffer root.

An Example of the resolution of a cubick equation of the foregoing form, $qx - x^3 = r$, by means of the expreffion $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$.

Let the equation that is to be resolved by means of this expreffion be $15x - x^3 = 4$.

Here q is $= 15$, and r is $= 4$; and consequently $\frac{q}{3}$ is $(= \frac{15}{3}) = 5$, and $\frac{r}{2}$ is $(= \frac{4}{2}) = 2$, and $\frac{q^3}{27}$ is $(= \frac{15^3}{27} = 5^3) = 125$, and $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, is $(= \frac{125}{2}) = 62.5$, and $\frac{rr}{4}$ is $(= \frac{4^2}{2^2} = 2^2) = 4$, which is less than 62.5, or $\frac{q^3}{54}$. Therefore this

this equation may be resolved by means of the foregoing expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c.$

Now, since $\frac{q^3}{27}$ is $= 125$, and $\frac{rr}{4}$ is $= 4$, we shall have $zz (= \frac{q^3}{27} - \frac{rr}{4} = 125 - 4) = 121$, and consequently $z (= \sqrt{121}) = 11$, and $\sqrt[3]{z} (= \sqrt[3]{11}) = 2.223,980,090,569,361, \&c.$ and $2\sqrt[3]{z} (= 2 \times 2.223,980,090,569,361, \&c) = 4.447,960,181,138,722, \&c.$

And we shall have $ee (= \frac{rr}{4}) = 4$, and consequently $e = 2$, and $\frac{e}{z} = \frac{2}{11} = 0.181,818,181,818$, and $\frac{e^2}{z^2} = \frac{4}{121}$.

Therefore $\frac{e^2}{z^3}$ will be $(= \frac{e}{z} \times \frac{ee}{zz} = 0.181,818,181,818 \times \frac{4}{121} = \frac{0.727,272,727,272}{121}) = 0.006,010,518,407$;

And $\frac{e^5}{z^5}$ will be $(= \frac{e^3}{z^3} \times \frac{e^2}{z^2} = 0.006,010,518,407 \times \frac{4}{121} = \frac{0.024,042,073,672}{121}) = 0.000,198,694,823$;

And $\frac{e^7}{z^7}$ will be $(= \frac{e^5}{z^5} \times \frac{e^2}{z^2} = 0.000,198,694,823 \times \frac{4}{121} = \frac{0.000,794,779,292}{121}) = 0.000,006,568,423$;

And $\frac{e^9}{z^9}$ will be $(= \frac{e^7}{z^7} \times \frac{e^2}{z^2} = 0.000,006,568,423 \times \frac{4}{121} = \frac{0.000,026,273,692}{121}) = 0.000,000,217,137$;

And $\frac{e^{11}}{z^{11}}$ will be $(= \frac{e^9}{z^9} \times \frac{e^2}{z^2} = 0.000,000,217,137 \times \frac{4}{121} = \frac{0.000,000,868,548}{121}) = 0.000,000,007,178$;

And $\frac{e^{13}}{z^{13}}$ will be $(= \frac{e^{11}}{z^{11}} \times \frac{e^2}{z^2} = 0.000,000,007,178 \times \frac{4}{121} = \frac{0.000,000,028,712}{121}) = 0.000,000,000,237$.

Therefore $\frac{Be}{z}$ will be $(= B \times 0.181,818,181,818 = \frac{1}{3} \times 0.181,818,181,818 = \frac{0.181,818,181,818}{3}) = 0.060,606,060,606$;

And $\frac{De^3}{z^3}$ will be $(= D \times 0.006,010,518,407 = \frac{5}{81} \times 0.006,010,518,407 = \frac{0.030,052,592,035}{81}) = 0.000,371,019,654$;

And $\frac{Fe^5}{z^5}$ will be $(= F \times 0.000,198,694,823 = \frac{22}{729} \times 0.000,198,694,823 = \frac{0.004,371,286,106}{729}) = 0.000,005,996,277$;

And $\frac{He^7}{z^7}$ will be ($= H \times 0.000,006,568,423 = \frac{374}{19683} \times 0.000,006,568,423 = \frac{0.002,456,590,202}{19683}$) $= 0.000,000,124,807$;

And $\frac{Ke^9}{z^9}$ will be ($= K \times 0.000,000,217,137 = \frac{21505}{1,594,323} \times 0.000,000,217,137 = \frac{0.004,669,531,185}{1,594,323}$) $= 0.000,000,002,928$;

And $\frac{Me^{11}}{z^{11}}$ will be ($= M \times 0.000,000,007,178 = \frac{147,407}{14,348,907} \times 0.000,000,007,178 = \frac{0.001,058,087,446}{14,348,907}$) $= 0.000,000,000,073$;

And $\frac{Oe^{13}}{z^{13}}$ will be ($= O \times 0.000,000,000,237 = \frac{3,174,920}{387,420,489} \times 0.000,000,000,237 = \frac{0.000,752,456,040}{387,420,489}$) $= 0.000,000,000,001$.

Therefore the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ will be $= 0.060,606,060,606 - 0.000,371,019,654 + 0.000,005,996,277, - 0.000,000,124,807 + 0.000,000,002,928 - 0.000,000,000,073 + 0.000,000,000,001 - \&c = 0.060,612,059,812 - 0.000,371,144,534 = 0.060,240,915,278$; and consequently the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \&c$ will be $= 4.447,960,181,138, \&c \times 0.060,240,915,278 = 0.267,949,192,431, \&c$. Therefore this number $0.267,949,192,431, \&c$ is the lesser root of the proposed equation $15x - x^3 = 4$. Q. E. I.

The foregoing number $0.267,949,192,431$, agrees with the true value of the lesser root of the equation $15x - x^3 = 4$ in all its twelve figures, the said root being equal to $0.267,949,192,431,122, \&c$, or $2 - 1.732,050,807,568,877, \&c$, or $2 - \sqrt{3}$. For, if we suppose x to be $= 2 - \sqrt{3}$, we shall have $15x (= 15 \times [2 - \sqrt{3}] = 30 - 15\sqrt{3})$, and $x^3 (= 8 - 3 \times 4 \times \sqrt{3} + 3 \times 2 \times 3 - 3\sqrt{3} = 8 - 12\sqrt{3} + 18 - 3\sqrt{3}) = 26 - 15\sqrt{3}$, and $15x - x^3 (= 30 - 15\sqrt{3} - [26 - 15\sqrt{3}] = 30 - 15\sqrt{3} - 26 + 15\sqrt{3} = 30 - 26) = 4$.

Another Example of the resolution of a cubick equation by means of the same expression.

58. As a second example of the foregoing method of resolving cubick equations, let it be proposed to resolve the equation $90x - x^3 = 98$.

Here q , the co-efficient of x , is $= 90$, and the absolute term r is $= 98$.

Therefore $\frac{q}{3}$ will be ($= \frac{90}{3}$) $= 30$, and $\frac{q^3}{27}$ will be ($= \left[\frac{q}{3}\right]^3 = 30^3$) $= 27,000$,
5
and

and $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, will be $(= \frac{27,000}{2}) = 13,500$; and $\frac{r}{2}$ will be $(= \frac{98}{2}) = 49$, and $\frac{rr}{4}$ will be $(= 49^2) = 2401$, which is less than 13,500, or $\frac{q^3}{54}$. Therefore the equation $90x - x^3 = 98$ may be resolved by means of the foregoing expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \&c.$

Now, since $\frac{q^3}{27}$ is $= 27,000$, and $\frac{rr}{4}$ is $= 2401$, we shall have $zz (= \frac{q^3}{27} - \frac{rr}{4} = 27,000 - 2401) = 24599$, and $z (= \sqrt{24599}) = 156.840,683$, and $\sqrt[3]{z} (= \sqrt[3]{156.840,683}) = 5.392,865,326,078$, and $2 \sqrt[3]{z} (= 2 \times 5.392,865,326,078) = 10.785,730,652,156$.

And we shall have $ee (= \frac{rr}{4}) = 2401$, and consequently $e = 49$, and $\frac{e}{z} = \frac{49}{156.840,683} = 0.312,418,940,435$, and $\frac{e^2}{z^2} = \frac{2401}{24599}$.

Therefore $\frac{e^3}{z^3}$ will be $(= \frac{e}{z} \times \frac{e^2}{z^2} = 0.312,418,940,435 \times \frac{2401}{24599} = \frac{750.117,875,984,435}{24599}) = 0.030,493,836,171$;

And $\frac{e^5}{z^5}$ will be $(= \frac{e^3}{z^3} \times \frac{e^2}{z^2} = 0.030,493,836,171 \times \frac{2401}{24599} = \frac{73.115,600,646,571}{24599}) = 0.002,972,299,713$;

And $\frac{e^7}{z^7}$ will be $(= \frac{e^5}{z^5} \times \frac{e^2}{z^2} = 0.002,972,299,713 \times \frac{2401}{24599} = \frac{7.136,491,610,913}{24599}) = 0.000,290,113,078$;

And $\frac{e^9}{z^9}$ will be $(= \frac{e^7}{z^7} \times \frac{e^2}{z^2} = 0.000,290,113,078 \times \frac{2401}{24599} = \frac{0.696,561,500,278}{24599}) = 0.000,028,316,659$;

And $\frac{e^{11}}{z^{11}}$ will be $(= \frac{e^9}{z^9} \times \frac{e^2}{z^2} = 0.000,028,316,659 \times \frac{2401}{24599} = \frac{0.067,988,298,259}{24599}) = 0.000,002,763,864$;

And $\frac{e^{13}}{z^{13}}$ will be $(= \frac{e^{11}}{z^{11}} \times \frac{e^2}{z^2} = 0.000,002,763,864 \times \frac{2401}{24599} = \frac{0.006,636,037,464}{24599}) = 0.000,000,269,768$;

And $\frac{e^{15}}{z^{15}} (= \frac{e^{13}}{z^{13}} \times \frac{e^2}{z^2} = 0.000,000,269,768 \times \frac{2401}{24599} = \frac{0.000,647,712,968}{24599}) = 0.000,000,026,330$.

Therefore $\frac{B e}{z}$ will be $(= B \times 0.312,418,940,435 = \frac{1}{3} \times 0.312,418,940,435 = \frac{0.312,418,940,435}{3}) = 0.104,139,646,811$;

And $\frac{D e^3}{z^3}$ will be ($= D \times 0.030,493,836,171 = \frac{5}{81} \times 0.030,493,836,171$
 $= \frac{0.152,469,180,855}{81}$) $= 0.001,882,335.566$;

And $\frac{F e^5}{z^5}$ will be ($= F \times 0.002,972,299,713 = \frac{22}{729} \times 0.002,972,299,713$
 $= \frac{0.065,390,593,686}{729}$) $= 0.000,089,699,031$;

And $\frac{H e^7}{z^7}$ will be ($= H \times 0.000,290,113,078 = \frac{374}{19683} \times 0.000,290,113,078$
 $= \frac{0.108,502,291,172}{19683}$) $= 0.000,005,512,487$;

And $\frac{K e^9}{z^9}$ will be ($= K \times 0.000,028,316,659 = \frac{21505}{1,594,323} \times 0.000,028,316,659$
 $= \frac{0.608,949,751,795}{1,594,323}$) $= 0.000,000,381,948$;

And $\frac{M e^{11}}{z^{11}}$ will be ($= M \times 0.000,002,763,864 = \frac{147,407}{14,348,907} \times 0.000,002,763,864$
 $= \frac{0.407,412,900,648}{14,348,907}$) $= 0.000,000,028,393$;

And $\frac{O e^{13}}{z^{13}}$ will be ($= O \times 0.000,000,269,768 = \frac{3,174,920}{387,420,489} \times 0.000,000,269,768$
 $= \frac{0.856,491,818,560}{387,420,489}$) $= 0.000,000,002,210$;

And $\frac{Q e^{15}}{z^{15}}$ will be ($= Q \times 0.000,000,026,330 = \frac{70,664,648}{10,460,353,203} \times 0.000,000,026,330$
 $= \frac{1.860,600,181,840}{10,460,353,203}$) $= 0.000,000,000,177$.

Therefore the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}}$
 &c will be $= 0.104,139,646,811 - 0.001,882,335,566 + 0.000,089,699,031$
 $- 0.000,005,512,487 + 0.000,000,381,948 - 0.000,000,028,393 +$
 $0.000,000,002,210 - 0.000,000,000,177 + \&c = 0.104,229,730,000 -$
 $0.001,887,876,623 = 0.102,341,853,377$; and consequently the expression
 $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} +$
 &c will be $= 10.785,730,652,156 \times 0.102,341,853,377 = 1.103,831,664,966$, &c.
 Therefore this last number 1.103,831,664,966, &c, is the leffer root of the proposed equation $90x - x^3 = 98$. Q. E. I.

59. That this number 1.103,831,664,966 is nearly equal to the leffer root of the equation $90x - x^3 = 98$, will appear by substituting the first seven figures of it, to wit, 1.103,831, instead of x in the compound quantity $90x - x^3$. For we shall then have $90x (= 90 \times 1.103,831) = 99.344,790$, and $x^3 (= 1.103,831^3) = 1.344,955,018,877,185,191$, and consequently $90x - x^3 (= 99.344,790,000,000,000,000 - 1.344,955,018,877,185,191 = 97.999,834,981,122,814,809$; which is very nearly equal to the absolute term

term 98. Therefore 1.103,831 must be very nearly equal to one of the roots of the equation $90x - x^3 = 98$.

If we were to make the number 1.103,831 the first step of a further approximation to the true value of x in this equation, according to Mr. Raphson's method of approximation, we should find the first eleven figures of the said true value to be 1.103,832,911,1.

60. These two examples are, as I apprehend, sufficient to illustrate and confirm the foregoing method of resolving cubick equations of the foregoing form, $qx - x^3 = r$, when the absolute term r is less than $\sqrt[3]{2} \times \frac{q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, by means of the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c$ *ad infinitum*: and therefore I shall not add to the length of this discourse (which is already longer than I could have wished) by applying the said expression to the resolution of any more examples.

Another expression of the value of the lesser root of the cubick equation $qx - x^3 = r$, derived from the foregoing expression of it.

61. But there is another expression for the value of the lesser root of the cubick equation $qx - x^3 = r$ in the case here supposed, which, as it may be easily derived from the foregoing expression for it, to wit, $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c$, ought not, I think, to be omitted. This expression does not consist entirely of an infinite series (as the foregoing expression does), but partly of a finite algebraick expression, and partly of an infinite series; and fewer terms of the infinite series are necessary to be computed and added together, in order to obtain the value of the expression to any proposed degree of exactness, than of the infinite series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c$ contained in the foregoing expression. It is as follows, to wit, $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the infinite series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}} + \frac{Ve^{19}}{z^{19}} + \&c$; the terms of which series are

taken from the series that is equal to $1 + \frac{e}{z} \sqrt[3]{3}$, or the cube-root of the binomial quantity $1 + \frac{e}{z}$, to wit the series $1 + \frac{Be}{z} - \frac{Ce^2}{z^2} + \frac{De^3}{z^3} - \frac{Ee^4}{z^4} + \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} + \frac{He^7}{z^7} - \frac{Ie^8}{z^8} + \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} + \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} + \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} + \frac{Qe^{15}}{z^{15}} - \frac{Re^{16}}{z^{16}} + \frac{Se^{17}}{z^{17}} - \frac{Te^{18}}{z^{18}} + \&c$

$\frac{S e^{17}}{z^{17}} - \frac{T e^{18}}{z^{18}} + \frac{V e^{19}}{z^{19}} - \&c$, by beginning with the fourth term $\frac{D e^3}{z^3}$, and taking every fourth term from it. This expression may be derived from the foregoing expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \&c$ in the manner following.

The derivation of the expression of the value of the lesser root of the equation $qx - x^3 = r$ given in the preceeding article 61, from the former expression of it.

62. By the binomial theorem in the case of roots we have $\sqrt[3]{1 + \frac{e}{z}} =$ the series $1 + \frac{B e}{z} - \frac{C e^2}{z^2} + \frac{D e^3}{z^3} - \frac{E e^4}{z^4} + \frac{F e^5}{z^5} - \frac{G e^6}{z^6} + \frac{H e^7}{z^7} - \frac{I e^8}{z^8} + \frac{K e^9}{z^9} - \frac{L e^{10}}{z^{10}} + \frac{M e^{11}}{z^{11}} - \frac{N e^{12}}{z^{12}} + \frac{O e^{13}}{z^{13}} - \frac{P e^{14}}{z^{14}} + \frac{Q e^{15}}{z^{15}} - \frac{R e^{16}}{z^{16}} + \frac{S e^{17}}{z^{17}} - \frac{T e^{18}}{z^{18}} + \frac{V e^{19}}{z^{19}} - \&c$ *ad infinitum*; and by the residual theorem in the case of roots we have $\sqrt[3]{1 - \frac{e}{z}} =$ the series $1 - \frac{B e}{z} - \frac{C e^2}{z^2} - \frac{D e^3}{z^3} - \frac{E e^4}{z^4} - \frac{F e^5}{z^5} - \frac{G e^6}{z^6} - \frac{H e^7}{z^7} - \frac{I e^8}{z^8} - \frac{K e^9}{z^9} - \frac{L e^{10}}{z^{10}} - \frac{M e^{11}}{z^{11}} - \frac{N e^{12}}{z^{12}} - \frac{O e^{13}}{z^{13}} - \frac{P e^{14}}{z^{14}} - \frac{Q e^{15}}{z^{15}} - \frac{R e^{16}}{z^{16}} - \frac{S e^{17}}{z^{17}} - \frac{T e^{18}}{z^{18}} - \frac{V e^{19}}{z^{19}} - \&c$ *ad infinitum*. Therefore, if we subtract $\sqrt[3]{1 - \frac{e}{z}}$ from $\sqrt[3]{1 + \frac{e}{z}}$, and this latter series (which is equal to $\sqrt[3]{1 - \frac{e}{z}}$) from the former series (which is equal to $\sqrt[3]{1 + \frac{e}{z}}$) the remainders will be equal to each other; that is, $\sqrt[3]{1 + \frac{e}{z}} - \sqrt[3]{1 - \frac{e}{z}}$ will be = the series $\frac{2 B e}{z} + \frac{2 D e^3}{z^3} + \frac{2 F e^5}{z^5} + \frac{2 H e^7}{z^7} + \frac{2 K e^9}{z^9} + \frac{2 M e^{11}}{z^{11}} + \frac{2 O e^{13}}{z^{13}} + \frac{2 Q e^{15}}{z^{15}} + \frac{2 S e^{17}}{z^{17}} + \frac{2 V e^{19}}{z^{19}} + \&c$ *ad infinitum*. Therefore $\frac{1}{2} \times \sqrt[3]{1 + \frac{e}{z}} - \frac{1}{2} \times \sqrt[3]{1 - \frac{e}{z}}$ will be = the series $\frac{B e}{z} + \frac{D e^3}{z^3} + \frac{F e^5}{z^5} + \frac{H e^7}{z^7} + \frac{K e^9}{z^9} + \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} + \frac{Q e^{15}}{z^{15}} + \frac{S e^{17}}{z^{17}} + \frac{V e^{19}}{z^{19}} + \&c$ *ad infinitum*. Let the series $\frac{2 D e^3}{z^3} + \frac{2 H e^7}{z^7} + \frac{2 M e^{11}}{z^{11}} + \frac{2 Q e^{15}}{z^{15}} + \frac{2 V e^{19}}{z^{19}} + \&c$ *ad infinitum* be subtracted from both sides. And we shall then have $\frac{1}{2} \sqrt[3]{1 + \frac{e}{z}} - \frac{1}{2} \sqrt[3]{1 - \frac{e}{z}} -$ the series $\frac{2 D e^3}{z^3} + \frac{2 H e^7}{z^7} + \frac{2 M e^{11}}{z^{11}} + \frac{2 Q e^{15}}{z^{15}} + \frac{2 V e^{19}}{z^{19}} + \&c$ *ad infinitum* = the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \frac{S e^{17}}{z^{17}} - \frac{V e^{19}}{z^{19}} + \&c$ *ad infinitum*. Therefore (multiplying both sides into $2 \sqrt[3]{z}$) we shall

shall have $\sqrt[3]{z} \times \sqrt[3]{1 + \frac{e}{z}} = \sqrt[3]{z} \times \sqrt[3]{1 - \frac{e}{z}} = 2 \sqrt[3]{z} \times$ the series $\frac{2 D e^3}{z^3} + \frac{2 H e^7}{z^7} + \frac{2 M e^{11}}{z^{11}} + \frac{2 Q e^{15}}{z^{15}} + \frac{2 V e^{19}}{z^{19}} + \&c \text{ ad infinitum} = 2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \frac{S e^{17}}{z^{17}} - \frac{V e^{19}}{z^{19}} + \&c \text{ ad infinitum}$, or $\sqrt[3]{z} \times \sqrt[3]{1 + \frac{e}{z}} = \sqrt[3]{z} \times \sqrt[3]{1 - \frac{e}{z}} = 4 \sqrt[3]{z} \times$ the series $\frac{D e^3}{z^3} + \frac{H e^7}{z^7} + \frac{M e^{11}}{z^{11}} + \frac{Q e^{15}}{z^{15}} + \frac{V e^{19}}{z^{19}} + \&c \text{ ad infinitum} = 2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \frac{S e^{17}}{z^{17}} - \frac{V e^{19}}{z^{19}} + \&c \text{ ad infinitum}$. But, because $z \times \sqrt[3]{1 + \frac{e}{z}}$ is $= z + e$, we shall have $\sqrt[3]{z} \times \sqrt[3]{1 + \frac{e}{z}} = \sqrt[3]{z + e}$; and because $z \times \sqrt[3]{1 - \frac{e}{z}}$ is $= z - e$, we shall have $\sqrt[3]{z} \times \sqrt[3]{1 - \frac{e}{z}} = \sqrt[3]{z - e}$. Therefore $\sqrt[3]{z + e} - \sqrt[3]{z - e} = 4 \sqrt[3]{z} \times$ the series $\frac{D e^3}{z^3} + \frac{H e^7}{z^7} + \frac{M e^{11}}{z^{11}} + \frac{Q e^{15}}{z^{15}} + \frac{V e^{19}}{z^{19}} + \&c \text{ ad infinitum}$ will be $=$ the expression $2 \sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \frac{S e^{17}}{z^{17}} - \frac{V e^{19}}{z^{19}} + \&c \text{ ad infinitum}$, and consequently will also be equal to the lesser root of the cubick equation $qx - x^3 = r$ in the case here supposed, or when the absolute term r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$.

Q. E. D.

An application of the last expression of the value of the lesser root of the equation $qx - x^3 = r$ to the resolution of the above-mentioned numeral equations $15x - x^3 = 4$ and $90x - x^3 = 98$.

63. This new expression $\sqrt[3]{z + e} - \sqrt[3]{z - e} = 4 \sqrt[3]{z} \times$ the series $\frac{D e^3}{z^3} + \frac{H e^7}{z^7} + \frac{M e^{11}}{z^{11}} + \frac{Q e^{15}}{z^{15}} + \frac{V e^{19}}{z^{19}} + \&c$ may be applied to the resolution of the two foregoing equations $15x - x^3 = 4$ and $90x - x^3 = 98$ in the manner following.

In the equation $15x - x^3 = 4$, we have seen above that e is $= 2$, and z is $= 11$, and $\frac{D e^3}{z^3}$ is $= 0.000,371,019,654$, and $\frac{H e^7}{z^7}$ is $= 0.000,000,124,807$, and $\frac{M e^{11}}{z^{11}}$ is $= 0.000,000,000,073$, and $\sqrt[3]{z}$ is $= \sqrt[3]{11} = 2.223,980,090,569,361, \&c$. Therefore the series $\frac{D e^3}{z^3} + \frac{H e^7}{z^7} + \frac{M e^{11}}{z^{11}} + \&c$ is $(= 0.000,371,019,654 + 0.000,000,124,807 + 0.000,000,000,073 + \&c) = 0.000,371,$

144,534, &c, and $4\sqrt[3]{z} \times$ the said series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \&c$ is ($= 4 \times 2.223,980,090,569,361, \&c \times 0.000,371,144,534 = 8.895,920,362,277,444, \times 0.000,371,144,534 = 0.003,301,672,217,358, \&c$. And $z + e$ is ($= 11 + 2 = 13$, and $z - e$ is ($= 11 - 2 = 9$; and consequently $\sqrt[3]{z + e}$ is ($= \sqrt[3]{13} = 2.351,334,687,721$, and $\sqrt[3]{z - e}$ is ($= \sqrt[3]{9} = 2.080,083,823,052$. Therefore $\sqrt[3]{z + e} - \sqrt[3]{z - e}$ will be ($= 2.351,334,687,721 - 2.080,083,823,052 = 0.271,250,864,669$, and $\sqrt[3]{z + e} - \sqrt[3]{z - e} - 4\sqrt[3]{z} \times$ the series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \&c$ will be ($= 0.271,250,864,669 - 0.003,301,672,217 = 0.267,949,192,452$. And consequently this last number 0.267,949,192,452 will be equal to the lesser of the two roots of the equation $15x - x^3 = 4$. Q. E. I.

This number 0.267,949,192,452 is exact in the first ten figures 0.267,949,192,4, the more exact value of the lesser root of the said equation being 0.267,949,192,431,122, &c, or $2 - 1.732,050,807,568,877, \&c$, or $2 - \sqrt{3}$, as was shewn above in art. 57.

In the other equation $90x - x^3 = 98$ we have seen above that e is $= 49$, and z is ($= \sqrt{24599} = 156.840,683, \&c$, and $\frac{e}{z}$ is ($= \frac{49}{156.840,683} = 0.312,418,940,435$, and $\frac{De^3}{z^3}$ is $= 0.001,882,335,566$, and $\frac{He^7}{z^7}$ is $= 0.000,005,512,487$, and $\frac{Me^{11}}{z^{11}}$ is $= 0.000,000,028,393$, and $\frac{Qe^{15}}{z^{15}}$ is $= 0.000,000,000,177$, and $\sqrt[3]{z}$ is ($= \sqrt[3]{156.840,683} = 5.392,865,326,078$. Therefore the series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}}$ will be ($= 0.001,882,335,566 + 0.000,005,512,487 + 0.000,000,028,393 + 0.000,000,000,177 = 0.001,887,876,623$, and $4\sqrt[3]{z} \times$ the said series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}}$ will be ($= 4 \times 5.392,865,326,078 \times 0.001,887,876,623 = 21.571,461,304,312 \times 0.001,887,876,623 = 0.040,724,257,520$.

And $z + e$ will be ($= 156.840,683 + 49 = 205.840,683$; and $z - e$ will be ($= 156.840,683 - 49 = 107.840,683$; and consequently $\sqrt[3]{z + e}$ will be ($= \sqrt[3]{205.840,683} = 5.904,417,671,968$, and $\sqrt[3]{z - e}$ will be ($= \sqrt[3]{107.840,683} = 4.759,860,337,980$. Therefore $\sqrt[3]{z + e} - \sqrt[3]{z - e}$ will be ($= 5.904,417,671,968 - 4.759,860,337,980 = 1.144,557,333,988$; and $\sqrt[3]{z + e} - \sqrt[3]{z - e} - 4\sqrt[3]{z} \times$ the series $\frac{De^3}{z^3} + \frac{He^7}{z^7} + \frac{Me^{11}}{z^{11}} + \frac{Qe^{15}}{z^{15}}$ will be ($= 1.144,557,333,988 - 0.040,724,257,520 = 1.103,833,076,468$. Therefore this last number 1.103,833,076,468 will be equal to the lesser of the two roots of the equation $90x - x^3 = 98$. Q. E. I.

This value of the lesser root of this equation is exact in the six first figures 1.103,83, and exceeds the true value of the said root (which is 1.103,832,911,1,)

911,1,) by only the small quantity 0.000,000,165,3, which is somewhat less than the difference whereby the former value found for this lesser root in art. 58 by means of the expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} \&c$, to wit, 1.103,831,664,9, falls short of the said true value, that difference being 0.000,001,246,2.

A third expression for the value of the lesser root of the equation $qx - x^3 = r$, derived from the expression obtained for it above in art. 55.

64. We may also derive another expression for the value of the lesser root of the equation $qx - x^3 = r$ in the case here supposed, from the foregoing expression $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c$, in the manner following.

The quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ is $= \sqrt[3]{z} \times \sqrt[3]{1 + \frac{e}{z}} - \sqrt[3]{z} \times \sqrt[3]{1 - \frac{e}{z}} = \sqrt[3]{z} \times$ the series $1 + \frac{Be}{z} - \frac{Ce^2}{z^2} + \frac{De^3}{z^3} - \frac{Ee^4}{z^4} + \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} + \frac{He^7}{z^7} - \frac{Ie^8}{z^8} + \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} + \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} + \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} + \frac{Qe^{15}}{z^{15}} - \&c -$
 $\sqrt[3]{z} \times$ the series $1 - \frac{Be}{z} - \frac{Ce^2}{z^2} - \frac{De^3}{z^3} - \frac{Ee^4}{z^4} - \frac{Fe^5}{z^5} - \frac{Ge^6}{z^6} - \frac{He^7}{z^7} - \frac{Ie^8}{z^8} - \frac{Ke^9}{z^9} - \frac{Le^{10}}{z^{10}} - \frac{Me^{11}}{z^{11}} - \frac{Ne^{12}}{z^{12}} - \frac{Oe^{13}}{z^{13}} - \frac{Pe^{14}}{z^{14}} - \frac{Qe^{15}}{z^{15}} - \&c = \sqrt[3]{z} \times$ the series $\frac{2Be}{z} + \frac{2De^3}{z^3} + \frac{2Fe^5}{z^5} + \frac{2He^7}{z^7} + \frac{2Ke^9}{z^9} + \frac{2Me^{11}}{z^{11}} + \frac{2Oe^{13}}{z^{13}} + \frac{2Qe^{15}}{z^{15}} + \&c = 2\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{De^3}{z^3} + \frac{Fe^5}{z^5} + \frac{He^7}{z^7} + \frac{Ke^9}{z^9} + \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} + \frac{Qe^{15}}{z^{15}} + \&c$. Therefore, if we add $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c$ to both sides, we shall have $\sqrt[3]{z+e} - \sqrt[3]{z-e} + 2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c = 2\sqrt[3]{z} \times$ the series $\frac{2Be}{z} + \frac{2Fe^5}{z^5} + \frac{2Ke^9}{z^9} + \frac{2Oe^{13}}{z^{13}} + \&c = 4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$. Therefore, if we subtract $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ from both sides, we shall have $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c = 4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c + \sqrt[3]{z-e} - \sqrt[3]{z+e}$. But $2\sqrt[3]{z} \times$ the series $\frac{Be}{z} - \frac{De^3}{z^3} + \frac{Fe^5}{z^5} - \frac{He^7}{z^7} + \frac{Ke^9}{z^9} - \frac{Me^{11}}{z^{11}} + \frac{Oe^{13}}{z^{13}} - \frac{Qe^{15}}{z^{15}} + \&c$ has been shewn above to be

equal to the leffer root of the equation $qx - x^3 = r$ in the case here supposed, or when $\frac{rr}{4}$ is less than $\frac{q^3}{54}$. Therefore the expression $4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ *ad infinitum* $+ \sqrt[3]{z-e} - \sqrt[3]{z+e}$, or the excess of the quantity $4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ *ad infinitum* above the quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$, will also be equal to the leffer root of the equation $qx - x^3 = r$ in the same case of it. Q. E. D.

An application of the last, or third, expression of the leffer root of the equation $qx - x^3 = r$ to the resolution of the above-mentioned numeral equations $15x - x^3 = 4$ and $90x - x^3 = 98$.

65. This last expression, $4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ *ad infinitum*, may be applied to the resolution of the two foregoing equations $15x - x^3 = 4$ and $90x - x^3 = 98$ in the manner following.

In the equation $15x - x^3 = r$ we have seen above that e is $= 2$, and z is $= 11$, and $\frac{Be}{z}$ is $= 0.060,606,060,606$, and $\frac{Fe^5}{z^5}$ is $= 0.000,005,996,277$, and $\frac{Ke^9}{z^9}$ is $= 0.000,000,002,928$, and $\frac{Oe^{13}}{z^{13}}$ is $= 0.000,000,000,001$, and $\sqrt[3]{z}$ is $= \sqrt[3]{11} = 2.223,980,090,569,361$, &c. Therefore the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ will be $(= 0.060,606,060,606 + 0.000,005,996,277 + 0.000,000,002,928 + 0.000,000,000,001 + \&c) = 0.060,612,059,812$, &c, and $4\sqrt[3]{z} \times$ the said series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ will be $(= 4 \times 2.223,980,090,569,361, \&c \times 0.060,612,059,812, \&c = 8.895,920,362,277,444 \times 0.060,612,059,812, \&c) = 0.539,200,057,081$, &c.

And $z + e$ is $(= 11 + 2) = 13$, and $z - e$ is $(= 11 - 2) = 9$; and consequently $\sqrt[3]{z+e}$ is $(= \sqrt[3]{13}) = 2.351,334,687,721$, and $\sqrt[3]{z-e}$ is $(= \sqrt[3]{9}) = 2.080,083,823,052$. Therefore $\sqrt[3]{z+e} - \sqrt[3]{z-e}$ is $(= 2.351,334,687,721 - 2.080,083,823,052) = 0.271,250,864,669$; and the expression $4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c - \sqrt[3]{z-e} + \sqrt[3]{z+e}$, or the excess of the quantity $4\sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ above the quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$, will be $(= 0.539,200,057,081, \&c - 0.271,250,864,669, \&c) = 0.267,949,192,412$. Therefore this last number, $0.267,949,192,412$, will be equal to the leffer root of the equation $15x - x^3 = 4$. Q. E. I.

This

This number 0.267,949,192,412 is exact in the first ten figures 0.267,949 192,4, the more exact value of the lesser root of the said equation being 0.267,949,192,431,122, &c, or $2 - 1.732,050,807,568,877$, &c, or $2 - \sqrt{3}$, as was shewn above in art. 57.

In the other equation $90x - x^3 = 98$ we have seen above that e is $= 49$, and z is $(= \sqrt{24599}) = 156.840,683$, &c, and $\frac{Be}{z}$ is $= 0.104,139,646,811$, and $\frac{Fe^5}{z^5}$ is $= 0.000,089,699,031$, and $\frac{Ke^9}{z^9}$ is $= 0.000,000,381,948$, and $\frac{Oe^{13}}{z^{13}}$ is $= 0.000,000,002,210$, and $\sqrt[3]{z}$ is $(= \sqrt[3]{156.840,683}) = 5.392,865,326,078$. Therefore the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ will be $(= 0.104,139,646,811 + 0.000,089,699,031 + 0.000,000,381,948 + 0.000,000,002,210 + \&c) = 0.104,229,729,000$, and $4 \sqrt[3]{z} \times$ the said series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ will be $(= 4 \times 5.392,865,326,078 \times 0.104,229,729,000 = 21.571,461,304,312 \times 0.104,229,729,000) = 2.248,387,565,882$, &c.

And $z + e$ is $(= 156.840,683 + 49) = 205.840,683$, and $z - e$ is $(= 156.840,683 - 49) = 107.840,683$; and consequently $\sqrt[3]{z + e}$ is $(= \sqrt[3]{205.840,683}) = 5.904,417,671,968$, and $\sqrt[3]{z - e}$ is $(= \sqrt[3]{107.840,683}) = 4.759,860,337,980$, and $\sqrt[3]{z + e} - \sqrt[3]{z - e}$ is $(= 5.904,417,671,968 - 4.759,860,337,980) = 1.144,557,333,988$. Therefore the expression $4 \sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c - \sqrt[3]{z - e} + \sqrt[3]{z + e}$, or the excess of the quantity $4 \sqrt[3]{z} \times$ the series $\frac{Be}{z} + \frac{Fe^5}{z^5} + \frac{Ke^9}{z^9} + \frac{Oe^{13}}{z^{13}} + \&c$ above the quantity $\sqrt[3]{z + e} - \sqrt[3]{z - e}$, will be $(= 2.248,387,565,882 - 1.144,557,333,988) = 1.103,830,231,894$. Therefore this last number 1.103,830,231,894 will be equal to the lesser of the two roots of the equation $90x - x^3 = 98$. Q. E. I.

This number, 1.103,830,231,894, is exact in the first six figures 1.103,83, the more exact value of the lesser root of the said equation being 1.103,832,911,1.

A short view of the three expressions that have been here obtained for the value of the lesser root of the cubick equation $qx - x^3 = r$.

66. It appears, therefore, that the lesser root of the cubick equation $qx - x^3 = r$, in the case of it that has been here supposed, or when the absolute term r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{54}$, or $\frac{q^3}{2 \times 27}$, will be equal to either of the three following expressions; to wit,

3 K 2

First,

First, to $2\sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \&c \text{ ad infinitum}$, which is an expression wholly transcendental;

And, 2dly, to the expression $\sqrt[3]{z+e} - \sqrt[3]{z-e} - 4\sqrt[3]{z} \times$ the series $\frac{D e^3}{z^3} + \frac{H e^7}{z^7} + \frac{M e^{11}}{z^{11}} + \frac{Q e^{15}}{z^{15}} + \&c \text{ ad infinitum}$; which is an expression partly algebräick and partly transcendental, and of which the greater part (to wit, $\sqrt[3]{z+e} - \sqrt[3]{z-e}$) is algebräick;

And, 3dly, to the expression $4\sqrt[3]{z} \times$ the series $\frac{B e}{z} + \frac{F e^5}{z^5} + \frac{K e^9}{z^9} + \frac{O e^{13}}{z^{13}} + \&c \text{ ad infinitum} + \sqrt[3]{z-e} - \sqrt[3]{z+e}$; which is also an expression partly transcendental, and partly algebräick, but of which the greater part is transcendental.

And we may observe that the algebräick part of the two latter expressions, to wit, the quantity $\sqrt[3]{z+e} - \sqrt[3]{z-e}$, is similar to the algebräick expression, $\sqrt[3]{s+e} - \sqrt[3]{s-e}$, given by Cardan's first rule, for the root of the cubick equation $qy + y^3 = r$.

67. The first of the three foregoing expressions of the value of the lesser root of the equation $qx - x^3 = r$, to wit, the expression $2\sqrt[3]{z} \times$ the series $\frac{B e}{z} - \frac{D e^3}{z^3} + \frac{F e^5}{z^5} - \frac{H e^7}{z^7} + \frac{K e^9}{z^9} - \frac{M e^{11}}{z^{11}} + \frac{O e^{13}}{z^{13}} - \frac{Q e^{15}}{z^{15}} + \&c \text{ ad infinitum}$, is given by Monsieur Clairaut in his Elements of Algebra, Sect. X, page 288: and I have been informed that all the three expressions were published in the year 1738, in the Memoirs of the French Academy of Sciences, by Monsieur Nicole. But they are obtained by both those learned gentlemen by the intervention of *negative quantities* (or quantities less than nothing, or quantities resulting from the supposed subtraction of a greater quantity from a lesser), and of the square-roots of negative quantities, or what are called in books of algebra *impossible quantities*; both which sorts of quantities I have throughout this discourse taken care to avoid mentioning.

Of the trisection of a circular arc by means of either of the three foregoing expressions for the value of the lesser root of the cubick equation $qx - x^3 = r$.

68. The foregoing expressions of the value of the lesser root of the cubick equation $qx - x^3 = r$ may be applied to the trisection of a circular arc, or to the finding of the chord of the third part of an arc of which the chord is given, in a circle of which the diameter is given, provided the given chord be less than the chord of the fourth part of the circumference of the circle, or of an arch of 90 degrees.

For, if a be the radius of a circle, and consequently $2a$ its diameter, and k be the chord of any arc in it that is not greater than the semicircumference, and x be the chord of the third part of the arc of which k is the chord, the relation between the chords k and x will be expressed by this equation, $3aax - x^3 = aak$; as is demonstrated in many books of mathematicks, and amongst the rest, in my Differtation on the Use of the Negative Sign in Algebra, published in the year 1758, pages 183, 184, art. 220, 221. Now let q be put $= 3aa$, and $r = aak$; and the equation $3aax - x^3 = aak$ will be converted into the equation $qx - x^3 = r$, in which $\frac{\sqrt{q}}{\sqrt{3}}$ represents a , or the radius of the circle, and $\frac{r}{q}$, or $r \times \frac{3}{q}$, or $\frac{3r}{q}$, represents $\frac{aak}{aa}$, or k , the given chord of the greater arch, which is to be trisected. Therefore, by computing either of the three foregoing expressions of the value of the lesser root of the equation $qx - x^3 = r$, we shall (if r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or aak is less than $\sqrt{2} \times a^3$, or k is less than $\sqrt{2} \times a$, or k is less than the chord of a quadrantal arc, or an arc of 90 degrees) obtain the value of the chord of the third part of the arch of which $\frac{3r}{q}$, or k , is the chord, in the said circle, of which $\frac{\sqrt{q}}{\sqrt{3}}$, or a , is the radius.

Q. E. I.

Of the analogy, or harmony, between circular arcs and logarithms, or between the measures of angles and the measures of ratios.

69. This application of the series obtained in the foregoing articles to the trisection of a circular arc is an instance of the analogy, or harmony, as Mr. Cotes calls it, that subsists between logarithms and circular arcs, or between the measures of ratios and the measures of angles. For from the expression $\sqrt[3]{s+e} - \sqrt[3]{s-e}$ given by Cardan for the root of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$ (the value of which expression is to be obtained by extracting the cube-roots of the given quantities $s+e$ and $s-e$, that is, by trisecting the ratios of $s+e$ to 1, and of $s-e$ to 1), we have derived the three expressions set down in art. 66, by either of which we may trisect the circular arc of which the given quantity $\frac{3r}{q}$, or k , is the chord, in a circle of which the given quantity $\frac{\sqrt{q}}{\sqrt{3}}$, or a , is the radius (provided the said arc is less than a quadrant) or may find the chord of the third part of the said arc by finding the lesser root of the equation $qx - x^3 = r$. And consequently problems that require the trisection of a circular arc, or of an angle, may, by means of the method here explained, be solved by the trisection of a ratio, or by the help of a table of logarithms.

A SCHO-

A SCHOLIUM.

70. That very learned and ingenious mathematician and philosopher of the latter part of the last century, Mr. Leibnitz, of Hanover, seems to have been the first person who took notice of this connection between circular arcs and logarithms, or the measures of angles and the measures of ratios, which was afterwards more fully insisted upon and illustrated by Mr. Cotes in his *Harmonia Mensurarum*. For in the *Commercium Epistolicum de Analyfi promotâ* (first printed by order of the Royal Society of London, in the year 1712, and afterwards published by Tonson and Watts in the year 1722) there is a very learned letter of Mr. Leibnitz to Mr. Henry Oldenburgh (at that time secretary to the Royal Society), dated on the 27th day of August in the year 1676, and directed to be communicated to Mr. Newton, at that time professor of mathematicks in the university of Cambridge (afterwards better known by the name of Sir Isaac Newton); in which there is the following curious passage: “*Imaginariorum quantitatum in Realium Radicum expressiones ingredientium sublationem, frustra putem sperari; imò quæri. Neque enim illæ ullo modo vel calculis vel constructionibus obsunt: et veræ realèsque sunt quantitates, si inter se conjunguntur, ob destructiones virtuales. Quod multis elegantibus exemplis et argumentis deprehendi.*

*Exempli gratiâ, $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$ est $= \sqrt{6}$. Tametsi enim neque ex binomio $1 + \sqrt{-3}$, neque ex binomio $1 - \sqrt{-3}$, radix extrahetur; nec proinde sic destruetur [quantitas] imaginaria $\sqrt{-3}$; supponenda tamen est destructa esse virtualitèr; quod actu appareret, si fieri posset extractio. Aliâ tamen viâ hæc summa reperitur esse $\sqrt{6}$. Unde in cubicis binomiis, ubi realitas ejusmodi formularum (tunc cum extractio ex singulis binomiis fieri nequit), ad oculum ostendi non potest, mente tamen intelligitur. Quarè frustra Cartesius aliique expressiones Cardanicas pro particularibus habuere. Si quis posset invenire quadraturam circuli et ejus partium ex datâ hyperbolæ et ejus partium quadraturâ, is posset eas tollere; modo in ipsam quadraturam imaginariæ illæ rursus ingrediantur. See the *Commercium Epistolicum*, &c, pages 137, 138.*

In this passage Mr. Leibnitz says that $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$ is $= \sqrt{6}$. Now, if we for a moment suppose it possible for such quantities as $\sqrt{1 + \sqrt{-3}}$ and $\sqrt{1 - \sqrt{-3}}$ to exist, and (having made this supposition) we treat them as other algebraïck quantities, it may be shewn in the following manner that their sum is $= \sqrt{6}$.

Since $(a + b)^2$ is $= aa + 2ab + bb$, it follows (by substituting $\sqrt{1 + \sqrt{-3}}$ for a , and $\sqrt{1 - \sqrt{-3}}$ for b) that the square of the quantity $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$ will be $= 1 + \sqrt{-3} + 2 \times \sqrt{1 + \sqrt{-3}} \times \sqrt{1 - \sqrt{-3}} + 1 - \sqrt{-3} = 2 + 2 \times \sqrt{1 + \sqrt{-3}} \times \sqrt{1 - \sqrt{-3}} = 2 + 2 \times \sqrt{(1 + \sqrt{-3})(1 - \sqrt{-3})} = 2 + 2 \times \sqrt{1 - (-3)} = 2 + 2 \times \sqrt{1 + 3} = 2 + 2 \times \sqrt{4} = 2 + 2 \times 2 = 2 + 4 = 6$. Q. E. D.

And in the same collection of letters called *Commercium Epistolicum*, &c, there is another letter of Mr. Leibnitz to Mr. Oldenburgh, dated on the 21st day of June

June in the following year 1677, in which there is the following passage relating to the same subject, which is still more curious and remarkable than the former.

*Antequam finiam, adjiciam usum pulcherrimum serierum, qui imprimis Collinio nostro non erit ingratus. Scis magnam esse difficultatem circa extrahendas radices ex binomiis cubicis, quando eas ingreditur quantitas imaginaria, orta ex radice quadraticâ negativæ quantitatis; ut $\sqrt[3]{a + \sqrt{-bb}} = M$, et $\sqrt[3]{a - \sqrt{-bb}} = N$; ubi utraque quantitas M et N est singulatim impossibilis, summa autem, ut alibi ostendi, est quantitas possibilis et realis, æqualis cuidem quæsitæ z. Ut verò ea eximatur, et ut extrahatur radix, nempe, ut inveniatur $\frac{z}{2} + e \times \sqrt{-bb} = \sqrt[3]{a + \sqrt{-bb}}$, et $\frac{z}{2} - e \times \sqrt{-bb} = \sqrt[3]{a - \sqrt{-bb}}$ (unde fit $\sqrt[3]{a + \sqrt{-bb}} + \sqrt[3]{a - \sqrt{-bb}} = z$) non potest adhiberi methodus Schotenii, Geometriæ Cartesianæ subjeeta, quia opus est ad eam ut valor ipsius $a + \sqrt{-bb}$ exhibeatur saltè approxi-
mando; quod notis methodis impossibile est. Quis enim valorem ipsius $-bb$ propè verum dabit? Necesse est enim invenire $b \times \sqrt{-1}$; quis autem exprimat $\sqrt{-1}$ appropinquando? Scripsi olim Collinio me remedium invenisse, quod etià ad omnes gradus superiores valeat. Id ecce hic uno verbo!*

*Ex binomiis $a + \sqrt{-bb}$ extraho radicem per seriem infinitam, sive per theorema Newtonianum, sive etià more meo priore, instituendo calculum secundùm naturam cujusque gradûs, cum, scilicet, nondum theorema generale abstraxissem: Quæ radix ponatur esse $l + m \times \sqrt{-bb} + n + p \times \sqrt{-bb} \&c$. Extrahatur jam et radix ex binomio altero, $\sqrt[3]{a - \sqrt{-bb}}$; fiet illa $l - m \times \sqrt{-bb} + n - p \times \sqrt{-bb} \&c$; ut facîle demonstrari potest ex calculo. Ergo, addendo hæc duo extracta, destruentur imaginariæ quantitates, et fiet $z = 2l + 2n + \&c$, quæ sunt eæ seriei portiones in quibus nulla reperitur imaginaria. Invento ergò valore ipsius z, quantum satis est, propinquo, quemadmodum Schotenius postulat, reliqua methodo Schotenianâ, perinde ac in illis binomiorum extrahendorum generibus, transigentur. See the *Commercium Epistolicum*, &c, pages 202, and 203.*

In the former of these passages Mr. Leibnitz seems to have entertained an opinion of the possibility of so extracting the roots of two impossible binomial quantities, that the impossible part of one of the roots should be equal to the impossible part of the other root, and should be marked with a contrary sign + or —, and consequently that the sum of the said two impossible roots should be a possible quantity; but he does not seem to have then found out a method of doing this: but in the second passage he informs Mr. Oldenburgh that he had found out a method of doing it, and he describes the method to him: which consists in extracting, in an infinite series, by means of the binomial theorem, or otherwise, the cube root of the binomial quantity $a + \sqrt{-bb}$, and in extracting, in an infinite series, by means of the residual theorem, or otherwise, the cube-root of the residual quantity $a - \sqrt{-bb}$, and in adding together the two serieses obtained by these extractions, in which serieses the impossible parts will mutually destroy each other, so as to leave a series consisting of only real terms, to wit, the series $2l + 2n + \&c$, for the value of the sum of the said two roots, or of the quantity $\sqrt[3]{a + \sqrt{-bb}} + \sqrt[3]{a - \sqrt{-bb}}$.

And

And *Monsieur Nicole* and *Monsieur Clairaut* have done nothing more than expatiate upon this method here given by Mr. Leibnitz, of obtaining these cube-roots by means of the binomial and residual theorems. But both Mr. Leibnitz and those other authors, in explaining this method of proceeding, all equally suppose, for a while, the existence of impossible quantities, such as $\sqrt{-bb}$, though they contrive to get rid of them in the conclusion: and this, I think, renders their methods of treating this subject obscure and unsatisfactory. But now in the present discourse, I have shewn how Cardan's rule for resolving the cubick equations $y^3 + qy = r$, or $qy + y^3 = r$ (which is true in all the cases of that equation, or in all the relative magnitudes of the co-efficient q and the absolute term r) may be extended (by means of Sir Isaac Newton's binomial and residual theorems, in the case of roots) to the resolution of the cubick equation $qx - x^3 = r$, when r is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by finding its lesser root, from which the root of the opposite equation $x^3 - qx = r$ may be derived by means of a quadratick equation, as is shewn in my *Dissertation on the Use of the Negative Sign in Algebra*, art. 218, pages 182, 183; and in the paper published in the *Philosophical Transactions* for the year 1778, I have shewn how Cardan's other rule for resolving the cubick equation $y^3 - qy = r$, or $x^3 - qx = r$ (which is true only when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) may be extended (by means of the same binomial and residual theorems in the case of roots) to the resolution of the equation $y^3 - qy = r$, or $x^3 - qx = r$, when r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$; and consequently, by both these discourses together, I have enabled the reader to extend one or other of Cardan's two rules to the resolution of the equation $x^3 - qx = r$, when r is of any magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, whether greater or less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is of any magnitude less than $\frac{q^3}{27}$, whether greater or less than $\frac{q^3}{54}$; and this *without any mention of either impossible or negative quantities*: which, I hope, will be considered as a valuable improvement on Mr. Leibnitz's and *Monsieur Nicole*'s method of effecting the same purpose, by all such cultivators of Algebra as are fond of seeing it treated in a scientific manner, or with perspicuity in the ideas and accuracy in the reasonings throughout all the various processes of it, instead of being used as a sort of manual exercise, or mechanic art, by which conclusions are obtained we know not how, like corn that is put into a mill, and ground to meal, without the owner's comprehension of the manner in which the operation is performed.

A
M E T H O D
OF EXTENDING
C A R D A N ' s R U L E

FOR RESOLVING THE CUBICK EQUATION

$$y^3 - qy = r,$$

IN THE FIRST CASE OF IT,

or when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$, to the other case of the same equation, in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which the said rule is not naturally fitted to resolve; provided that the absolute term r (though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) be greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{rr}{4}$ (though less than $\frac{q^3}{27}$) be greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$.

BY FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

N. B. *This Discourse was first published in the Philosophical Transactions for the Year 1778.*

CARDAN'S RULE

FOR RESOLVING THE CUBIC EQUATION

$$x^3 - px = q$$

IN THE FIRST CASE OF IT,

or when r is equal to, or greater than, $\frac{27q^2}{4p^3}$, or $\frac{q^2}{4}$ is equal to, or greater than, $\frac{q^2}{27}$, to the other case of the same equation, in which r is less than $\frac{27q^2}{4p^3}$, or $\frac{q^2}{4}$ is less than $\frac{q^2}{27}$, and which the said rule is not naturally fitted to resolve; provided that the absolute term r (though less than $\frac{27q^2}{4p^3}$) be greater than $\sqrt{2} \times \frac{q}{\sqrt[3]{p}}$, or that $\frac{q^2}{4}$ (though less than $\frac{q^2}{27}$) be greater than $\frac{1}{2} \times \frac{q^2}{27}$, or than $\frac{q^2}{24}$.

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M. D. This Discourse was first published in the Philosophical Transactions for the Year 1778.

A

M E T H O D

OF EXTENDING

CARDAN'S RULE, &c.

ART. 1. **I**T is well known to all persons conversant with Algebra, that Cardan's rule for resolving the cubick equation $y^3 - qy = r$ is only fitted to resolve it when $\frac{rr}{4}$ is equal to, or greater than $\frac{q^3}{27}$, or when the absolute term r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and that it is of no use in the resolution of the other case of this equation, in which $\frac{rr}{4}$ is of any magnitude less than $\frac{q^3}{27}$, or r is of any magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$. For in this case the residual quantity $\frac{rr}{4} - \frac{q^3}{27}$, or the excess of $\frac{rr}{4}$ above $\frac{q^3}{27}$ becomes (according to the usual language of algebraists) a negative quantity, and consequently its square-root (which enters into Cardan's expressions of the root of the equation $y^3 - qy = r$) becomes impossible, and those expressions (which are $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} +$

$$\frac{q}{3\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}, \text{ and } \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}},$$

and $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} - \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$) become impossible, or, according to what appears to me a more correct way of speaking (as I never could form any idea of a negative quantity, nor could ever understand by the sign $-$ or *minus*, any thing more than the subtraction of a lesser quantity from a greater) the quantity $\frac{rr}{4} - \frac{q^3}{27}$ itself becomes impossible, or the supposition that $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$ (which is one of the foundations of Cardan's rule above-mentioned) is no longer true, and consequently the rule itself, which is built upon it, can no longer take place.

2. Nevertheless it is possible by the help of Sir Isaac Newton's binomial and residual theorems in the case of roots (which have been demonstrated in the foregoing discourses of this volume) to extend this rule of Cardan to the latter case of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and which it is not, of itself, fitted to resolve; or, to speak with more accuracy, it is possible, by means of the said two theorems of Sir Isaac Newton, to derive from one of the expressions of the value of y given by the said rule of Cardan for the resolution of the equation $y^3 - qy = r$ in the first case of the said equation, or when $\frac{rr}{4}$ is equal to, or greater than $\frac{q^3}{27}$, another expression somewhat different from the former, but bearing a great resemblance to it, that shall exhibit the true value of y in the second case of that equation, or when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, provided it be not also less than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. And this may be done by a train of just and clear reasonings, and without any mention of impossible, or even of negative quantities. To show how this may be effected is the design of the following pages.

3. That the whole of this matter may be seen at one view, it will be convenient to set forth the foundation and investigation of Cardan's rule for resolving the equation $y^3 - qy = r$ in the first case of it, or when the absolute term r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$; which may be done in the manner following.

Observations preparatory to the investigation of Cardan's rule for resolving the cubick equation $y^3 - qy = r$, when r is equal to, or greater than, $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is equal to, or greater than, $\frac{q^3}{27}$.

4. Previously to the investigation of this celebrated rule, it will be proper to make the following observations.

OBSERVATION 1. In the cubick equation $y^3 - qy = r$ (which is a proposition affirming that y^3 , or the cube of the unknown quantity y , is greater than qy , or the product of the multiplication of y by the known co-efficient q , and that the excess, or difference, is equal to the known quantity r) it is evident that, since y^3 is greater than qy , $\frac{y^3}{y}$, or yy , must be greater than $\frac{qy}{y}$ or q , and consequently that y must be greater than $\sqrt{q}$.

OBS. 2. While y increases from $\sqrt{q}$ *ad infinitum*, y^3 will increase continually from $q\sqrt{q}$ *ad infinitum*, and qy will also increase continually from the same quantity $q\sqrt{q}$ *ad infinitum*.

OBS. 3.

Obs. 3. And, while y increases from $\sqrt[3]{q}$ *ad infinitum*, the excess of y^3 above qy will increase continually from nothing *ad infinitum*, without ever decreasing.

For, if we put $\dot{y}$ (or the letter y with a point placed over it) to denote the increment which y receives in any given portion of time, either small or great, during its increase, $q\dot{y}$ will be the increment which qy will receive in the same time, and $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ will be the increment which y^3 will receive in the same time; because, when y is increased to $y + \dot{y}$, qy will be increased to $q \times y + \dot{y}$, or to $qy + q\dot{y}$, and y^3 will be increased from y^3 to $y + \dot{y}$, or $y^3 + 3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$. Now, since y^2 is always greater than q during the whole increase of y from being equal to $\sqrt[3]{q}$ *ad infinitum*, it follows that $y^2 \times \dot{y}$, or $y^2\dot{y}$, will be greater than $q \times \dot{y}$, or $q\dot{y}$, during that whole increase. But $3y^2\dot{y}$ is greater than $y^2\dot{y}$, and $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ is still greater than $3y^2\dot{y}$. Therefore, *a fortiori*, $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ must be greater than $q\dot{y}$ during the whole increase of y ; that is, the increment of y^3 will be greater than the contemporary increment of qy during the whole increase of y . Therefore the residual quantity $y^3 - qy$, or the excess of y^3 above qy , will continually increase from 0, without ever decreasing, while y increases from $\sqrt[3]{q}$ to any greater magnitude.

Further, since $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ is the increment of y^3 , and $q\dot{y}$ is the increment of qy , and $y^2\dot{y}$ is greater than $q\dot{y}$, it follows that the excess of the increment of y^3 above the increment of qy will be greater than the excess of the increment of y^3 above $y^2\dot{y}$, or than the excess of $3y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ above $y^2\dot{y}$, or than the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$. But the excess of the increment of y^3 above the increment of qy is the increment of the residual quantity $y^3 - qy$. Therefore the increment of the residual quantity $y^3 - qy$ is greater than the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$. But it is evident that the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$ will increase continually *ad infinitum*, while y increases *ad infinitum*; so that no quantity can be assigned, how great soever, which the said quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$, or either of its two first members $2y^2\dot{y}$ and $3y\dot{y}^2$, may not, by increasing the quantity y continually, be made to exceed. Therefore the increment of the residual quantity $y^3 - qy$ (which increment is greater than the quantity $2y^2\dot{y} + 3y\dot{y}^2 + \dot{y}^3$) will increase continually *ad infinitum*, or so as to become greater than any finite quantity, how great soever. And consequently the residual quantity itself (which receives the said continually-increasing increments) will increase continually from 0 *ad infinitum*, or so as to become greater than any finite quantity, how great soever. Q. E. D.

Obs. 4. Since the compound quantity $y^3 - qy$ increases continually at the same time as y increases; and, when y is equal to $\frac{2\sqrt[3]{q}}{3\sqrt[3]{3}}$, the said compound quantity is equal to $(\frac{8q\sqrt[3]{q}}{3\sqrt[3]{3}} - \frac{2q\sqrt[3]{q}}{\sqrt[3]{3}})$, or $\frac{8q\sqrt[3]{q}}{3\sqrt[3]{3}} - \frac{6q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$; it follows that, if y is greater than $\frac{2\sqrt[3]{q}}{\sqrt[3]{3}}$, the compound quantity $y^3 - qy$ will be greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, and, if y is less than $\frac{2\sqrt[3]{q}}{\sqrt[3]{3}}$, the said compound quantity will be less than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$; and *converse*, if the compound quantity $y^3 - qy$, or, its equal, the absolute term r , is greater,

greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of y will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, and, if $y^3 - qy$, or r , is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the value of y will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$; or if $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, y will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, y will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$.

OBS. 5. When r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and consequently (by the last observation) y is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, yy will be greater than $\frac{4q}{3}$, and consequently $\frac{yy}{4}$ will be greater than $\frac{q}{3}$. But $\frac{yy}{4}$ is the square of $\frac{y}{2}$. Therefore, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, the square of $\frac{y}{2}$, or of half the unknown quantity y , will be greater than $\frac{q}{3}$. But (by Euclid's Elements, Book II, Prop. 5) it is always possible to divide a line, as y , into two unequal parts of such magnitudes that the rectangle under the said parts shall be equal to any quantity that is less than the square of it's half. Therefore, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it is possible to divide the line, or root, y into two unequal parts of such magnitudes that their rectangle, or product, shall be equal to $\frac{q}{3}$.

This last observation is the foundation of Cardan's rule for the resolution of the cubick equation $y^3 - qy = r$, in the first case of it, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$; the investigation of which rule I shall now proceed to explain by the solution of the following problem.

PROBLEM I.

5. To resolve the cubick equation $y^3 - qy = r$, when the absolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

SOLUTION.

Since r is supposed to be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and consequently (by Observation 5) $\frac{yy}{4}$ is greater than $\frac{q}{3}$, it is possible for y to be divided into two unequal parts of such magnitudes that their rectangle, or product, shall be equal to $\frac{q}{3}$. Now, let it be conceived to be so divided; and let the greater of its two parts be called v , and the lesser be called z . Then will vz be $= \frac{q}{3}$, and consequently $3vz$ will be $= q$, and $3vz \times v + z$ will be $= q \times v + z$.

Now,

Now, since $v + z$ is $= y$, we shall have $y^3 = \overline{v + z}^3 = v^3 + 3v^2z + 3vz^2 + z^3 = v^3 + z^3 + 3v^2z + 3vz^2 = v^3 + z^3 + 3vz \times v + z$; and we shall have $qy = q \times v + z$. Therefore $y^3 - qy$ will be $= v^3 + z^3 + 3vz \times v + z - q \times v + z$; that is (because $3vz \times v + z$ is equal to $q \times v + z$), $y^3 - qy$ will be $= v^3 + z^3$. Therefore the absolute term r (which is equal to $y^3 - qy$) will be $= v^3 + z^3$.

But, since $3vz$ is $= q$, we shall have $z = \frac{q}{3v}$, and consequently $z^3 = \frac{q^3}{27v^3}$. Therefore $v^3 + z^3$ will be $= v^3 + \frac{q^3}{27v^3}$, and consequently r (which has been shewn to be equal $v^3 + z^3$) will be $= v^3 + \frac{q^3}{27v^3}$. Therefore (multiplying both sides of the equation by v^3) rv^3 will be $= v^6 + \frac{q^3}{27}$; and subtracting v^6 from both sides) $rv^3 - v^6$ will be $= \frac{q^3}{27}$.

But $rv^3 - v^6$ is the product of the multiplication of $r - v^3$ into v^3 , which are together equal to r . Therefore (by El. II, 5,) $rv^3 - v^6$ must be less than the square of half of r , that is, than $\frac{rr}{4}$, and consequently may be subtracted from $\frac{rr}{4}$. Let it be so subtracted; and, let its equal, $\frac{q^3}{27}$, be also subtracted from the same quantity $\frac{rr}{4}$. And the remainders will be equal to each other; that is, $\frac{rr}{4} - [rv^3 - v^6]$, or $\frac{rr}{4} - rv^3 + v^6$, will be equal to $\frac{rr}{4} - \frac{q^3}{27}$. Therefore the square-root of the trinomial quantity $\frac{rr}{4} - rv^3 + v^6$ will be equal to the square-root of $\frac{rr}{4} - \frac{q^3}{27}$. But the square-root of the trinomial quantity $\frac{rr}{4} - rv^3 + v^6$ is the difference of the simple quantities $\frac{r}{2}$ and v^3 , that is, either $\frac{r}{2} - v^3$ or $v^3 - \frac{r}{2}$, according as $\frac{r}{2}$ or v^3 is the greater quantity. But it has been already shewn that v^3 and z^3 together are equal to r ; and v is supposed to be greater than z , and consequently v^3 is greater than z^3 . Therefore v^3 must be greater than the half of $v^3 + z^3$, and consequently than the half of r , or than $\frac{r}{2}$. Therefore the difference of $\frac{r}{2}$ and v^3 must be $v^3 - \frac{r}{2}$, and not $\frac{r}{2} - v^3$; and consequently $v^3 - \frac{r}{2}$ must be the square-root of the trinomial quantity $\frac{rr}{4} - rv^3 + v^6$. We shall therefore have $v^3 - \frac{r}{2} = \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$; and consequently (adding $\frac{r}{2}$ to both sides) v^3 will be $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and therefore (extracting the cube-roots of both sides) v will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$.

But

$$qy = e + s + q \times \sqrt[3]{e + s} + \frac{qq}{3 \times \sqrt[3]{e + s}} + \frac{q^3}{27 \times \sqrt[3]{e + s}} - q \times \sqrt[3]{e + s} - \frac{qq}{3 \times \sqrt[3]{e + s}} - \frac{q^3}{27 \times \sqrt[3]{e + s}}$$

$$\frac{qq}{3 \times \sqrt[3]{e + s}} = e + s + \frac{q^3}{27 \times \sqrt[3]{e + s}} = \frac{27 \times e + s^2 + q^3}{27 \times \sqrt[3]{e + s}} = \frac{27 \times ee + 2es + ss + q^3}{27 \times \sqrt[3]{e + s}}$$

$$= \frac{27e^2 + 54es + 27s^2 + q^3}{27e + 27s}.$$

But, because ss is $= \frac{rr}{4} - \frac{q^3}{27}$, or $ee - \frac{q^3}{27}$, we shall have $ss + \frac{q^3}{27} = ee$, and $\frac{q^3}{27} = ee - ss$, and consequently $q^3 = 27e^2 - 27s^2$.

Therefore the fraction $\frac{27e^2 + 54es + 27s^2 + q^3}{27e + 27s}$ will be $= \frac{27e^2 + 54es + 27s^2 + 27e^2 - 27s^2}{27e + 27s} = \frac{54e^2 + 54es}{27e + 27s}$; and consequently the compound quantity $y^3 - qy$ (which has been shewn to be equal to the said fraction) will also be equal to the fraction $\frac{54e^2 + 54es}{27e + 27s}$, and therefore $= \frac{2e^2 + 2es}{e + s} = \frac{2e \times \sqrt[3]{e + s}}{e + s} = 2e = 2 \times \frac{r}{2} = r$. Therefore $\sqrt[3]{e + s} + \frac{q}{3 \times \sqrt[3]{e + s}}$ must be equal to the root y of the equation $y^3 - qy = r$. Q. E. D.

Another expression for the root of the foregoing equation
 $y^3 - qy = r$.

8. By resuming the solution of the foregoing problem we may find another expression for the root y of the equation $y^3 - qy = r$, to wit, the expression

$$\sqrt[3]{\frac{r}{2}} - \sqrt[3]{\frac{rr}{4} - \frac{q^3}{27}} + \frac{q}{3 \sqrt[3]{\frac{r}{2}} - \sqrt[3]{\frac{rr}{4} - \frac{q^3}{27}}}, \text{ or } \sqrt[3]{e - s} + \frac{q}{3 \sqrt[3]{e - s}},$$

or $\sqrt[3]{e - s} + \frac{q}{3 \times \sqrt[3]{e - s}}$. This expression may be found in the following manner.

The investigation of the said second expression for the value of the root y of the cubick equation $y^3 - qy = r$.

9. In art. 5 we supposed the line y to be divided into two unequal parts, v and z , of which v was supposed to be the greater; and we first found the value of the greater part v , and then determined that of the lesser part z by its relation to v , which is expressed by the equation $3vz = q$. But we may with the same

ease first determine the value of the lesser part z , and then derive from the said value of z the value of the greater part v by means of the same equation $3vz = q$, which expresses their relation to each other; whereby we should obtain the second expression of the value of $v + z$, or $z + v$, or y , which was set forth in

the foregoing article, to wit, $\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$,

or $\sqrt[3]{e - s} + \frac{q}{3\sqrt[3]{e - s}}$, or $e - s \sqrt[3]{\frac{1}{3}} + \frac{q}{3 \times e - s \sqrt[3]{\frac{1}{3}}}$. This may be done in the manner following.

Since $3vz$ is $= q$, and consequently v is $= \frac{q}{3z}$, and v^3 is $= \frac{q^3}{27z^3}$, and it has been shewn in art. 5 that r is $= v^3 + z^3$, it follows that r will be $= \frac{q^3}{27z^3} + z^3$. Therefore rz^3 will be $= \frac{q^3}{27} + z^6$, and $rz^3 - z^6$ will be $= \frac{q^3}{27}$. Therefore, if we subtract both sides of this equation from $\frac{rr}{4}$ (which is supposed to be greater than $\frac{q^3}{27}$, and consequently must also be greater than, its equal, $rz^3 - z^6$) the remainders will be equal to each other; that is, $\frac{rr}{4} - [rz^3 - z^6]$, or $\frac{rr}{4} - rz^3 + z^6$, will be equal to $\frac{rr}{4} - \frac{q^3}{27}$. Therefore the square-root of the trinomial quantity $\frac{rr}{4} - rz^3 + z^6$ will be equal to the square-root of $\frac{rr}{4} - \frac{q^3}{27}$. But the square-root of the trinomial quantity $\frac{rr}{4} - rz^3 + z^6$ is the difference of the simple quantities $\frac{r}{2}$ and z^3 , that is, either $\frac{r}{2} - z^3$ or $z^3 - \frac{r}{2}$, according as $\frac{r}{2}$ or z^3 is the greater quantity. But, because r is equal to $v^3 + z^3$, and v^3 is greater than z^3 , it follows that z^3 must be less than one half of $v^3 + z^3$, or than one half of r , or than $\frac{r}{2}$; and consequently the difference between $\frac{r}{2}$ and z^3 will be $\frac{r}{2} - z^3$, and not $z^3 - \frac{r}{2}$. Therefore $\frac{r}{2} - z^3$ will be the square-root of the trinomial quantity $\frac{rr}{4} - rz^3 + z^6$, and consequently will be equal to the square-root of $\frac{rr}{4} - \frac{q^3}{27}$, or $\frac{r}{2} - z^3$ will be $= \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore (adding z^3 to both sides) $\frac{r}{2}$ will be $= z^3 + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and (subtracting $\sqrt{\frac{rr}{4} - \frac{q^3}{27}}$ from both sides) z^3 will be $= \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$, and consequently (extracting the cube-roots of both sides) z will be $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$. Therefore $z + v$ will be $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + v = \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3z} =$

$\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or the root y of the equation $y^3 - qy = r$ will be equal to the expression $\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \frac{q}{3\sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}}$, or to the expression $\sqrt[3]{e-s} + \frac{q}{3\sqrt[3]{e-s}}$, or $\sqrt[3]{e-s} \frac{1}{3} + \frac{q}{e-s} \frac{1}{3}$.

Q. E. I.

A synthetick demonstration of the truth of the foregoing expression.

10. Here again we may demonstrate synthetically, that this expression $\sqrt[3]{e-s} \frac{1}{3} + \frac{q}{3\sqrt[3]{e-s}}$ is equal to the true value of y in the proposed equation $y^3 - qy = r$, by substituting it for y in the compound quantity $y^3 - qy$, which forms the left-hand side of the said equation. For, if we make this substitution, we shall find that the value of $y^3 - qy$ thence arising will be equal to the absolute term r . This may be shewn in the manner following.

If y is supposed to be equal to $\sqrt[3]{e-s} \frac{1}{3} + \frac{q}{3\sqrt[3]{e-s}}$, we shall have y^3

$$\begin{aligned} & (= e-s + 3 \times \sqrt[3]{e-s} \frac{1}{3} \times \frac{q}{3\sqrt[3]{e-s}} + 3 \times \sqrt[3]{e-s} \frac{1}{3} \times \frac{qq}{9\sqrt[3]{e-s}} + \frac{q^3}{27 \times \sqrt[3]{e-s}}) = e-s + q \times \sqrt[3]{e-s} \frac{1}{3} + \frac{qq}{3 \times \sqrt[3]{e-s}} + \frac{q^3}{27 \times \sqrt[3]{e-s}}, \text{ and } qy (= q \\ & \times \sqrt[3]{e-s} \frac{1}{3} + \frac{q}{3 \times \sqrt[3]{e-s}}) = q \times \sqrt[3]{e-s} \frac{1}{3} + \frac{qq}{3 \times \sqrt[3]{e-s}}, \text{ and consequently } y^3 \\ & - qy (= e-s + q \times \sqrt[3]{e-s} \frac{1}{3} + \frac{qq}{3 \times \sqrt[3]{e-s}} + \frac{q^3}{27 \times \sqrt[3]{e-s}} - q \times \sqrt[3]{e-s} \frac{1}{3} \\ & - \frac{qq}{3 \times \sqrt[3]{e-s}}) = e-s + \frac{q^3}{27 \times \sqrt[3]{e-s}} = \frac{27 \times \sqrt[3]{e-s}^2 + q^3}{27 \times \sqrt[3]{e-s}} = \frac{27 \times e^2 - 27s + q^3}{27e - 27s} \\ & = \frac{27e^2 - 54es + 27s^2 + q^3}{27e - 27s}. \end{aligned}$$

But it has been shewn in art. 7 that q^3 is $= 27e^2 - 27s$. Therefore the fraction $\frac{27e^2 - 54es + 27s^2 + q^3}{27e - 27s}$ is $= \frac{27e^2 - 54es + 27s^2 + 27e^2 - 27s^2}{27e - 27s} = \frac{54e^2 - 54es}{27e - 27s}$; and consequently the compound quantity $y^3 - qy$ (which has been shewn to be equal to the said fraction) will also be equal to the fraction $\frac{54e^2 - 54es}{27e - 27s} =$

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$$\frac{54 \times e^2 - es}{27 \times e - s}$$

$\frac{54 \times \overline{e^2 - es}}{27 \times \overline{e - s}} = 2 \times \frac{e^2 - es}{e - s} = 2 \times \frac{e \times e - s}{e - s} = 2 \times e = 2 \times \frac{r}{2} = r$; that is, the value of the compound quantity $y^3 - qy$ arising from the substitution of the expression $\overline{e - s}^{\frac{1}{3}} + \frac{q}{3 \times \overline{e - s}^{\frac{1}{3}}}$ in its terms instead of y , is equal to the absolute term r of the cubick equation $y^3 - qy = r$. Therefore the said expression $\overline{e - s}^{\frac{1}{3}} + \frac{q}{3 \times \overline{e - s}^{\frac{1}{3}}}$ must be equal to the value of y in the said equation.

Q. E. D.

A third expression for the root of the foregoing equation
 $y^3 - qy = r$.

11. We may also, by resuming the solution of the foregoing problem contained in art. 5, find a third expression for the root of this equation $y^3 - qy = r$, to wit, the expression $\sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, or $\sqrt[3]{e + s} + \sqrt[3]{e - s}$, or $\overline{e + s}^{\frac{1}{3}} + \overline{e - s}^{\frac{1}{3}}$. This expression may be found in the following manner.

The investigation of the said third expression for the value of the root y of the cubick equation $y^3 - qy = r$.

12. Since $v^3 + z^3$ is $= r$, it follows that z^3 will be $= r - v^3$. But v^3 is shewn in art. 5 to be $= \frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore $r - v^3$ will be $= r - \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}} = \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Consequently z^3 (which is equal to $r - v^3$) will be $= \frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}$. Therefore v will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, and z will be $= \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, and consequently $v + z$, or y , will be $= \sqrt[3]{\frac{r}{2} + \sqrt{\frac{rr}{4} - \frac{q^3}{27}}} + \sqrt[3]{\frac{r}{2} - \sqrt{\frac{rr}{4} - \frac{q^3}{27}}}$, or $\sqrt[3]{e + s} + \sqrt[3]{e - s}$, or $\overline{e + s}^{\frac{1}{3}} + \overline{e - s}^{\frac{1}{3}}$.

Q. E. I.

A syn.

A synthetick demonstration of the truth of the foregoing expression.

13. Here again we may demonstrate synthetically that this expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ is equal to the true value of y in the proposed equation $y^3 - qy = r$, by substituting it for y in the compound quantity $y^3 - qy$, which forms the first, or left-hand, side of the said equation. For, if we make this substitution, we shall find that the value of $y^3 - qy$ thence arising will be equal to the absolute term r of the said equation. This may be shewn in the following manner.

If y is supposed to be $= \sqrt[3]{e+s} + \sqrt[3]{e-s}$, we shall have $y^3 (= e + s + 3 \times \sqrt[3]{e+s}^2 \times \sqrt[3]{e-s} + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s}^2 + e - s = 2e + 3 \times \sqrt[3]{e+s}^2 \times \sqrt[3]{e-s} + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s}^2 = 2e + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s} (\sqrt[3]{e+s}^2 + \sqrt[3]{e-s}^2) = 2e + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s} (ee - ss) + 3 \times \sqrt[3]{e-s} \times \sqrt[3]{e+s} (ee - ss) = 2e + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s} (ee - ss) + 3 \times \sqrt[3]{e-s} \times \sqrt[3]{e+s} (ee - ss)$, and $qy (= q \times (\sqrt[3]{e+s} + \sqrt[3]{e-s})) = q \times \sqrt[3]{e+s} + q \times \sqrt[3]{e-s}$, and consequently $y^3 - qy = 2e + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s} (ee - ss) + 3 \times \sqrt[3]{e-s} \times \sqrt[3]{e+s} (ee - ss) - q \times \sqrt[3]{e+s} - q \times \sqrt[3]{e-s}$.

But it has been shewn in art. 7 that q^3 is $= 27e^2 - 27s^2 = 27 \times ee - ss$.

Therefore q will be $= 3 \times \sqrt[3]{ee - ss}$, or $3 \times \sqrt[3]{ee - ss}$. Consequently $q \times \sqrt[3]{e+s}$ will be $= 3 \times \sqrt[3]{ee - ss} \times \sqrt[3]{e+s}$, and $q \times \sqrt[3]{e-s}$ will be $= 3 \times \sqrt[3]{ee - ss} \times \sqrt[3]{e-s}$. Therefore $y^3 - qy$ (which is equal to $2e + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s} (ee - ss) + 3 \times \sqrt[3]{e-s} \times \sqrt[3]{e+s} (ee - ss) - q \times \sqrt[3]{e+s} - q \times \sqrt[3]{e-s}$) will be $= 2e + 3 \times \sqrt[3]{e+s} \times \sqrt[3]{e-s} (ee - ss) + 3 \times \sqrt[3]{e-s} \times \sqrt[3]{e+s} (ee - ss) - 3 \times \sqrt[3]{ee - ss} \times \sqrt[3]{e+s} - 3 \times \sqrt[3]{ee - ss} \times \sqrt[3]{e-s} = 2e = 2 \times \frac{r}{2} = r$; that is, the value of the compound quantity $y^3 - qy$ arising

from the substitution of the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ in its terms instead of y , is equal to the absolute term r of the cubick equation $y^3 - qy = r$, and consequently,

consequently the said expression $\sqrt[3]{e + s} + \sqrt[3]{e - s}$ must be equal to the root y of the said equation. Q. E. D.

14. I do not remember to have seen these substitutions, or synthetick demonstrations of the expressions given by the foregoing rule of Cardan for the root of the cubick equation $y^3 - qy = r$, in any book of Algebra.

15. I will now, by way of illustration of the expressions that have been above investigated, subjoin an example of the resolution of a numeral equation of the foregoing form $y^3 - qy = r$ by each of the three expressions above-mentioned, by which it will appear that they will all three bring out the same number for the root of the proposed equation.

An example of the resolution of a cubick equation of the foregoing form, $y^3 - qy = r$, by means of each of the three foregoing expressions.

16. Let it be required to find the value of y in the equation $y^3 - 3y = 18$.

In this equation q is $= 3$, and r is $= 18$. Therefore $\sqrt{q}$ is $= \sqrt{3}$, and $\frac{2q\sqrt{q}}{3\sqrt{3}}$ is $= \frac{2 \times 3 \times \sqrt{3}}{3\sqrt{3}} = 2$, which is much less than 18, or r . Therefore this equation comes under the first case of the general equation $y^3 - qy = r$, and consequently may be resolved by Cardan's rule, or by either of the three foregoing expressions.

The resolution of the equation $y^3 - 3y = 18$ by means of the first of the three foregoing expressions, to wit, the expression $\sqrt[3]{e + s} + \frac{q}{3\sqrt[3]{e + s}}$.

17. Now, since q is $= 3$, $\frac{q}{3}$ will be $(= \frac{3}{3}) = 1$, and consequently $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, will also be $= 1$. And, since r is $= 18$, we shall have $\frac{r}{2}$, or e , $= 9$, and $\frac{rr}{4}$, or ee , $= 81$, and consequently $\frac{rr}{4} - \frac{q^3}{27} (= 81 - 1) = 80$; that is, ss will be $= 80$. Therefore s will be $(= \sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5}) = 4\sqrt{5}$, and $e + s$ will be $= 9 + 4\sqrt{5} = \frac{9 \times 8}{8} + \frac{8 \times 4 \times \sqrt{5}}{8} = \frac{9 \times 8 + 8 \times 4 \times \sqrt{5}}{8} = \frac{72 + 32\sqrt{5}}{8} = \frac{27 + 27\sqrt{5} + 45 + 5\sqrt{5}}{8} = \frac{3^3 + 3 \times 9\sqrt{5} + 3 \times 3 \times 5 + 5\sqrt{5}}{8}$

$$\frac{3^3 + 3 \times 9\sqrt{5} + 3 \times 3 \times 5 + 5\sqrt{5}}{8} = \frac{3^3 + 3 \times 3^2 \times \sqrt{5} + 3 \times 3 \times 5 + 5\sqrt{5}}{8} = \frac{(3 + \sqrt{5})^3}{2^3},$$

and consequently $\sqrt[3]{e+s}$ will be $= \frac{3+\sqrt{5}}{2}$. Therefore $3 \times \sqrt[3]{e+s}$ will

be $= 3 \times \frac{3+\sqrt{5}}{2}$, and $\frac{q}{3\sqrt[3]{e+s}}$ will be $= \frac{3}{3 \times \frac{3+\sqrt{5}}{2}} = \frac{1}{\frac{3+\sqrt{5}}{2}} = 1 \times$

$\frac{2}{3+\sqrt{5}}$, and $\sqrt[3]{e+s} + \frac{q}{3\sqrt[3]{e+s}}$ will be $= \frac{3+\sqrt{5}}{2} + \frac{2}{3+\sqrt{5}}$

$= \frac{(3+\sqrt{5}) \times (3+\sqrt{5})}{2 \times (3+\sqrt{5})} + \frac{2 \times 2}{2 \times (3+\sqrt{5})} = \frac{9+6\sqrt{5}+5}{2 \times (3+\sqrt{5})} + \frac{4}{2 \times (3+\sqrt{5})} = \frac{14+6\sqrt{5}}{2 \times (3+\sqrt{5})}$

$+ \frac{4}{2 \times (3+\sqrt{5})} = \frac{18+6\sqrt{5}}{2 \times (3+\sqrt{5})} = \frac{9+3\sqrt{5}}{3+\sqrt{5}} = \frac{3 \times (3+\sqrt{5})}{3+\sqrt{5}} = 3$. Therefore 3 is

the value of y in the equation $y^3 - 3y = 18$ obtained by means of the expression $\sqrt[3]{e+s} + \frac{q}{3\sqrt[3]{e+s}}$.

Q. E. I.

18. And, upon trial, we shall find 3 to be the true value of y in the said equation $y^3 - 3y = 18$. For, if y be supposed to be $= 3$, we shall have $y^3 = 27$, and $3y = 3 \times 3 = 9$, and consequently $y^3 - 3y = 27 - 9 = 18$, or the absolute term of the proposed equation $y^3 - 3y = 18$. Therefore 3 is the true value of y in the said equation.

Q. E. D.

And thus we see that the first of the three foregoing expressions, to wit, the expression $\sqrt[3]{e+s} + \frac{q}{3\sqrt[3]{e+s}}$, has given us the true value of y in the proposed equation $y^3 - 3y = 18$.

The resolution of the same equation $y^3 - 3y = 18$ by means of the second of the three foregoing expressions, to wit, the expression $\sqrt[3]{e-s} + \frac{q}{3\sqrt[3]{e-s}}$.

19. Now, since r is $= 18$, and consequently $\frac{r}{2}$, or e , is $= 9$, and s has been shewn to be $= \sqrt{80}$, or $4\sqrt{5}$, we shall have $e-s = 9 - 4\sqrt{5} = \frac{72-32\sqrt{5}}{8} = \frac{27-27\sqrt{5}+45-5\sqrt{5}}{8}$, and consequently $\sqrt[3]{e-s} = \frac{3-\sqrt{5}}{2}$. Therefore

$\frac{q}{3\sqrt[3]{e-s}}$ will be $= \frac{3}{3 \times \frac{3-\sqrt{5}}{2}} = \frac{1}{\frac{3-\sqrt{5}}{2}} = \frac{2}{3-\sqrt{5}}$, and $\sqrt[3]{e-s} +$

$\frac{q}{3\sqrt[3]{e-s}}$ will be $= \frac{3-\sqrt{5}}{2} + \frac{2}{3-\sqrt{5}} = \frac{(3-\sqrt{5}) \times (3-\sqrt{5})}{2 \times (3-\sqrt{5})} + \frac{2 \times 2}{2 \times (3-\sqrt{5})} = \frac{9-6\sqrt{5}+6}{2 \times (3-\sqrt{5})}$

$\frac{9 - 6\sqrt{5} + 5}{2 \times 3 - \sqrt{5}} + \frac{4}{2 \times 3 - \sqrt{5}} = \frac{14 - 6\sqrt{5}}{2 \times 3 - \sqrt{5}} + \frac{4}{2 \times 3 - \sqrt{5}} = \frac{18 - 6\sqrt{5}}{2 \times 3 - \sqrt{5}} =$
 $\frac{9 - 3\sqrt{5}}{3 - \sqrt{5}} = \frac{3 \times 3 - \sqrt{5}}{3 - \sqrt{5}} = 3.$ Therefore, according to this second expression
 $\sqrt[3]{e - s} + \frac{q}{3\sqrt[3]{e - s}},$ the root of the cubick equation $y^3 - 3y = 18$ is 3; as
 it was found to be by the first expression $\sqrt[3]{e + s} + \frac{q}{3\sqrt[3]{e + s}}.$

Q. E. I.

The resolution of the same equation $y^3 - 3y = 18$ by means of the third, or last, of the three foregoing expressions, to wit, the expression $\sqrt[3]{e + s} + \sqrt[3]{e - s}.$

20. Since $\sqrt[3]{e + s}$ has been shewn in art. 17 to be $= \frac{3 + \sqrt{5}}{2}$, and $\sqrt[3]{e - s}$ has been shewn in art. 19 to be $= \frac{3 - \sqrt{5}}{2}$, we shall have $\sqrt[3]{e + s} + \sqrt[3]{e - s} = \frac{3 + \sqrt{5}}{2} + \frac{3 - \sqrt{5}}{2} = \frac{6}{2} = 3.$ Therefore by this expression, as well as by both the former, the value of y in the equation $y^3 - 3y = 18$ comes out to be 3.

A SCHOLIUM.

21. I have dwelt the longer on the foregoing explanation of Cardan's rule for resolving the cubick equation $y^3 - qy = r$ in its first case, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, because I have observed that it has been delivered by many writers of Algebra with an uncommon degree of obscurity*, and has been made the subject of much mysterious and fantastick reasoning, or rather discourfing, concerning negative and impossible quantities. But now, I hope, it will be found intelligible, and even easy, to every reader who is acquainted with the first rudiments of Algebra.

22. This method of resolving cubick equations of this form $y^3 - qy = r$, in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, or r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and a similar method of resolving cubick equations of the first form, $y^3 + qy = r$, in

* See Newton's *Arithmetica Universalis*, the 2d edition, in the year 1722, pages 279, 280, 281; and Mac Laurin's *Algebra*, Part II. chap. viii. pages 223, 224, 225, and Part I. chap. xiv. the Supplement, pages 127, 128, 129, 130; and Clairaut's *Elémens d'Algèbre*, pages 282, 283, 297, and Saunderson's *Algebra*, Vol. II. pages 692, 693, 694, 695, 707.

all cases of that form, or whatever may be the relative magnitudes of the coefficient q and the absolute term r , are usually known by the name of *Cardan's Rules*, because they were first published by him in his Treatise of Algebra, intitled, *Ars magna, quam vulgò Cossam vocant, seu REGULAS ALGEBRAICAS*, in A. D. 1545, and not because they were of his invention. For the rule for resolving cubick equations of the first form, $y^3 + qy = r$ was first discovered (as Cardan himself informs us) by one *Scipio Ferreus*, or *Ferreus*, of *Bononia*, or *Bologna*, in Italy, about 30 years before the publication of Cardan's treatise above mentioned; and the other rule for resolving cubick equations of the second form $y^3 - qy = r$, in the first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$ (and which has been the subject of the foregoing pages) was invented by another Italian named *Nicholas Tartaglia*, or *Tartalea*, who was Professor of Mathematicks at Venice, and distinguished himself very much by his ingenious discoveries. He was a cotemporary and a friend of Cardan, and communicated this invention to him under a strict promise that he would keep it secret; and when Cardan afterwards published it in the treatise above mentioned, Tartalea complained bitterly of his breach of promise, and would never afterwards be reconciled to him. He died in A. D. 1557. See Montucla's *Histoire des Mathematiques*, Vol. I. pages 462, and 479, 480, 481.

*End of the investigation and illustration of Cardan's Rule
for resolving the cubick equation $y^3 - qy = r$ in the
first case of it, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$
is greater than $\frac{q^3}{27}$.*

23. I shall now proceed to convert the last of the three foregoing expressions of the value of the root y in the cubick equation $y^3 - qy = r$, to wit, the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, into an expression involving in it an infinite series of terms; which may be done by means of Sir Isaac Newton's two theorems for finding the root of a binomial quantity, as $1+x$, and the root of a residual quantity, as $1-x$; both which theorems have been demonstrated in some of the preceeding discourses contained in this volume of tracts.

P R O B L E M II.

24. To convert the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ (which has been shewn to be equal to the root y of the equation $y^3 - qy = r$ in the first case of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) into an expression containing an infinite converging series, by means of Sir Isaac Newton's binomial and residual theorems in the case of roots.

SOLUTION.

Since $e + s$ is $= e \times \sqrt[3]{1 + \frac{s}{e}}$, and $e - s$ is $= e \times \sqrt[3]{1 - \frac{s}{e}}$, it follows that $\sqrt[3]{e + s}$ will be $= \sqrt[3]{e} \times \sqrt[3]{1 + \frac{s}{e}}$, and $\sqrt[3]{e - s}$ will be $= \sqrt[3]{e} \times \sqrt[3]{1 - \frac{s}{e}}$, and consequently $\sqrt[3]{e + s} + \sqrt[3]{e - s}$ will be $= \sqrt[3]{e} \times \sqrt[3]{1 + \frac{s}{e} + 1 - \frac{s}{e}} = \sqrt[3]{e} \times \sqrt[3]{2}$.

But, by the binomial theorem in the case of roots, $\sqrt[3]{1 + \frac{s}{e}}$ is equal to the infinite series $1 + \frac{1}{3} A \frac{s}{e} - \frac{2}{6} B \frac{s^2}{e^2} + \frac{5}{9} C \frac{s^3}{e^3} - \frac{8}{12} D \frac{s^4}{e^4} + \frac{11}{15} E \frac{s^5}{e^5} - \frac{14}{18} F \frac{s^6}{e^6} + \frac{17}{21} G \frac{s^7}{e^7} - \frac{20}{24} H \frac{s^8}{e^8} + \frac{23}{27} I \frac{s^9}{e^9} - \frac{26}{30} K \frac{s^{10}}{e^{10}} + \frac{29}{33} L \frac{s^{11}}{e^{11}} - \frac{32}{36} M \frac{s^{12}}{e^{12}} + \frac{35}{39} N \frac{s^{13}}{e^{13}} - \frac{38}{42} O \frac{s^{14}}{e^{14}} + \frac{41}{45} P \frac{s^{15}}{e^{15}} - \frac{44}{48} Q \frac{s^{16}}{e^{16}} + \frac{47}{51} R \frac{s^{17}}{e^{17}} - \frac{50}{54} S \frac{s^{18}}{e^{18}} + \&c \text{ ad infinitum}$, or (if, for the sake of brevity, we omit the several generating fractions $\frac{1}{3}, \frac{2}{6}, \frac{5}{9}, \frac{8}{12}, \frac{11}{15}, \frac{14}{18}, \frac{17}{21}, \&c$) to the series $1 + \frac{B s}{e} - \frac{C s^2}{e^2} + \frac{D s^3}{e^3} - \frac{E s^4}{e^4} + \frac{F s^5}{e^5} - \frac{G s^6}{e^6} + \frac{H s^7}{e^7} - \frac{I s^8}{e^8} + \frac{K s^9}{e^9} - \frac{L s^{10}}{e^{10}} + \frac{M s^{11}}{e^{11}} - \frac{N s^{12}}{e^{12}} + \frac{O s^{13}}{e^{13}} - \frac{P s^{14}}{e^{14}} + \frac{Q s^{15}}{e^{15}} - \frac{R s^{16}}{e^{16}} + \frac{S s^{17}}{e^{17}} - \frac{T s^{18}}{e^{18}} + \&c \text{ ad infinitum}$. And, by the residual theorem in the case of roots, $\sqrt[3]{1 - \frac{s}{e}}$ is equal to the series $1 - \frac{1}{3} A \frac{s}{e} - \frac{2}{6} B \frac{s^2}{e^2} - \frac{5}{9} C \frac{s^3}{e^3} - \frac{8}{12} D \frac{s^4}{e^4} - \frac{11}{15} E \frac{s^5}{e^5} - \frac{14}{18} F \frac{s^6}{e^6} - \frac{17}{21} G \frac{s^7}{e^7} - \frac{20}{24} H \frac{s^8}{e^8} - \frac{23}{27} I \frac{s^9}{e^9} - \frac{26}{30} K \frac{s^{10}}{e^{10}} - \frac{29}{33} L \frac{s^{11}}{e^{11}} - \frac{32}{36} M \frac{s^{12}}{e^{12}} - \frac{35}{39} N \frac{s^{13}}{e^{13}} - \frac{38}{42} O \frac{s^{14}}{e^{14}} - \frac{41}{45} P \frac{s^{15}}{e^{15}} - \frac{44}{48} Q \frac{s^{16}}{e^{16}} - \frac{47}{51} R \frac{s^{17}}{e^{17}} - \frac{50}{54} S \frac{s^{18}}{e^{18}} - \&c \text{ ad infinitum}$, or to the series $1 - \frac{B s}{e} - \frac{C s^2}{e^2} - \frac{D s^3}{e^3} - \frac{E s^4}{e^4} - \frac{F s^5}{e^5} - \frac{G s^6}{e^6} - \frac{H s^7}{e^7} - \frac{I s^8}{e^8} - \frac{K s^9}{e^9} - \frac{L s^{10}}{e^{10}} - \frac{M s^{11}}{e^{11}} - \frac{N s^{12}}{e^{12}} - \frac{O s^{13}}{e^{13}} - \frac{P s^{14}}{e^{14}} - \frac{Q s^{15}}{e^{15}} - \frac{R s^{16}}{e^{16}} - \frac{S s^{17}}{e^{17}} - \frac{T s^{18}}{e^{18}} - \&c \text{ ad infinitum}$, which consists of the very same terms as the series which is equal to $\sqrt[3]{1 + \frac{s}{e}}$, but with the sign $-$ prefixed to every term after the first term 1 instead of every other term.

Now, if these two serieses (which are equal to $\sqrt[3]{1 + \frac{s}{e}}$ and $\sqrt[3]{1 - \frac{s}{e}}$) be added together their sum will be the series $2 - \frac{2 C s^2}{e^2} - \frac{2 E s^4}{e^4} - \frac{2 G s^6}{e^6} - \frac{2 I s^8}{e^8} - \frac{2 L s^{10}}{e^{10}} - \frac{2 N s^{12}}{e^{12}} - \frac{2 P s^{14}}{e^{14}} - \frac{2 R s^{16}}{e^{16}} - \frac{2 T s^{18}}{e^{18}} - \&c \text{ ad infinitum}$, in which all the terms, after the first term, 2, are marked with the sign $-$, or subtracted from the said first term. Therefore $\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}}$ will be equal to the
said

faid series $2 - \frac{2Cs^2}{e^2} - \frac{2Es^4}{e^4} - \frac{2Gs^6}{e^6} - \frac{2Is^8}{e^8} - \frac{2Ls^{10}}{e^{10}} - \frac{2Ns^{12}}{e^{12}} - \frac{2Ps^{14}}{e^{14}} - \frac{2Rs^{16}}{e^{16}} - \frac{2Ts^{18}}{e^{18}} - \&c \text{ ad infinitum}$. Therefore $\sqrt[3]{e} \times \left[\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}} \right]$ will be $= \sqrt[3]{e} \times$ the faid series $2 - \frac{2Cs^2}{e^2} - \frac{2Es^4}{e^4} - \frac{2Gs^6}{e^6} - \frac{2Is^8}{e^8} - \frac{2Ls^{10}}{e^{10}} - \frac{2Ns^{12}}{e^{12}} - \frac{2Ps^{14}}{e^{14}} - \frac{2Rs^{16}}{e^{16}} - \frac{2Ts^{18}}{e^{18}} - \&c \text{ ad infinitum} = 2\sqrt[3]{e} \times$ the series $1 - \frac{Cs^2}{e^2} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c \text{ ad infinitum}$. Therefore the expreffion $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ (which is $= \sqrt[3]{e} \times \left[\sqrt[3]{1 + \frac{s}{e}} + \sqrt[3]{1 - \frac{s}{e}} \right]$) will be equal to $2\sqrt[3]{e} \times$ the series $1 - \frac{Cs^2}{e^2} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c \text{ ad infinitum}$.

Q. E. F.

25. It follows from the foregoing article, that the root y of the cubick equation $y^3 - qy = r$ (which has been shewn above to be equal to the expreffion $\sqrt[3]{e+s} + \sqrt[3]{e-s}$) will be equal to $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Cs^2}{e^2} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c \text{ ad infinitum}$.

26. This series will always be a converging one; because $\frac{rr}{4} - \frac{q^3}{27}$, or ss , is always less than $\frac{rr}{4}$, or ee , and consequently the fractions $\frac{ss}{ee}, \frac{s^4}{e^4}, \frac{s^6}{e^6}, \frac{s^8}{e^8}, \frac{s^{10}}{e^{10}}, \frac{s^{12}}{e^{12}}, \frac{s^{14}}{e^{14}}, \frac{s^{16}}{e^{16}}, \frac{s^{18}}{e^{18}}, \&c$ (which are the literal parts of the terms of the faid series), will form a decreasing progression, and therefore the terms themselves, which are produced by the multiplication of the faid fractions into the numeral co-efficients $C, E, G, I, L, N, P, R, T, \&c$ (which likewise are a decreasing progression), will also form a decreasing progression.

27. If we compute the several numeral co-efficients of the terms of the series $1 + \frac{Bs}{e} - \frac{Cs^2}{e^2} + \frac{Ds^3}{e^3} - \frac{Es^4}{e^4} + \frac{Fs^5}{e^5} - \frac{Gs^6}{e^6} + \frac{Hs^7}{e^7} - \frac{Is^8}{e^8} + \frac{Ks^9}{e^9} - \frac{Ls^{10}}{e^{10}} + \frac{Ms^{11}}{e^{11}} - \frac{Ns^{12}}{e^{12}} + \frac{Os^{13}}{e^{13}} - \frac{Ps^{14}}{e^{14}} + \frac{Qs^{15}}{e^{15}} - \frac{Rs^{16}}{e^{16}} + \frac{Ss^{17}}{e^{17}} - \frac{Ts^{18}}{e^{18}} + \&c \text{ ad infinitum}$, which is equal to $\sqrt[3]{1 + \frac{s}{e}}$, we shall find them to be as follows, to wit:

$$A = 1,$$

$$\text{and } B (= \frac{1}{3} A = \frac{1}{3} \times 1) = \frac{1}{3},$$

$$\text{and } C (= \frac{1}{6} B = \frac{1}{6} \times \frac{1}{3} = \frac{1}{3} \times \frac{1}{3}) = \frac{1}{9},$$

$$\text{and } D (= \frac{5}{9} C = \frac{5}{9} \times \frac{1}{9}) = \frac{5}{81},$$

3 N 2

and

$$\text{and } E (= \frac{8}{12} D = \frac{8}{12} \times \frac{5}{81} = \frac{2}{3} \times \frac{5}{81}) = \frac{10}{243},$$

$$\text{and } F (= \frac{11}{15} E = \frac{11}{15} \times \frac{10}{243} = \frac{11}{3 \times 5} \times \frac{2 \times 5}{243} = \frac{11 \times 2}{3 \times 243}) = \frac{22}{729},$$

$$\text{and } G (= \frac{14}{18} F = \frac{14}{18} \times \frac{22}{729} = \frac{7}{9} \times \frac{22}{729}) = \frac{154}{6561},$$

$$\text{and } H (= \frac{17}{21} G = \frac{17}{21} \times \frac{154}{6561} = \frac{17}{3 \times 7} \times \frac{7 \times 22}{6561} = \frac{17 \times 22}{3 \times 6561}) = \frac{374}{19,683},$$

$$\text{and } I (= \frac{20}{24} H = \frac{20}{24} \times \frac{374}{19,683} = \frac{5}{6} \times \frac{374}{19,683} = \frac{5}{3 \times 2} \times \frac{187 \times 2}{19,683} = \frac{5 \times 187}{3 \times 19,683}) \\ = \frac{935}{59,049},$$

$$\text{and } K (= \frac{23}{27} I = \frac{23}{27} \times \frac{935}{59,049}) = \frac{21,505}{1,594,323},$$

$$\text{and } L (= \frac{26}{30} K = \frac{26}{30} \times \frac{21,505}{1,594,323} = \frac{13}{15} \times \frac{21,505}{1,594,323} = \frac{13}{3 \times 5} \times \frac{5 \times 4301}{1,594,323} = \\ \frac{13 \times 4301}{3 \times 1,594,323}) = \frac{55,913}{4,782,969},$$

$$\text{and } M (= \frac{29}{33} L = \frac{29}{33} \times \frac{55,913}{4,782,969}) = \frac{147,407}{14,348,907},$$

$$\text{and } N (= \frac{32}{36} M = \frac{32}{36} \times \frac{147,407}{14,348,907} = \frac{8}{9} \times \frac{147,407}{14,348,907}) = \frac{1,179,256}{129,140,163},$$

$$\text{and } O (= \frac{35}{39} N = \frac{35}{39} \times \frac{1,179,256}{129,140,163} = \frac{35}{3 \times 13} \times \frac{13 \times 90,712}{129,140,163} = \frac{35 \times 90,712}{3 \times 129,140,163}) \\ = \frac{3,174,920}{387,420,489},$$

$$\text{and } P (= \frac{38}{42} O = \frac{38}{42} \times \frac{3,174,920}{387,420,489} = \frac{19}{21} \times \frac{3,174,920}{387,420,489} = \frac{19}{3 \times 7} \times \frac{3,174,920}{387,420,489} \\ = \frac{19}{3 \times 7} \times \frac{7 \times 453,560}{387,420,489} = \frac{19 \times 453,560}{3 \times 387,420,489}) = \frac{8,617,640}{1,162,261,467},$$

$$\text{and } Q (= \frac{41}{45} \times P = \frac{41}{45} \times \frac{8,617,640}{1,162,261,467} = \frac{41}{5 \times 9} \times \frac{8,617,640}{1,162,261,467} = \frac{41}{5 \times 9} \times \\ \frac{5 \times 1,723,528}{1,162,261,467} = \frac{41 \times 1,723,528}{9 \times 1,162,261,467}) = \frac{70,664,648}{10,460,353,203},$$

$$\text{and } R (= \frac{44}{48} Q = \frac{44}{48} \times \frac{70,664,648}{10,460,353,203} = \frac{11}{12} \times \frac{70,664,648}{10,460,353,203} = \frac{11}{3 \times 4} \times \\ \frac{4 \times 17,666,162}{10,460,353,203} = \frac{11 \times 17,666,162}{3 \times 10,460,353,203}) = \frac{194,327,782}{31,381,059,609},$$

$$\text{and } S (= \frac{47}{51} R = \frac{47}{51} \times \frac{194,327,782}{31,381,059,609} = \frac{47}{3 \times 17} \times \frac{17 \times 11,431,046}{31,381,059,609} = \\ \frac{47 \times 11,431,046}{3 \times 31,381,059,609}) = \frac{537,259,162}{94,143,178,827},$$

$$\text{and } T (= \frac{50}{54} S = \frac{50}{54} \times \frac{537,259,162}{94,143,178,827} = \frac{25}{27} \times \frac{537,259,162}{94,143,178,827}) = \\ \frac{13,431,479,050}{2,541,865,828,329}.$$

Note,

Note, these values of the co-efficients B, C, D, E, F, G, H, I, K, L, M, N, O, P, Q, R, S, and T, are expressed in the smallest possible numbers.

28. It follows from art. 25 and 27 that the root y of the equation $y^3 - qy = r$ is equal to the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{s^2}{9e^2} - \frac{10s^4}{243e^4} - \frac{154s^6}{6561e^6} - \frac{935s^8}{59049e^8} - \frac{55,913s^{10}}{4,782,969e^{10}} - \frac{1,179,256s^{12}}{129,140,163e^{12}} - \frac{8,617,640s^{14}}{1,162,261,467e^{14}} - \frac{194,327,782s^{16}}{31,381,059,609e^{16}} - \frac{13,431,479,050s^{18}}{2,541,865,828,329e^{18}} - \&c \text{ ad infinitum}.$

29. It has been observed in art. 26 that the series $1 - \frac{css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c \text{ ad infinitum}$ will always converge; from whence it follows that the expression $2\sqrt[3]{e} \times$ the said series $1 - \frac{css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c \text{ ad infinitum}$ will always truly exhibit the value of the root y of the cubick equation $y^3 - qy = r$ in the aforefaid first case of it, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$. And, when ss is considerably less than ee , or $\frac{rr}{4} - \frac{q^3}{27}$ is considerably less than $\frac{rr}{4}$, or $\frac{rr}{4}$ is very little greater than $\frac{q^3}{27}$, the convergency of the terms of this series will be sufficient to make it useful. But in other cases, or when $\frac{rr}{4}$ is much greater than $\frac{q^3}{27}$ (as, for example, when $\frac{rr}{4}$ is triple, or quadruple, or quintuple, of $\frac{q^3}{27}$, or of some still greater magnitude), the terms of this series will decrease so slowly as to render it very unfit for practice. And, indeed, in the most favourable cases, this expression of the value of the root y of the equation $y^3 - qy = r$ will be less convenient in practice than the expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, from which it is derived. But, though its merit in a practical view be but small, yet, as it is the foundation of the method, which is here intended to be explained, of extending Cardan's rule to the second case of the equation $y^3 - qy = r$, I shall now proceed to illustrate the truth of it by applying it to the resolution of a single numeral equation of the foregoing form $y^3 - qy = r$, in the first case of the said equation, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, in which I have taken care to choose such numbers for q and r as shall make $\frac{rr}{4}$ be but little greater than $\frac{q^3}{27}$, and consequently shall give us only a small quantity for the value of the fraction $\frac{ss}{ee}$, by the continual multiplication of which the terms of the above series are generated.

An example of the resolution of a cubick equation of the foregoing form $y^3 - qy = r$, in the first case of it, (or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) by means of the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \frac{P s^{14}}{e^{14}} - \frac{R s^{16}}{e^{16}} - \frac{T s^{18}}{e^{18}} - \&c \text{ ad infinitum}$, or $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{1}{9} \times \frac{ss}{ee} - \frac{10}{243} \times \frac{s^4}{e^4} - \frac{154}{6561} \times \frac{s^6}{e^6} - \frac{935}{59,049} \times \frac{s^8}{e^8} - \frac{55,913}{4,782,969} \times \frac{s^{10}}{e^{10}} - \frac{1,179,256}{129,140,163} \times \frac{s^{12}}{e^{12}} - \frac{8,617,640}{1,162,261,467} \times \frac{s^{14}}{e^{14}} - \frac{194,327,782}{31,381,059,609} \times \frac{s^{16}}{e^{16}} - \frac{13,431,479,050}{2,541,865,828,329} \times \frac{s^{18}}{e^{18}} - \&c \text{ ad infinitum}$.

30. Let it be required to resolve the equation $y^3 - 300y = 2108$ by means of this expression.

Here q is $= 300$, and r is $= 2108$. Therefore $\frac{q}{3}$ is $= 100$, and $\frac{\sqrt{q}}{\sqrt{3}}$ is $(= \sqrt{\frac{q}{3}} = \sqrt{100}) = 10$, and consequently $\frac{q\sqrt{q}}{3\sqrt{3}}$ is $(= 100 \times 10) = 1000$, and $\frac{2q\sqrt{q}}{3\sqrt{3}}$ is $(= 2 \times 1000) = 2000$, which is less than 2108 , or r . Therefore this equation comes under the first case of the general equation $y^3 - qy = r$, and consequently may be resolved either by one of the three expressions obtained by Cardan's second rule above explained, or by the foregoing transcendental expression $2\sqrt[3]{e} \times$ the series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \text{ ad infinitum}$, which was derived from the third, or last, of them.

31. Now, since r is $= 2108$, we shall have $\frac{r}{2}$, or e , $= 1054$, and $\frac{rr}{4}$, or ee , $(= 1054^2) = 1,110,916$. And, since q is $= 300$, we shall have $\frac{q}{3} = 100$, and $\frac{q^3}{27}$, or the cube of $\frac{q}{3}$, $(= 100^3) = 1,000,000$, and consequently ss , or $\frac{rr}{4} - \frac{q^3}{27} (= 1,110,916 - 1,000,000) = 110,916$. Therefore the fraction $\frac{ss}{ee}$ will be $(= \frac{110,916}{1,110,916}) = 0.099,841,932,2$; and $\frac{s^4}{e^4}$ will be $(= 0.099,841,932,2^2) = 0.009,968,411,4$; and $\frac{s^6}{e^6}$ will be $(= \frac{s^4}{e^4} \times \frac{ss}{ee} = 0.009,968,411,4 \times 0.099,841,932,2) = 0.000,995,265,4$;

and

$$\text{and } \frac{s^8}{e^8} \text{ will be } (= \frac{s^6}{e^6} \times \frac{s^2}{e^2} = 0.000,995,265,4 \times 0.099,841,932,2) = 0.000,099,369,2;$$

$$\text{and } \frac{s^{10}}{e^{10}} \text{ will be } (= \frac{s^8}{e^8} \times \frac{s^2}{e^2} = 0.000,099,369,2 \times 0.099,841,932,2) = 0.000,009,921,2;$$

$$\text{and } \frac{s^{12}}{e^{12}} \text{ will be } (= \frac{s^{10}}{e^{10}} \times \frac{s^2}{e^2} = 0.000,009,921,2 \times 0.099,841,932,2) = 0.000,000,990,5;$$

$$\text{and } \frac{s^{14}}{e^{14}} \text{ will be } (= \frac{s^{12}}{e^{12}} \times \frac{s^2}{e^2} = 0.000,000,990,5 \times 0.099,841,932,2) = 0.000,000,098,8;$$

$$\text{and } \frac{s^{16}}{e^{16}} \text{ will be } (= \frac{s^{14}}{e^{14}} \times \frac{s^2}{e^2} = 0.000,000,098,8 \times 0.099,841,932,2) = 0.000,000,009,8;$$

$$\text{and } \frac{s^{18}}{e^{18}} \text{ will be } (= \frac{s^{16}}{e^{16}} \times \frac{s^2}{e^2} = 0.000,000,009,8 \times 0.099,841,932,2) = 0.000,000,000,0;$$

$$\text{And consequently } \frac{c s^2}{e^2} \text{ will be } (= C \times 0.099,841,932,2 = \frac{1}{9} \times 0.099,841,932,2 = \frac{0.099,841,932,2}{9}) = 0.011,093,548,0;$$

$$\text{and } \frac{E s^4}{e^4} \text{ will be } (= E \times 0.009,968,411,4 = \frac{10}{243} \times 0.009,968,411,4 = \frac{10 \times 0.009,968,411,4}{243} = \frac{0.099,684,114,0}{223}) = 0.000,410,222,6;$$

$$\text{and } \frac{G s^6}{e^6} \text{ will be } (= G \times 0.000,995,265,4 = \frac{154}{6561} \times 0.000,995,265,4 = \frac{154 \times 0.000,995,265,4}{6561} = \frac{0.153,270,871,6}{6561}) = 0.000,023,360,9;$$

$$\text{and } \frac{I s^8}{e^8} \text{ will be } (= I \times 0.000,099,369,2 = \frac{935}{59,049} \times 0.000,099,369,2 = \frac{935 \times 0.000,099,369,2}{59,049} = \frac{0.092,910,202,0}{59,049}) = 0.000,001,573,4;$$

$$\text{and } \frac{L s^{10}}{e^{10}} \text{ will be } (= L \times 0.000,009,921,2 = \frac{55,913}{4,782,969} \times 0.000,009,921,2 = \frac{55,913 \times 0.000,009,921,2}{4,782,969} = \frac{0.554,724,055,6}{4,782,969}) = 0.000,000,115,9;$$

$$\text{and } \frac{N s^{12}}{e^{12}} \text{ will be } (= N \times 0.000,000,990,5 = \frac{1,179,256}{129,140,163} \times 0.000,000,990,5 = \frac{1,179,256 \times 0.000,000,990,5}{129,140,163} = \frac{1.168,053,068,0}{129,140,163}) = 0.000,000,009,0;$$

$$\text{and } \frac{P s^{14}}{e^{14}} \text{ will be } (= P \times 0.000,000,098,8 = \frac{8,617,640}{1,162,261,467} \times 0.000,000,098,8 = \frac{8,617,640 \times 0.000,000,098,8}{1,162,261,467} = \frac{0.851,422,832,0}{1,162,261,467}) = 0.000,000,000,7;$$

$$\text{and } \frac{R s^{16}}{e^{16}} \text{ will be } (= R \times 0.000,000,009,8 = \frac{194,327,782}{31,381,059,609} \times 0.000,000,009,8 = \frac{194,327,782 \times 0.000,000,009,8}{31,381,059,609} = \frac{1.904,412,263,6}{31,381,059,609}) = 0.000,000,000,0.$$

Therefore

Therefore $\frac{C s^2}{e^2} + \frac{E s^4}{e^4} + \frac{G s^6}{e^6} + \frac{I s^8}{e^8} + \frac{L s^{10}}{e^{10}} + \frac{N s^{12}}{e^{12}} + \frac{P s^{14}}{e^{14}} + \frac{R s^{16}}{e^{16}} + \&c$ will be $= 0.011,093,548,0 + 0.000,410,222,6 + 0.000,023,360,9 + 0.000,001,573,4 + 0.000,000,115,9 + 0.000,000,009,0 + 0.000,000,000,7 + 0.000,000,000,0 = 0.011,528,830,5 + \&c$; and consequently the series $1 - \frac{C s^2}{e^2} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \frac{P s^{14}}{e^{14}} - \frac{R s^{16}}{e^{16}} - \&c$ will be $= 1.000,000,000,0 - 0.011,528,830,5 \&c = 0.988,471,169,5 - \&c$.

Further, since e is $= 1054$, we shall have $\sqrt[3]{e} (= \sqrt[3]{1054}) = 10.176,853,833,7$, and consequently $2 \sqrt[3]{e} (= 2 \times 10.176,853,833,7) = 20.353,707,667,4$. Therefore $2 \sqrt[3]{e} \times$ the series $1 - \frac{C s^2}{e^2} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \frac{P s^{14}}{e^{14}} - \frac{R s^{16}}{e^{16}} - \&c$ *ad infinitum* will be $= 20.353,707,667,4 \times 0.988,471,169,5 = 20.119,053,221,6$. Therefore the root of the proposed equation $y^3 - 300y = 2108$ is $= 20.119,053,221,6$. Q. E. I.

32. This value of y is true to nine places of figures. For its true value is somewhat greater than $20.119,053,2$, as will appear by substituting $20.119,053,2$ instead of y in the compound quantity $y^3 - 300y$. For, if we suppose y to be $= 20.119,053,2$, we shall have $y^2 = 404.776,301,664,430,24$, and $y^3 = 8143.715,947,285,920,546,248,768$, and $300y (= 300 \times 20.119,053,2) = 6035.715,960,0$, and consequently $y^3 - 300y (= 8143.715,947,285,920,546,248,768 - 6035.715,960,0) = 2107.999,987,285,920,546,248,768$, which is somewhat less than the absolute term, 2108 , of the proposed equation $y^3 - 300y = 2108$; and consequently $20.119,053,2$ must be somewhat less than the true value of y in the said equation. And, if we were to prosecute the value of y somewhat further by means of Mr. Raphson's method of approximation, by supposing y to be $= 20.119,053,2 + z$, and substituting this quantity instead of y in the equation $y^3 - 300y = 2108$, and resolving the new equation that would result from such substitution, as if it was a simple equation, (to wit, by omitting all the terms that involve either the square, or cube, of z) we should find z to be equal to the fraction $\frac{0.000,012,714,079,453,751,232}{914.328,904,993,290,72}$, or to $0.000,000,013$. And consequently the first eleven figures of $20.119,053,2 + z$, or of the true value of y in the proposed equation $y^3 - 300y = 2108$, would be $20.119,053,2 + 0.000,000,013$, or $20.119,053,213$.

33. It appears therefore from the foregoing example, that this expression $2 \sqrt[3]{e} \times$ the infinite series $1 - \frac{C s^2}{e^2} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \frac{P s^{14}}{e^{14}} - \frac{R s^{16}}{e^{16}} - \frac{T s^{18}}{e^{18}} - \&c$ does truly exhibit the root y of the cubick equation $y^3 - qy = r$ in that case of it which falls under Cardan's second rule above explained, or in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

Of the second case of the cubick equation $y^3 - qy = r$, in which the absolute term r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$.

34. We must now proceed to consider the other case of the cubick equation $y^3 - qy = r$, in which r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$. This case (as we have already seen) cannot be resolved by the aforesaid rule of Cardan, because it is impossible in this case to divide the line, or root, y into two such parts v and z , that the product, or rectangle, under the said parts, to wit, the product vz , shall be equal to $\frac{q}{3}$, and consequently that $3vz$ shall be $= q$, and $3vz \times v + z$ shall be $= q \times v + z$, which is a fundamental step in the solution of Problem 1 given above in art 5. And on this account this case of the equation $y^3 - qy = r$ has obtained amongst Algebraists the name of *the irreducible case*; and particularly it is often so denominated by the French writers of Algebra. Monsieur Montucla in his *Histoire des Mathématiques*, Tom. I. page 482, speaks of it in these words: *On doit à Cardan la remarque de la limitation d'un cas des équations cubiques, où il arrive que l'extraction de la racine quarrée qui entre dans la formule, n'est pas possible. C'est ce que nous appellons le cas irréductible; dont la difficulté a donné et donne encore la torture aux Analystes.* It may, however, be resolved by means of a certain transcendental expression (or expression containing an infinite series of terms) which bears a great resemblance to the foregoing transcendental expression which we have shewn to be equal to the value of y in the first case of the equation $y^3 - qy = r$, to wit, the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{cs^2}{e^2} - \frac{es^4}{e^4} - \frac{gs^6}{e^6} - \frac{is^8}{e^8} - \frac{ls^{10}}{e^{10}} - \frac{ns^{12}}{e^{12}} - \frac{ps^{14}}{e^{14}} - \frac{rs^{16}}{e^{16}} - \frac{ts^{18}}{e^{18}} - \&c$, and which was derived from the finite expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ by the help of Sir Isaac Newton's binomial and residual theorems. To assign such a transcendental expression, and to demonstrate that it will be equal to the root y of the cubick equation $y^3 - qy = r$ in the second case of it, or when r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, with a certain limitation which we shall mention presently, is the chief object of the remaining part of this discourse.

35. In order to preserve the two cases of the cubick equation $y^3 - qy = r$ (in the first of which the absolute term r is supposed to be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the second of which it is supposed to be less than the said quantity), distinct from each other, it will be convenient to denote the root of it in these two cases, and likewise the absolute term of the equation, by different letters. I shall therefore henceforward denote the root of this equation in the first case of it (or when the absolute term is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$), by the letter y , and in

the second case by the letter x , and shall denote the absolute term of the equation in the first case by the letter r (as in the foregoing articles) and in the second case by the letter t ; so that the two cases of the equation $y^3 - qy = r$ will now be expressed by the two separate equations $y^3 - qy = r$ and $x^3 - qx = t$, in the former of which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the latter of which t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the letter q being supposed to denote the same quantity in both equations. And I shall denote $\frac{r}{2}$ (as before) by the letter e , and $\frac{rr}{4} - \frac{q^3}{27}$ (as before) by the letters ss , but shall put the letter g for $\frac{t}{2}$, and the letters zz for $\frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$. With this notation the proposition which I shall now endeavour to demonstrate, will be as follows.

PROP. 3. A THEOREM.

36. If in the foregoing transcendental expression, to wit, $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{css}{ee} - \frac{ess^4}{e^4} - \frac{ess^6}{e^6} - \frac{iss^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c$ *ad infinitum* (which has been shewn to be equal to the root y of the cubick equation $y^3 - qy = r$), we make the following changes, to wit, first, insert the letter g every where instead of the letter e , and, secondly, insert the letter z every where instead of the letter s , and, 3dly, change the sign $-$ into the sign $+$ in the second, and fourth, and sixth, and eighth, and every following even term of the infinite series contained in the said expression, the new transcendental expression that will be thereby obtained, to wit, the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ *ad infinitum* will be equal to the root x of the cubick equation $x^3 - qx = t$; provided that the absolute term t of this equation (though it be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) be greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{tt}{4}$, or gg , (though it be less than $\frac{q^3}{27}$) be greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$.

37. This limitation is necessary to the end that the series $1 + \frac{czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ (which forms a part of the expression of the value of the root x) may be a converging series. For if t is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{54}$, the compound quantity $\frac{q^3}{27} - \frac{tt}{4}$ will be greater than $\frac{q^3}{27} - \frac{q^3}{54}$, or than $\frac{2q^3}{54} - \frac{q^3}{54}$, or than $\frac{q^3}{54}$, and, *à fortiori*, greater than

than $\frac{tt}{4}$; that is, zz will be greater than gg ; and consequently the terms of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ will diverge, instead of converging, and the said expression $2\sqrt[3]{g} \times$ the said series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$, will consequently become useless. But, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and yet greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, and yet greater than $\frac{q^3}{54}$, the terms of the aforesaid series will converge, or decrease, and the aforesaid transcendental expression will be equal to the root x of the equation $x^3 - qx = t$.

38. Now, in order to demonstrate this proposition, it will be necessary to make some further observations on the former transcendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Ts^{18}}{e^{18}} - \&c$ *ad infinitum*, which has been shewn to be equal to the root of the equation $y^3 - qy = r$.

Observations on the expression $2\sqrt[3]{e} \times$ the infinite series

$1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$
ad infinitum, which is equal to the root of the equation $y^3 - qy = r$.

39. Since the expression $2\sqrt[3]{e} \times$ the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ *ad infinitum*, is equal to the root of the cubick equation $y^3 - qy = r$, it follows that, if we were, first, to raise the said expression to its cube by multiplying it twice into itself, and then were to multiply it into the co-efficient q , the said cube would be greater than the said product, and their difference would be equal to r , to whatever number of terms the said series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ may be continued. For, if this difference were not equal to r , it would not be true that the said expression $2\sqrt[3]{e} \times$ the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ *ad infinitum* was equal to the root of the equation $y^3 - qy = r$. These multiplications may be performed in the manner following.

3 O 2

40. The

40. The cube of the expression $2\sqrt[3]{e} \times$ the series $1 - \frac{C s^3}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum* is equal to the product of the multiplication of the cube of $2\sqrt[3]{e}$ into the cube of the said series, that is, to the product of the multiplication of $8e$ into the cube of the said series. We must therefore raise the said series to its cube by multiplying it twice into itself; which may be done as follows.

*The multiplication of the series $1 - \frac{C s^3}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum* twice into itself in order to obtain its cube.*

$$\begin{array}{r}
 1 - \frac{C s^3}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \\
 1 - \frac{C s^3}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \\
 \hline
 1 - \frac{C s^3}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \\
 - \frac{C s^3}{ee} + \frac{C^2 s^4}{e^4} + \frac{CE s^6}{e^6} + \frac{CG s^8}{e^8} + \frac{CL s^{10}}{e^{10}} + \frac{CN s^{12}}{e^{12}} + \&c \\
 - \frac{E s^4}{e^4} + \frac{CE s^6}{e^6} + \frac{E^2 s^8}{e^8} + \frac{EG s^{10}}{e^{10}} + \frac{EN s^{12}}{e^{12}} + \&c \\
 - \frac{G s^6}{e^6} + \frac{CG s^8}{e^8} + \frac{EG s^{10}}{e^{10}} + \frac{G^2 s^{12}}{e^{12}} + \&c \\
 - \frac{I s^8}{e^8} + \frac{CI s^{10}}{e^{10}} + \frac{EI s^{12}}{e^{12}} + \&c \\
 - \frac{L s^{10}}{e^{10}} + \frac{CL s^{12}}{e^{12}} + \&c \\
 - \frac{N s^{12}}{e^{12}} + \&c \\
 \hline
 1 - \frac{2 C s^3}{ee} - \frac{2 E s^4}{e^4} - \frac{2 G s^6}{e^6} - \frac{2 I s^8}{e^8} - \frac{2 L s^{10}}{e^{10}} - \frac{2 N s^{12}}{e^{12}} - \&c \\
 + \frac{C^2 s^4}{e^4} + \frac{2 CE s^6}{e^6} + \frac{2 CG s^8}{e^8} + \frac{2 CI s^{10}}{e^{10}} + \frac{2 CL s^{12}}{e^{12}} + \&c \\
 + \frac{E^2 s^8}{e^8} + \frac{2 EG s^{10}}{e^{10}} + \frac{2 EI s^{12}}{e^{12}} + \&c \\
 + \frac{G^2 s^{12}}{e^{12}} + \&c \\
 \hline
 1 - \frac{C s^3}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \\
 1 - \frac{2 C s^3}{ee} - \frac{2 E s^4}{e^4} - \frac{2 G s^6}{e^6} - \frac{2 I s^8}{e^8} - \frac{2 L s^{10}}{e^{10}} - \frac{2 N s^{12}}{e^{12}} - \&c \\
 \hline
 \end{array}$$

+

$$\begin{aligned}
& + \frac{C^2 J^4}{e^4} + \frac{2 CE J^6}{e^6} + \frac{2 CG J^8}{e^8} + \frac{2 CI J^{10}}{e^{10}} + \frac{2 CL J^{12}}{e^{12}} + \&c \\
& + \frac{E^2 J^8}{e^8} + \frac{2 EG J^{10}}{e^{10}} + \frac{2 EI J^{12}}{e^{12}} + \&c \\
& + \frac{G^2 J^{12}}{e^{12}} + \&c \\
& - \frac{C J J}{e e} + \frac{2 C^2 J^4}{e^4} + \frac{2 CE J^6}{e^6} + \frac{2 CG J^8}{e^8} + \frac{2 CI J^{10}}{e^{10}} + \frac{2 CL J^{12}}{e^{12}} + \&c \\
& - \frac{C^3 J^6}{e^6} - \frac{2 C^2 E J^8}{e^8} - \frac{2 C^2 G J^{10}}{e^{10}} - \frac{2 C^2 I J^{12}}{e^{12}} - \&c \\
& - \frac{CE^2 J^{10}}{e^{10}} - \frac{2 CEG J^{12}}{e^{12}} - \&c \\
& - \frac{E J^4}{e^4} + \frac{2 CE J^6}{e^6} + \frac{2 E^2 J^8}{e^8} + \frac{2 EG J^{10}}{e^{10}} + \frac{2 EI J^{12}}{e^{12}} + \&c \\
& - \frac{C^2 E J^8}{e^8} - \frac{2 CE^2 J^{10}}{e^{10}} - \frac{2 CEG J^{12}}{e^{12}} - \&c \\
& - \frac{E^3 J^{12}}{e^{12}} - \&c \\
& - \frac{G J^6}{e^6} + \frac{2 CG J^8}{e^8} + \frac{2 EG J^{10}}{e^{10}} + \frac{2 G^2 J^{12}}{e^{12}} + \&c \\
& - \frac{C^2 G J^{10}}{e^{10}} - \frac{2 CEG J^{12}}{e^{12}} - \&c \\
& - \frac{I J^8}{e^8} + \frac{2 CI J^{10}}{e^{10}} + \frac{2 EI J^{12}}{e^{12}} + \&c \\
& - \frac{C^2 I J^{12}}{e^{12}} - \&c \\
& - \frac{L J^{10}}{e^{10}} + \frac{2 CL J^{12}}{e^{12}} + \&c \\
& - \frac{N J^{12}}{e^{12}} + \&c \\
\hline
& I - \frac{3 C J J}{e e} - \frac{3 E J^4}{e^4} - \frac{3 G J^6}{e^6} - \frac{3 I J^8}{e^8} - \frac{3 L J^{10}}{e^{10}} - \frac{3 N J^{12}}{e^{12}} - \&c \\
& + \frac{3 C^2 J^4}{e^4} + \frac{6 CE J^6}{e^6} + \frac{6 CG J^8}{e^8} + \frac{6 CI J^{10}}{e^{10}} + \frac{6 CL J^{12}}{e^{12}} + \&c \\
& - \frac{C^3 J^6}{e^6} + \frac{3 E^2 J^8}{e^8} + \frac{6 EG J^{10}}{e^{10}} + \frac{6 EI J^{12}}{e^{12}} + \&c \\
& - \frac{3 C^2 E J^8}{e^8} + \frac{3 C^2 G J^{10}}{e^{10}} + \frac{3 G^2 J^{12}}{e^{12}} + \&c \\
& - \frac{3 CE^2 J^{10}}{e^{10}} - \frac{3 C^2 I J^{12}}{e^{12}} - \&c \\
& - \frac{6 CEG J^{12}}{e^{12}} - \&c \\
& - \frac{E^3 J^{12}}{e^{12}} - \&c.
\end{aligned}$$

This last compound series (which, for the sake of brevity, we will denote by the Greek capital letter Γ) is the cube of the series $I - \frac{C J J}{e e} - \frac{E J^4}{e^4} - \frac{G J^6}{e^6} - \frac{I J^8}{e^8} - \frac{L J^{10}}{e^{10}} - \frac{N J^{12}}{e^{12}} - \&c$ *ad infinitum*. Therefore $8 e \times$ the cube of the series $I -$

$$\frac{C J J}{e e}$$

$\frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum*, will be equal to $8e \times$ the said last compound series, or to $8e \times$ the compound series Γ ; and consequently y^3 , or the cube of the root y of the equation $y^3 - qy = r$, will be equal to $8e \times$ the compound series Γ .

41. If the foregoing compound series Γ be actually multiplied into $8e$, the product thence arising will be the following compound series, to wit,

$$\begin{aligned}
 8e - \frac{24 C ss}{e} - \frac{24 E s^4}{e^3} - \frac{24 G s^6}{e^5} - \frac{24 I s^8}{e^7} - \frac{24 L s^{10}}{e^9} - \frac{24 N s^{12}}{e^{11}} - \&c \\
 + \frac{24 C^2 s^4}{e^3} + \frac{48 CE s^6}{e^5} + \frac{48 CG s^8}{e^7} + \frac{48 CI s^{10}}{e^9} + \frac{48 CL s^{12}}{e^{11}} + \&c \\
 - \frac{8 C^3 s^6}{e^5} + \frac{24 E^2 s^8}{e^7} + \frac{48 EG s^{10}}{e^9} + \frac{48 EI s^{12}}{e^{11}} + \&c \\
 - \frac{24 C^2 E s^8}{e^7} - \frac{24 C^2 G s^{10}}{e^9} + \frac{24 G^2 s^{12}}{e^{11}} + \&c \\
 - \frac{24 CE^2 s^{10}}{e^9} - \frac{24 C^2 I s^{12}}{e^{11}} - \&c \\
 - \frac{48 CEG s^{12}}{e^{11}} - \&c \\
 - \frac{8 E^3 s^{12}}{e^{11}} - \&c,
 \end{aligned}$$

which, for the sake of brevity, we will denote by the capital letter Δ . Then will y^3 be = to the compound series Δ .

42. We must now multiply the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum* (which is equal to the root y of the cubick equation $y^3 - qy = r$) into the co-efficient q .

Now, since ss is $= \frac{rr}{4} - \frac{q^3}{27}$, we shall have $ss + \frac{q^3}{27} = \frac{rr}{4}$, and $\frac{q^3}{27} = \frac{rr}{4} - ss = ee - ss$, and consequently $q^3 = 27 \times ee - ss = 27 \times ee \times \left[1 - \frac{ss}{ee}\right]$

and $q = 3 \times \sqrt[3]{ee} \times \sqrt[3]{1 - \frac{ss}{ee}} = 3 \times e^{\frac{2}{3}} \times \left[1 - \frac{ss}{ee}\right]^{\frac{1}{3}} =$ (by the residual

theorem in the case of roots) $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{B ss}{ee} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \frac{H s^{14}}{e^{14}} - \frac{I s^{16}}{e^{16}} - \frac{K s^{18}}{e^{18}} - \&c$ *ad infinitum*. Therefore

$q \times$ the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} -$

$\frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ will be equal to $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{B ss}{ee} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \frac{H s^{14}}{e^{14}} - \frac{I s^{16}}{e^{16}} - \frac{K s^{18}}{e^{18}} - \&c$ *ad infinitum* $\times$ the

expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$

$\frac{N s^{12}}{e^{12}} - \&c \text{ ad infinitum} = 6e \times \text{the infinite series } 1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c \text{ ad infinitum} \times \text{the series } 1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \text{ ad infinitum.}$ We must therefore multiply these two series together; which may be done as follows.

The multiplication of the infinite series $1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$ into the infinite series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$.

$$\begin{array}{r}
 1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c \\
 1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \\
 \hline
 1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c \\
 - \frac{C s s}{e e} + \frac{B C s^4}{e^4} + \frac{C^2 s^6}{e^6} + \frac{C D s^8}{e^8} + \frac{C E s^{10}}{e^{10}} + \frac{C F s^{12}}{e^{12}} + \&c \\
 - \frac{E s^4}{e^4} + \frac{B E s^6}{e^6} + \frac{C E s^8}{e^8} + \frac{D E s^{10}}{e^{10}} + \frac{E^2 s^{12}}{e^{12}} + \&c \\
 - \frac{G s^6}{e^6} + \frac{B G s^8}{e^8} + \frac{C G s^{10}}{e^{10}} + \frac{D G s^{12}}{e^{12}} + \&c \\
 - \frac{I s^8}{e^8} + \frac{B I s^{10}}{e^{10}} + \frac{C I s^{12}}{e^{12}} + \&c \\
 - \frac{L s^{10}}{e^{10}} + \frac{B L s^{12}}{e^{12}} + \&c \\
 - \frac{N s^{12}}{e^{12}} + \&c.
 \end{array}$$

Therefore the product of the multiplication of q into the expression $2 \sqrt[3]{e} \times$ the series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c \text{ ad infinitum}$ (which is equal to the root of the equation $y^3 - qy = r$) will be equal to $6e \times$ the compound series just now obtained, to wit, the series

$$\begin{aligned}
 I &= \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \&c \\
 &= \frac{Css}{ee} + \frac{BCs^4}{e^4} + \frac{C^2s^6}{e^6} + \frac{CDs^8}{e^8} + \frac{CEs^{10}}{e^{10}} + \frac{CFs^{12}}{e^{12}} + \&c \\
 &\quad - \frac{Es^4}{e^4} + \frac{BEs^6}{e^6} + \frac{CEs^8}{e^8} + \frac{DEs^{10}}{e^{10}} + \frac{E^2s^{12}}{e^{12}} + \&c \\
 &\quad - \frac{Gs^6}{e^6} + \frac{BGs^8}{e^8} + \frac{CGs^{10}}{e^{10}} + \frac{DGs^{12}}{e^{12}} + \&c \\
 &\quad - \frac{Is^8}{e^8} + \frac{BIs^{10}}{e^{10}} + \frac{CIs^{12}}{e^{12}} + \&c \\
 &\quad - \frac{Ls^{10}}{e^{10}} + \frac{BLs^{12}}{e^{12}} + \&c \\
 &\quad - \frac{Ns^{12}}{e^{12}} + \&c.
 \end{aligned}$$

Or if, for the sake of brevity, we denote this compound series by the Greek capital letter Λ , the said product will be equal to $6e \times$ into the compound series Λ . And consequently the product qy (which is equal to the said product) will also be equal to $6e \times$ the said compound series Λ .

43. If the foregoing compound series Λ be actually multiplied into $6e$, the product thence arising will be the following compound series, to wit,

$$\begin{aligned}
 6e &= \frac{6Bss}{e} - \frac{6Cs^4}{e^3} - \frac{6Ds^6}{e^5} - \frac{6Es^8}{e^7} - \frac{6Fs^{10}}{e^9} - \frac{6Gs^{12}}{e^{11}} - \&c \\
 &= \frac{6Css}{e} + \frac{6BCs^4}{e^3} + \frac{6C^2s^6}{e^5} + \frac{6CDs^8}{e^7} + \frac{6CEs^{10}}{e^9} + \frac{6CFs^{12}}{e^{11}} + \&c \\
 &\quad - \frac{6Es^4}{e^3} + \frac{6BEs^6}{e^5} + \frac{6CEs^8}{e^7} + \frac{6DEs^{10}}{e^9} + \frac{6E^2s^{12}}{e^{11}} + \&c \\
 &\quad - \frac{6Gs^6}{e^5} + \frac{6BGs^8}{e^7} + \frac{6CGs^{10}}{e^9} + \frac{6DGs^{12}}{e^{11}} + \&c \\
 &\quad - \frac{6Is^8}{e^7} + \frac{6BIs^{10}}{e^9} + \frac{6CIs^{12}}{e^{11}} + \&c \\
 &\quad - \frac{6CLs^{10}}{e^9} + \frac{6BLs^{12}}{e^{11}} + \&c \\
 &\quad - \frac{6Ns^{12}}{e^{11}} + \&c
 \end{aligned}$$

which for the sake of brevity, we will denote by the Greek capital letter Π . Then will qy be equal to the compound series Π .

44. Since the compound series Δ , obtained in art. 41, is equal to y^3 , and the compound series Π , obtained in the last foregoing article, 43, is equal to the product qy (which is less than y^3 by the difference r , which is the absolute term of the equation $y^3 - qy = r$) it follows that the compound series Π will be less than the compound series Δ , and that their difference will be equal to the difference of y^3 and qy , or to the absolute term r of the equation $y^3 - qy = r$; that is $\Delta - \Pi$ will be $= r$. But the excess of the first term of the compound series Δ above the first term of the compound series Π is also equal to r ; the first term of the compound series Δ being $8e$, and the first term of the compound series

series II being $6e$, and their difference consequently being equal to $8e - 6e$, or $2e$, or $2 \times \frac{r}{2}$, or r . Therefore the excess of the first term of the compound series Δ above the first term of the compound series II is equal to the excess of the whole compound series Δ above the whole compound series II. Therefore the difference between the whole compound series Δ and its first term $8e$ will be equal to the difference between the whole compound series II and its first term $6e$; that is (because each of these compound series Δ and II is less than its own first term, the second terms in both series, to wit, the term $-\frac{24cs^3}{e}$ and the term $-\frac{6bs^3}{e} - \frac{6cs^3}{e}$, being marked with the sign $-$, or subtracted from the first terms $8e$ and $6e$) the excess of the first term, $8e$, of the compound series Δ above the whole of the said compound series will be equal to the excess of the first term, $6e$, of the compound series II above the whole of the said compound series. But the excess of the first term, $8e$, of the compound series Δ above the whole of the said series is equal to a compound series consisting of all the terms of the said compound series Δ , except its first term $8e$, with the signs of the terms every where changed into their contraries, that is, to the following compound series, to wit,

$$\begin{aligned} & \frac{24cs^3}{e} + \frac{24es^4}{e^3} + \frac{24gs^6}{e^5} + \frac{24is^8}{e^7} + \frac{24ls^{10}}{e^9} + \frac{24ns^{12}}{e^{11}} + \&c \\ & - \frac{24c^2s^4}{e^3} - \frac{48ces^6}{e^5} - \frac{48ce^2s^8}{e^7} - \frac{48ci^2s^{10}}{e^9} - \frac{48cl^2s^{12}}{e^{11}} - \&c \\ & + \frac{8c^3s^6}{e^5} - \frac{24e^2s^8}{e^7} - \frac{48egs^{10}}{e^9} - \frac{48ei^2s^{12}}{e^{11}} - \&c \\ & + \frac{24c^2es^8}{e^7} + \frac{24c^2gs^{10}}{e^9} - \frac{24g^2s^{12}}{e^{11}} - \&c \\ & + \frac{24ce^2s^{10}}{e^9} + \frac{24c^2is^{12}}{e^{11}} + \&c \\ & + \frac{48ceg^2s^{12}}{e^{11}} + \&c \\ & + \frac{8e^3s^{12}}{e^{11}} + \&c; \end{aligned}$$

and the excess of the first term, $6e$, of the compound series II above the whole of the said series is equal to a compound series consisting of all the terms of the said compound series II, except its first term $6e$, with the signs of the terms every where changed into their contraries, that is, to the following compound series, to wit,

$$\begin{aligned}
& \frac{6Bss}{e} + \frac{6Cs^4}{e^3} + \frac{6Ds^6}{e^5} + \frac{6Es^8}{e^7} + \frac{6Fs^{10}}{e^9} + \frac{6Gs^{12}}{e^{11}} + \&c \\
& + \frac{6Css}{e} - \frac{6BCs^4}{e^3} - \frac{6C^2s^6}{e^5} - \frac{6CDs^8}{e^7} - \frac{6CEs^{10}}{e^9} - \frac{6CFs^{12}}{e^{11}} - \&c \\
& + \frac{6Es^4}{e^3} - \frac{6BES^6}{e^5} - \frac{6CES^8}{e^7} - \frac{6DES^{10}}{e^9} - \frac{6E^2s^{12}}{e^{11}} - \&c \\
& + \frac{6Gs^6}{e^5} - \frac{6EGs^8}{e^7} - \frac{6CGs^{10}}{e^9} - \frac{6DGs^{12}}{e^{11}} - \&c \\
& + \frac{6Is^8}{e^7} - \frac{6BIs^{10}}{e^9} - \frac{6CIs^{12}}{e^{11}} - \&c \\
& + \frac{6Ls^{10}}{e^9} - \frac{6BLs^{12}}{e^{11}} - \&c \\
& + \frac{6Ns^{12}}{e^{11}} - \&c.
\end{aligned}$$

Therefore, if, for the sake of brevity, we call the former of these two last-mentioned compound serieses $8e - \Delta$, and the latter $6e - \Pi$, we shall have the compound series $8e - \Delta =$ the compound series $6e - \Pi$.

Of the equality between each separate term of the compound series $8e - \Delta$, and the corresponding term of the compound series $6e - \Pi$.

45. In the foregoing article it has been shewn that the whole compound series $8e - \Delta$ is equal to the whole compound series $6e - \Pi$. But it is also true that each separate term of the former compound series (reckoning all the quantities in it that involve the same power of ss as one term) will be equal to the corresponding term of the latter compound series. Of this equality we will first give some examples in some of the first terms of these two compound serieses, and then will give a general proof that the same equality must also take place in all the following terms of the said serieses, to whatever number of terms they may be continued.

Examples of the said equality in the first six terms of the said serieses.

46. In the first place, then, the first term of the compound series $8e - \Delta$ is $\frac{24Css}{e}$, or (because C is $= \frac{1}{9}$), $24 \times \frac{1}{9} \times \frac{ss}{e} = 8 \times 3 \times \frac{1}{9} \times \frac{ss}{e} = \frac{8ss}{3e}$. And the first term of the compound series $6e - \Pi$ is $\frac{6Bss}{e} + \frac{6Css}{e} = 6 \times \frac{1}{3} \times \frac{ss}{e}$

$$\frac{ss}{e} + 6 \times \frac{1}{9} \times \frac{ss}{e} = \frac{6ss}{3e} + \frac{2ss}{3e} = \frac{8ss}{3e}. \text{ Therefore } \frac{6Bss}{e} + \frac{6Css}{e} \text{ are equal to } \frac{24Css}{e}.$$

Q. E. D.

47. Secondly, the next term of the compound series $8e - \Delta$ is $\frac{24Es^4}{e^3} - \frac{24C^2s^4}{e^3}$, which is $= 24 \times \frac{10}{243} \times \frac{s^4}{e^3} - 24 \times \frac{1}{9} \times \frac{1}{9} \times \frac{s^4}{e^3} = 8 \times 3 \times \frac{10}{81 \times 3} \times \frac{s^4}{e^3} - 24 \times \frac{1}{9} \times \frac{1}{9} \times \frac{s^4}{e^3} = \frac{80s^4}{81e^3} - \frac{24s^4}{81e^3} = \frac{56s^4}{81}$; and the next term of the compound series $6e - \Pi$ is $\frac{6Cs^4}{e^3} - \frac{6BCs^4}{e^3} + \frac{6Es^4}{e^3}$, which is $= 6 \times \frac{1}{9} \times \frac{s^4}{e^3} - 6 \times \frac{1}{3} \times \frac{1}{9} \times \frac{s^4}{e^3} + 6 \times \frac{10}{243} \times \frac{s^4}{e^3} = \frac{6 \times 9}{9 \times 9} \times \frac{s^4}{e^3} - \frac{6 \times 3}{9 \times 9} \times \frac{s^4}{e^3} + \frac{2 \times 10}{81} \times \frac{s^4}{e^3} = \frac{54s^4}{81e^3} - \frac{18s^4}{81e^3} + \frac{20}{81} \times \frac{s^4}{e^3} = \frac{74s^4}{81e^3} - \frac{18s^4}{81e^3} = \frac{56s^4}{81e^3}$. Therefore $\frac{6Cs^4}{e^3} - \frac{6BCs^4}{e^3} + \frac{6Es^4}{e^3}$, or the second term of the compound series $6e - \Pi$, is equal to $\frac{24Es^4}{e^3} - \frac{24C^2s^4}{e^3}$, or the second term of the compound series $8e - \Delta$.

Q. E. D.

48. The third term of the compound series $8e - \Delta$ is $\frac{24Gs^6}{e^5} - \frac{48CEs^6}{e^5} + \frac{8C^3s^6}{e^5}$, which is $= 24 \times \frac{154}{6561} \times \frac{s^6}{e^5} - 48 \times \frac{1}{9} \times \frac{10}{243} \times \frac{s^6}{e^5} + 8 \times \frac{1}{9} \times \frac{1}{9} \times \frac{s^6}{e^5} = \frac{8 \times 154}{2187} \times \frac{s^6}{e^5} - \frac{480}{2187} \times \frac{s^6}{e^5} + \frac{24}{2187} \times \frac{s^6}{e^5} = \frac{1232}{2187} \times \frac{s^6}{e^5} - \frac{480}{2187} \times \frac{s^6}{e^5} + \frac{24}{2187} \times \frac{s^6}{e^5} = \frac{1256}{2187} \times \frac{s^6}{e^5} - \frac{480}{2187} \times \frac{s^6}{e^5} = \frac{776s^6}{2187e^5}$; and the third term of the compound series $6e - \Pi$ is $\frac{6Ds^6}{e^5} - \frac{6C^2s^6}{e^5} - \frac{6BEs^6}{e^5} + \frac{6Gs^6}{e^5}$, which is $= 6 \times \frac{5}{81} \times \frac{s^6}{e^5} - 6 \times \frac{1}{81} \times \frac{s^6}{e^5} - 6 \times \frac{1}{3} \times \frac{10}{243} \times \frac{s^6}{e^5} + 6 \times \frac{154}{6561} \times \frac{s^6}{e^5} = \frac{6 \times 5 \times 27}{2187} \times \frac{s^6}{e^5} - \frac{6 \times 27}{2187} \times \frac{s^6}{e^5} - \frac{6 \times 10 \times 3}{2187} \times \frac{s^6}{e^5} + \frac{2 \times 154}{2187} \times \frac{s^6}{e^5} = \frac{810}{2187} \times \frac{s^6}{e^5} - \frac{162}{2187} \times \frac{s^6}{e^5} - \frac{180}{2187} \times \frac{s^6}{e^5} + \frac{308}{2187} \times \frac{s^6}{e^5} = \frac{1118}{2187} \times \frac{s^6}{e^5} - \frac{342}{2187} \times \frac{s^6}{e^5} = \frac{776s^6}{2187e^5}$. Therefore $\frac{6Ds^6}{e^5} - \frac{6C^2s^6}{e^5} - \frac{6BEs^6}{e^5} + \frac{6Gs^6}{e^5}$, or the third term of the compound series $6e - \Pi$, is equal to $\frac{24Gs^6}{e^5} - \frac{48CEs^6}{e^5} + \frac{8C^3s^6}{e^5}$, or the third term of the compound series $8e - \Delta$.

Q. E. D.

49. The fourth term of the compound series $8e - \Delta$ is $\frac{24Is^8}{e^7} - \frac{48CGs^8}{e^7} - \frac{24E^2s^8}{e^7} + \frac{24C^2Es^8}{e^7}$, which is $= 24 \times \frac{935}{59049} \times \frac{s^8}{e^7} - 48 \times \frac{1}{9} \times \frac{154}{6561} \times \frac{s^8}{e^7} - 24 \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^8}{e^7} + 24 \times \frac{1}{81} \times \frac{10}{243} \times \frac{s^8}{e^7} = \frac{8 \times 935}{19683} \times \frac{s^8}{e^7} - \frac{16 \times 154}{3 \times 6561} \times \frac{s^8}{e^7} - \frac{2400}{59049} \times \frac{s^8}{e^7} + \frac{240}{59049} \times \frac{s^8}{e^7}$

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 $\frac{s^8}{e^7}$

$$\begin{aligned}
& \frac{s^8}{e^7} - \frac{8 \times 10 \times 10}{81 \times 243} \times \frac{s^8}{e^7} + \frac{24 \times 10}{81 \times 243} \times \frac{s^8}{e^7} = \frac{7480}{19683} \times \frac{s^8}{e^7} - \frac{2464}{19683} \times \frac{s^8}{e^7} - \frac{800}{19683} \\
& \times \frac{s^8}{e^7} + \frac{240}{19683} \times \frac{s^8}{e^7} = \frac{7720}{19683} \times \frac{s^8}{e^7} - \frac{3264}{19683} \times \frac{s^8}{e^7} = \frac{4456 s^8}{19683 e^7}; \text{ and the fourth} \\
& \text{term of the compound series } 6e - \Pi \text{ is } \frac{6Es^8}{e^7} - \frac{6CDs^8}{e^7} - \frac{6CEs^8}{e^7} - \frac{6BGs^8}{e^7} + \\
& \frac{6Is^8}{e^7}, \text{ which is } = 6 \times \frac{10}{243} \times \frac{s^8}{e^7} - 6 \times \frac{1}{9} \times \frac{5}{81} \times \frac{s^8}{e^7} - 6 \times \frac{1}{9} \times \frac{10}{243} \times \\
& \frac{s^8}{e^7} - 6 \times \frac{1}{3} \times \frac{154}{6561} \times \frac{s^8}{e^7} + 6 \times \frac{935}{59,049} \times \frac{s^8}{e^7} = \frac{6 \times 81 \times 10}{81 \times 243} \times \frac{s^8}{e^7} - \\
& \frac{6 \times 27 \times 5}{9 \times 27 \times 81} \times \frac{s^8}{e^7} - \frac{6 \times 9 \times 10}{81 \times 243} \times \frac{s^8}{e^7} - \frac{6 \times 154}{3 \times 6561} \times \frac{s^8}{e^7} + \frac{2 \times 935}{19683} \times \frac{s^8}{e^7} = \frac{4860}{19683} \times \\
& \frac{s^8}{e^7} - \frac{810}{19683} \times \frac{s^8}{e^7} - \frac{540}{19683} \times \frac{s^8}{e^7} - \frac{924}{19683} \times \frac{s^8}{e^7} + \frac{1870}{19683} \times \frac{s^8}{e^7} = \frac{6730}{19683} \times \frac{s^8}{e^7} \\
& - \frac{2274}{19683} \times \frac{s^8}{e^7} = \frac{4456}{19683} \times \frac{s^8}{e^7}. \text{ Therefore } \frac{6Es^8}{e^7} - \frac{6CDs^8}{e^7} - \frac{6CEs^8}{e^7} - \frac{6BGs^8}{e^7} + \\
& \frac{6Is^8}{e^7} \text{ is } = \frac{24Is^8}{e^7} - \frac{48CGs^8}{e^7} - \frac{24E^2s^8}{e^7} + \frac{24C^2Es^8}{e^7}.
\end{aligned}$$

Q. E. D.

$$\begin{aligned}
& 50. \text{ The fifth term of the compound series } 8e - \Delta \text{ is } \frac{24Ls^{10}}{e^9} - \frac{48CIs^{10}}{e^9} - \\
& \frac{48EGs^{10}}{e^9} + \frac{24C^2Gs^{10}}{e^9} + \frac{24CE^2s^{10}}{e^9}, \text{ which is } = 24 \times \frac{55,913}{4,782,969} \times \frac{s^{10}}{e^9} - 48 \times \frac{1}{9} \\
& \times \frac{935}{59,049} \times \frac{s^{10}}{e^9} - 48 \times \frac{10}{243} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} + 24 \times \frac{1}{81} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} + 24 \times \\
& \frac{1}{9} \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{10}}{e^9} = \frac{8 \times 55,913}{1,594,323} \times \frac{s^{10}}{e^9} - 48 \times \frac{3}{27} \times \frac{935}{59,049} \times \frac{s^{10}}{e^9} - \\
& \frac{48 \times 10 \times 154}{1,594,323} \times \frac{s^{10}}{e^9} + 24 \times \frac{3}{243} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} + 24 \times \frac{3}{27} \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{10}}{e^9} \\
& = \frac{447,304}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{134,640}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{73,920}{1,594,323} \times \frac{s^{10}}{e^9} + \frac{11,088}{1,594,323} \times \frac{s^{10}}{e^9} + \\
& \frac{7200}{1,594,323} \times \frac{s^{10}}{e^9} = \frac{465,592}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{208,560}{1,594,323} \times \frac{s^{10}}{e^9} = \frac{257,032}{1,594,323} \times \frac{s^{10}}{e^9}; \text{ and} \\
& \text{the fifth term of the compound series } 6e - \Pi \text{ is } \frac{6Fs^{10}}{e^9} - \frac{6CEs^{10}}{e^9} - \frac{6DEs^{10}}{e^9} - \\
& \frac{6CGs^{10}}{e^9} - \frac{6BIs^{10}}{e^9} + \frac{6Ls^{10}}{e^9}, \text{ which is } = 6 \times \frac{22}{729} \times \frac{s^{10}}{e^9} - 6 \times \frac{1}{9} \times \frac{10}{243} \times \frac{s^{10}}{e^9} \\
& - 6 \times \frac{5}{81} \times \frac{10}{243} \times \frac{s^{10}}{e^9} - 6 \times \frac{1}{9} \times \frac{154}{6561} \times \frac{s^{10}}{e^9} - 6 \times \frac{1}{3} \times \frac{935}{59,049} \times \frac{s^{10}}{e^9} \\
& + \frac{6 \times 55,913}{4,782,969} \times \frac{s^{10}}{e^9} = \frac{6 \times 22 \times 2187}{729 \times 2187} \times \frac{s^{10}}{e^9} - \frac{6 \times 10 \times 729}{9 \times 243 \times 729} \times \frac{s^{10}}{e^9} - \frac{6 \times 5 \times 10 \times 81}{81 \times 81 \times 243} \\
& \times \frac{s^{10}}{e^9} - \frac{6 \times 27 \times 154}{9 \times 27 \times 6561} \times \frac{s^{10}}{e^9} - \frac{6 \times 9 \times 935}{3 \times 9 \times 59,049} \times \frac{s^{10}}{e^9} + \frac{2 \times 55,913}{1,594,323} \times \frac{s^{10}}{e^9} = \\
& \frac{288,684}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{43,740}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{24,300}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{24,948}{1,594,323} \times \frac{s^{10}}{e^9} - \\
& \frac{50,490}{1,594,323} \times \frac{s^{10}}{e^9} + \frac{111,826}{1,594,323} \times \frac{s^{10}}{e^9} = \frac{400,510}{1,594,323} \times \frac{s^{10}}{e^9} - \frac{143,478}{1,594,323} \times \frac{s^{10}}{e^9} = \\
& \frac{257,032}{1,594,323} \times \frac{s^{10}}{e^9}. \text{ Therefore } \frac{6Fs^{10}}{e^9} - \frac{6CEs^{10}}{e^9} - \frac{6DEs^{10}}{e^9} - \frac{6CGs^{10}}{e^9} - \frac{6BIs^{10}}{e^9} + \\
& \frac{6Ls^{10}}{e^9}
\end{aligned}$$

$\frac{6 L s^{10}}{e^9}$, or the fifth term of the compound series $6e - \Pi$, is equal to $\frac{24 L s^{10}}{e^9} - \frac{48 CI s^{10}}{e^9} - \frac{48 EG s^{10}}{e^9} + \frac{24 C^2 G s^{10}}{e^9} + \frac{24 CE^2 s^{10}}{e^9}$, or the fifth term of the compound series $8e - \Delta$. Q. E. D.

51. Lastly, the sixth term of the compound series $8e - \Delta$ is $\frac{24 N s^{12}}{e^{11}} - \frac{48 CL s^{12}}{e^{11}} - \frac{48 EI s^{12}}{e^{11}} - \frac{24 G^2 s^{12}}{e^{11}} + \frac{24 C^2 I s^{12}}{e^{11}} + \frac{48 CEG s^{12}}{e^{11}} + \frac{8 E^3 s^{12}}{e^{11}}$, which is $= 24 \times \frac{1,179,256}{129,140,163} \times \frac{s^{12}}{e^{11}} - 48 \times \frac{1}{9} \times \frac{55,913}{4,782,969} \times \frac{s^{12}}{e^{11}} - 48 \times \frac{10}{243} \times \frac{935}{59,049} \times \frac{s^{12}}{e^{11}} - 24 \times \frac{154}{6561} \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} + 24 \times \frac{1}{81} \times \frac{935}{59,049} \times \frac{s^{12}}{e^{11}} + 48 \times \frac{1}{9} \times \frac{10}{243} \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} + 8 \times \frac{10}{243} \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{12}}{e^{11}} = \frac{8 \times 1,179,256}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{48 \times 55,913}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{48 \times 10 \times 3 \times 935}{243 \times 3 \times 59,049} \times \frac{s^{12}}{e^{11}} - \frac{24 \times 154 \times 154}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{24 \times 9 \times 935}{81 \times 9 \times 59,049} \times \frac{s^{12}}{e^{11}} + \frac{48 \times 3 \times 10 \times 154}{9 \times 3 \times 243 \times 6561} \times \frac{s^{12}}{e^{11}} + \frac{8 \times 3 \times 1000}{3 \times 243 \times 243 \times 243} \times \frac{s^{12}}{e^{11}} = \frac{9,434,048}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{2,683,824}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{1,346,400}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{569,184}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{201,960}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{221,760}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{24000}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{9,881,768}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{4,599,408}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{5,282,360}{43,046,721} \times \frac{s^{12}}{e^{11}}$; and the sixth term of the compound series $6e - \Pi$ is $\frac{6 G s^{12}}{e^{11}} - \frac{6 CF s^{12}}{e^{11}} - \frac{6 E^2 s^{12}}{e^{11}} - \frac{6 DG s^{12}}{e^{11}} - \frac{6 CI s^{12}}{e^{11}} - \frac{6 BL s^{12}}{e^{11}} + \frac{6 N s^{12}}{e^{11}}$, which is $= 6 \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{1}{9} \times \frac{22}{729} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{10}{243} \times \frac{10}{243} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{5}{81} \times \frac{154}{6561} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{1}{9} \times \frac{935}{59,049} \times \frac{s^{12}}{e^{11}} - 6 \times \frac{1}{3} \times \frac{55,913}{4,782,969} \times \frac{s^{12}}{e^{11}} + \frac{6 \times 1,179,256}{129,140,163} \times \frac{s^{12}}{e^{11}} = \frac{6 \times 154 \times 6561}{6561 \times 6561} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 22 \times 6561}{9 \times 729 \times 6561} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 10 \times 10 \times 729}{243 \times 243 \times 729} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 5 \times 154 \times 81}{81 \times 81 \times 6561} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 81 \times 935}{9 \times 81 \times 59,049} \times \frac{s^{12}}{e^{11}} - \frac{6 \times 3 \times 55,913}{3 \times 3 \times 4,782,969} \times \frac{s^{12}}{e^{11}} + \frac{2 \times 1,179,256}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{6,062,364}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{866,052}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{437,400}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{374,220}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{454,410}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{1,006,434}{43,046,721} \times \frac{s^{12}}{e^{11}} + \frac{2,358,512}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{8,420,876}{43,046,721} \times \frac{s^{12}}{e^{11}} - \frac{3,138,516}{43,046,721} \times \frac{s^{12}}{e^{11}} = \frac{5,282,360}{43,046,721} \times \frac{s^{12}}{e^{11}}$. Therefore $\frac{6 G s^{12}}{e^{11}} - \frac{6 CF s^{12}}{e^{11}} - \frac{6 E^2 s^{12}}{e^{11}} - \frac{6 DG s^{12}}{e^{11}} - \frac{6 CI s^{12}}{e^{11}} - \frac{6 BL s^{12}}{e^{11}} + \frac{6 N s^{12}}{e^{11}}$, or the sixth term of the compound series $6e - \Pi$, is equal to $\frac{24 N s^{12}}{e^{11}} - \frac{48 CL s^{12}}{e^{11}} - \frac{48 EI s^{12}}{e^{11}} - \frac{24 G^2 s^{12}}{e^{11}} + \frac{24 C^2 I s^{12}}{e^{11}} + \frac{48 CEG s^{12}}{e^{11}} + \frac{8 E^3 s^{12}}{e^{11}}$, or the sixth term of the compound series $8e - \Delta$. Q. E. D.

A ge-

A general demonstration of the equality between the corresponding terms of the two compound serieses $8e - \Delta$ and $6e - \Pi$.

52. We have seen in the six preceeding articles that each of the first six compound terms (or vertical columns of simple terms involving the same powers of s and e) in the compound series $6e - \Pi$ is equal to the corresponding compound term, or vertical column of simple terms, in the compound series $8e - \Delta$. We now proceed to shew that the same equality must of necessity take place between all the correspondent terms of the said two serieses, as well as between the first six terms of them, to whatever number of terms the said serieses may be continued.

53. Now this equality of the correspondent terms of these two serieses will appear from this consideration, to wit, that the compound series $6e - \Pi$ is constantly equal to the compound series $8e - \Delta$ in all the different values of ss and ee that are possible, that is, when ee , or $\frac{rr}{4}$, is of any magnitude greater than $\frac{q^3}{27}$ (which is its least possible magnitude) and consequently when ss , or $\frac{rr}{4} - \frac{q^3}{27}$, is of any magnitude greater than $\frac{q^3}{27} - \frac{q^3}{27}$, or 0, or, in other words, of any magnitude, how small soever. For from hence it may be shewn of each of the compound terms of the compound series $6e - \Pi$ successively, beginning with the first term, that it is equal to the corresponding term of the compound series $8e - \Delta$. This may be done in the manner following.

54. In the first place, since the whole compound series $6e - \Pi$ is equal to the whole compound series $8e - \Delta$ (as has been shewn in art. 44) it follows that, if we divide all the terms of both serieses by the fraction $\frac{ss}{e}$, the quotients of these divisions will be equal to each other, that is, the compound series

$$\begin{aligned}
 & 6B + \frac{6css}{ee} + \frac{6Ds^4}{e^4} + \frac{6Es^6}{e^6} + \&c \\
 + & 6c - \frac{6BCss}{ee} - \frac{6C^2s^4}{e^4} - \frac{6CDs^6}{e^6} - \&c \\
 & + \frac{6Ess}{ee} - \frac{6BEs^4}{e^4} - \frac{6CES^6}{e^6} - \&c \\
 & + \frac{6Gs^4}{e^4} - \frac{6BGs^6}{e^6} - \&c \\
 & + \frac{6Is^6}{e^6} - \&c
 \end{aligned}$$

will be equal to the compound series

$$\begin{aligned}
& 24 C + \frac{24 E s s}{e e} + \frac{24 G s^4}{e^4} + \frac{24 I s^6}{e^6} + \&c \\
& - \frac{24 C^2 s s}{e e} - \frac{48 C E s^4}{e^4} - \frac{48 C G s^6}{e^6} - \&c \\
& + \frac{8 C^3 s^4}{e^4} - \frac{24 E^2 s^6}{e^6} - \&c \\
& + \frac{24 C^2 E s^6}{e^6} + \&c.
\end{aligned}$$

And this will be true, of how small a magnitude soever we may suppose ss to be taken. And consequently it will be true likewise, when ss is equal to 0, or when $\frac{rr}{4} - \frac{q^3}{27}$ is equal to 0, or when $\frac{rr}{4}$ is equal to $\frac{q^3}{27}$. But, when ss is = 0, all the terms of these two compound serieses that involve any power of ss , that is, all the terms, except the first terms, of the said serieses, will likewise be equal to 0, and the said two serieses will become equal to their first terms $6 B + 6 C$ and $24 C$ respectively. Therefore the said first terms must be equal to each other, that is, $6 B + 6 C$, the first term of the compound series $\frac{6e - \Pi}{\frac{ss}{e}}$, will be

equal to $24 C$, the first term of the compound series $\frac{8e - \Delta}{\frac{ss}{e}}$. Therefore, if we

multiply both these first terms into $\frac{ss}{e}$, it will follow that $\frac{6 B s s}{e} + \frac{6 C s s}{e}$, or the first term of the compound series $6e - \Pi$, will be equal to $\frac{24 C s s}{e}$, or the first term of the compound series $8e - \Delta$.

Q. E. D.

55. Secondly, since it has been shewn in the last article that $\frac{6 B s s}{e} + \frac{6 C s s}{e}$, or the first term of the compound series $6e - \Pi$, is equal to $\frac{24 s s}{e}$, or the first term of the compound series $8e - \Delta$, it follows that, if we subtract these first terms of these two serieses from the whole serieses, the remainders will be equal to each other, that is, the compound series

$$\begin{aligned}
& \frac{6 C s^4}{e^3} + \frac{6 D s^6}{e^5} + \frac{6 E s^8}{e^7} + \&c \\
& - \frac{6 B C s^4}{e^3} - \frac{6 C^2 s^6}{e^5} - \frac{6 C D s^8}{e^7} - \&c \\
& + \frac{6 E s^4}{e^3} - \frac{6 B E s^6}{e^5} - \frac{6 C E s^8}{e^7} - \&c \\
& - \frac{6 G s^6}{e^5} - \frac{6 B G s^8}{e^7} - \&c \\
& + \frac{6 I s^8}{e^7} - \&c
\end{aligned}$$

will be equal to the compound series

2

$$\frac{24 E s^4}{e^3}$$

$$\begin{aligned}
& \frac{24 E s^4}{e^3} + \frac{24 G s^6}{e^5} + \frac{24 I s^8}{e^7} + \&c \\
& - \frac{24 C^2 s^4}{e^3} - \frac{48 C E s^6}{e^5} - \frac{48 C G s^8}{e^7} - \&c \\
& + \frac{8 C^3 s^6}{e^5} - \frac{24 E^2 s^8}{e^7} - \&c \\
& + \frac{24 C^2 E s^8}{e^7} + \&c.
\end{aligned}$$

Now let all the terms of these two last compound serieses be divided by the fraction $\frac{s^4}{e^3}$; and it is evident that the quotients thence arising must be equal to each other, that is, the compound series

$$\begin{aligned}
& 6 C + \frac{6 D s s}{e e} + \frac{6 E s^4}{e^4} + \&c \\
& - 6 B C - \frac{6 C^2 s s}{e e} - \frac{6 C D s^4}{e^4} - \&c \\
& + 6 E - \frac{6 B E s s}{e e} - \frac{6 C E s^4}{e^4} - \&c \\
& + \frac{6 G s s}{e e} - \frac{6 B G s^4}{e^4} - \&c \\
& + \frac{6 I s^4}{e^4} - \&c
\end{aligned}$$

will be equal to the compound series

$$\begin{aligned}
& 24 E + \frac{24 G s s}{e e} + \frac{24 I s^4}{e^4} + \&c \\
& - 24 C^2 - \frac{48 C E s s}{e e} - \frac{48 C G s^4}{e^4} - \&c \\
& + \frac{8 C^3 s s}{e e} - \frac{24 C^2 s^4}{e^4} - \&c \\
& + \frac{24 C^2 E s^4}{e^4} + \&c.
\end{aligned}$$

And this will be true, of how small a magnitude soever we may suppose ss to be taken. And consequently it will likewise be true, when ss is $= 0$, or when $\frac{rr}{4} - \frac{q^3}{27}$ is $= 0$, or when $\frac{rr}{4}$ is $= \frac{q^3}{27}$. But in this case all the terms of these two compound serieses that involve any power of ss , that is, all the terms except the first, must likewise be equal to 0. Consequently those first terms of the said two serieses (being equal to the said whole serieses respectively) must be equal to each other, that is, $6 C - 6 B C + 6 E$, or the first term of the compound series derived from the compound series $6 e - \Pi$, will be equal to $24 E - 24 C^2$, or the first term of the compound series derived from the compound series $8 e - \Delta$. Therefore, if we multiply both these first terms into the fraction $\frac{s^4}{e^3}$, we shall have $\frac{6 C s^4}{e^3} - \frac{6 B C s^4}{e^3} + \frac{6 E s^4}{e^3}$ equal to $\frac{24 E s^4}{e^3} - \frac{24 C^2 s^4}{e^3}$; that is, the second term of the compound series $6 e - \Pi$ will be equal to the second term of the compound series $8 e - \Delta$.

Q. E. D.

56. Thirdly, since it has been shewn in art. 54, that $\frac{6 B s s}{e} + \frac{6 C s s}{e}$, or the first term of the compound series $6 e - \Pi$, is equal to $\frac{24 s s}{e}$, or the first term of the

the compound series $8e - \Delta$; and it has been shewn in art. 55, that $\frac{6cs^4}{e^3} - \frac{6BCs^4}{e^3} + \frac{6Es^4}{e^3}$, or the second term of the compound series $6e - \Pi$, is equal to $\frac{24Es^4}{e^3} - \frac{24C^2s^4}{e^3}$, or the second term of the compound series $8e - \Delta$; it follows that, if we subtract these two first terms of these two serieses from the whole serieses, the remainders will be equal to each other, that is, the compound series

$$\begin{aligned} & \frac{6Ds^6}{e^5} + \frac{6Es^8}{e^7} + \&c \\ & - \frac{6C^2s^6}{e^5} - \frac{6CDs^8}{e^7} - \&c \\ & - \frac{6BEs^6}{e^5} - \frac{6CEs^8}{e^7} - \&c \\ & + \frac{6Gs^6}{e^5} - \frac{6BGs^8}{e^7} - \&c \\ & + \frac{6Is^8}{e^7} - \&c \end{aligned}$$

will be equal to the compound series

$$\begin{aligned} & \frac{24Gs^6}{e^5} + \frac{24Is^8}{e^7} + \&c \\ & - \frac{48CEs^6}{e^5} - \frac{48CGs^8}{e^7} - \&c \\ & + \frac{8C^3s^6}{e^5} - \frac{24E^2s^8}{e^7} - \&c \\ & + \frac{24C^2Es^8}{e^7} + \&c. \end{aligned}$$

Now let all the terms of these two last compound serieses be divided by the fraction $\frac{s^6}{e^5}$; and it is evident that the quotients thence arising must be equal to each other; that is, the compound series

$$\begin{aligned} & 6D + \frac{6Es^2}{ee} + \&c \\ & - 6C^2 - \frac{6CDs^2}{ee} - \&c \\ & - 6BE - \frac{6CEs^2}{ee} - \&c \\ & + 6G - \frac{6BGs^2}{ee} - \&c \\ & + \frac{6Is^2}{ee} - \&c \end{aligned}$$

will be equal to the compound series

$$\begin{aligned} & 24G + \frac{24Is^2}{ee} + \&c \\ & - 48CE - \frac{48CGs^2}{ee} - \&c \\ & + 8C^3 - \frac{24E^2s^2}{ee} - \&c \\ & + \frac{24C^2Es^2}{ee} + \&c. \end{aligned}$$

And this will be true, of how small a magnitude soever we may suppose ss to be taken; and consequently it will likewise be true, when ss is $= 0$, or when $\frac{rr}{4} - \frac{q^3}{27}$ is $= 0$, or when $\frac{rr}{4}$ is $= \frac{q^3}{27}$. But in this case all the terms of these two compound serieses that involve any power of ss , that is, all the terms except the first, must likewise be equal to 0. Consequently those first terms of the said two serieses (being equal to the said whole serieses respectively) must be equal to each other; that is, $6D - 6C^2 - 6BE + 6G$, the first term of the compound series derived from the compound series $6e - \Pi$, must be equal to $24G - 48CE + 8C^3$, the first term of the compound series derived from the compound series $8e - \Delta$. Therefore, if we multiply both these first terms into the fraction $\frac{s^6}{e^5}$, we shall have $\frac{6Ds^6}{e^5} - \frac{6C^2s^6}{e^5} - \frac{6BEs^6}{e^5} + \frac{6Gs^6}{e^5}$ equal to $\frac{24Gs^6}{e^5} - \frac{48CEs^6}{e^5} + \frac{8C^3s^6}{e^5}$; that is, the third term of the compound series $6e - \Pi$ will be equal to the third term of the compound series $8e - \Delta$. Q. E. D.

57. And in the same manner we may shew that the fourth term of the compound series $6e - \Pi$ is equal to the fourth term of the compound series $8e - \Delta$, and the fifth to the fifth, and the sixth to the sixth, and the seventh to the seventh, and every following term of the one series to the corresponding term of the latter series, to whatever number of terms the said serieses may be continued.

The reduction of the compound series $8e - \Delta$ to a simple series $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c$ ad infinitum.

58. Now let the several numeral co-efficients of the fractions $\frac{ss}{e}$, $\frac{s^4}{e^3}$, $\frac{s^6}{e^5}$, $\frac{s^8}{e^7}$, $\frac{s^{10}}{e^9}$, $\frac{s^{12}}{e^{11}}$, &c in the terms of the compound series $8e - \Delta$ (which is set down above in art. 44) be reduced (by performing the necessary arithmetical operations of multiplication, addition, and subtraction) to single numbers, so as to convert the said compound series into a simple series, or series of simple, or single, terms. This reduction has been already made for the first six terms of this series in art. 46, 47, 48, 49, 50, and 51; in which it has been shewn that the first term of the said compound series, to wit, $\frac{24Css}{e}$, is $= \frac{8ss}{3e}$; and that its second term, $\frac{24Es^4}{e^3} - \frac{24C^2s^4}{e^3}$, is $= \frac{56s^4}{81e^3}$; and that its third term, $\frac{24Gs^6}{e^5} - \frac{48CEs^6}{e^5} + \frac{8C^3s^6}{e^5}$, is $= \frac{776s^6}{2187e^5}$; and that its fourth term, $\frac{24Is^8}{e^7} - \frac{48CGs^8}{e^7} - \frac{24E^2s^8}{e^7}$

$\frac{24 E^2 s^8}{e^7} + \frac{24 C^2 E s^8}{e^7}$, is $= \frac{4456 s^8}{19,683 e^7}$; and that its fifth term, $\frac{24 L s^{10}}{e^9} - \frac{48 C I s^{10}}{e^9} -$
 $\frac{48 E G s^{10}}{e^9} + \frac{24 C^2 G s^{10}}{e^9} + \frac{24 C E^2 s^{10}}{e^9}$, is $= \frac{257,032 s^{10}}{1,594,323 e^9}$; and that its sixth term, $\frac{24 N s^{12}}{e^{11}} -$
 $\frac{48 C L s^{12}}{e^{11}} - \frac{48 E I s^{12}}{e^{11}} - \frac{24 G^2 s^{12}}{e^{11}} + \frac{24 C^2 I s^{12}}{e^{11}} + \frac{48 C E G s^{12}}{e^{11}} + \frac{8 E^3 s^{12}}{e^{11}}$, is $=$
 $\frac{5,282,360 s^{12}}{43,046,721 e^{11}}$; so that the first six terms of the said compound series $8e - \Delta$ are
 equal to the six following simple quantities, to wit, $\frac{8 s^3}{3e} + \frac{56 s^4}{81 e^3} + \frac{776 s^6}{2187 e^5} +$
 $\frac{4456 s^8}{19683 e^7} + \frac{257,032 s^{10}}{1,594,323 e^9} + \frac{5,282,360 s^{12}}{43,046,721 e^{11}}$. And let these co-efficients of the fractions
 $\frac{s^3}{e}, \frac{s^4}{e^3}, \frac{s^6}{e^5}, \frac{s^8}{e^7}, \frac{s^{10}}{e^9}, \frac{s^{12}}{e^{11}}$, &c, in the said simple series be (for the sake of bre-
 vity) denoted by the capital letters P, Q, R, S, T, V, &c. Then will the
 compound series $8e - \Delta$ be equal to the simple series $\frac{P s^3}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7}$
 $+ \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}.$

*The reduction of the compound series $6e - \Pi$ in like man-
ner to a series of simple terms.*

59. And, in like manner, let the several numeral co-efficients of the same
 fractions $\frac{s^3}{e}, \frac{s^4}{e^3}, \frac{s^6}{e^5}, \frac{s^8}{e^7}, \frac{s^{10}}{e^9}, \frac{s^{12}}{e^{11}}$ &c in the terms of the other compound series
 $6e - \Pi$ (which is also set down above in art. 44) be reduced (by performing the
 necessary operations of multiplication, addition, and subtraction) to single num-
 bers, so as to convert the said compound series into a simple series, or series of
 simple, or single, terms. This reduction has been already made for the first six
 terms of the series in art. 46, 47, 48, 49, 50, and 51; in which it has been
 shewn that the first term of the said compound series, to wit, $\frac{6 B s^3}{e} + \frac{6 C s^3}{e}$, is
 $= \frac{8 s^3}{3e}$, as well as the first term of the former compound series $8e - \Delta$;
 and that the second term of this compound series, to wit, $\frac{6 C s^4}{e^3} - \frac{6 B C s^4}{e^3} + \frac{6 E s^4}{e^3}$,
 is $= \frac{56 s^4}{81 e^3}$, as well as the second term of the said former compound series; and
 that the third term of this compound series, to wit, $\frac{6 D s^6}{e^5} - \frac{6 C^2 s^6}{e^5} - \frac{6 B E s^6}{e^5} +$
 $\frac{6 G s^6}{e^5}$, is $= \frac{776 s^6}{2187 e^5}$, as well as the third term of the said former compound series;
 and that the fourth term of this compound series, to wit, $\frac{6 E s^8}{e^7} - \frac{6 C D s^8}{e^7} - \frac{6 C E s^8}{e^7}$
 $- \frac{6 B G s^8}{e^7} + \frac{6 I s^8}{e^7}$, is $= \frac{4456 s^8}{19683 e^7}$, as well as the fourth term of the said former

compound series; and that the fifth term of this compound series, to wit, $\frac{6 F s^{10}}{e^9}$ — $\frac{6 C E s^{10}}{e^9}$ — $\frac{6 D E s^{10}}{e^9}$ — $\frac{6 C G s^{10}}{e^9}$ — $\frac{6 B I s^{10}}{e^9}$ + $\frac{6 L s^{10}}{e^9}$, is = $\frac{257,032 s^{10}}{1,594,323 e^9}$, as well as the fifth term of the said former compound series; and that the sixth term of this compound series, to wit, $\frac{6 G s^{12}}{e^{11}}$ — $\frac{6 C F s^{12}}{e^{11}}$ — $\frac{6 E^2 s^{12}}{e^{11}}$ — $\frac{6 D G s^{12}}{e^{11}}$ — $\frac{6 C I s^{12}}{e^{11}}$ — $\frac{6 B L s^{12}}{e^{11}}$ + $\frac{6 N s^{12}}{e^{11}}$, is = $\frac{5,282,360 s^{12}}{43,046,721 e^{11}}$, as well as the sixth term of the said former compound series; so that the first six terms of the compound series $6 e - \Pi$ are equal to the six following simple quantities, to wit, $\frac{8 s s}{3 e} + \frac{56 s^4}{81 e^3} + \frac{776 s^6}{2187 e^5} + \frac{4456 s^8}{19683 e^7} + \frac{257,032 s^{10}}{1,594,323 e^9} + \frac{5,282,360 s^{12}}{43,046,721 e^{11}}$, as well as the first six terms of the former compound series $8 e - \Delta$. And it is evident, from art. 57, that if the seventh, and eighth, and ninth, and other following terms of the compound series $6 e - \Pi$ were in the same manner to be reduced to simple terms, the said simple terms would be equal to the simple terms to which the seventh, eighth, ninth, and other following terms of the said former compound series $8 e - \Delta$ would be reduced. And consequently the same simple series $\frac{P s s}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, will be equal to both the compound serieses $8 e - \Delta$ and $6 e - \Pi$.

The values of y^3 and qy expressed by means of the simple series $\frac{P s s}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$.

60. We have seen, in art. 41, that the compound series Δ is = y^3 ; and we have seen, in art. 43, that the compound series Π is = qy . Therefore $8 e - \Delta$ is = $8 e - y^3$, and $6 e - \Pi$ is = $6 e - qy$. We shall therefore have $8 e - y^3$ (= $8 e - \Delta$) = the simple series $\frac{P s s}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, and consequently (adding Δ to both sides) $8 e = y^3$ + the series $\frac{P s s}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, and $y^3 = 8 e - \frac{P s s}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$; and we shall also have $6 e - qy$ (= $6 e - \Pi$) = the same series $\frac{P s s}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, and consequently (adding qy to both sides)

fides) $6e = qy + \frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c \text{ ad infinitum},$
 and $qy = 6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c \text{ ad infinitum}.$

Examination of the expression $2\sqrt[3]{g} \times$ the infinite series

$$1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$$

ad infinitum.

61. We must now turn our attention to the other transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$, in which g is put for $\frac{t}{2}$, or half the absolute term of the equation $x^3 - qx = t$, as e was before put for $\frac{r}{2}$, or half the absolute term of the equation $y^3 - qy = r$, and in which zz is put for $\frac{q^3}{27} - \frac{tt}{4}$, or the difference between $\frac{q^3}{27}$ and the square of half the absolute term t , as ss was before put for $\frac{rr}{4} - \frac{q^3}{27}$, or the difference between $\frac{q^3}{27}$ and the square of half the absolute term r . This expression we have asserted above, in art. 36, to be equal to the root of the equation $x^3 - qx = t$, in which the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or in which $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. This assertion we must now endeavour to prove.

62. Now "that this expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is equal to the root x of the equation $x^3 - qx = t$, when t is of the magnitude here supposed," will be evident, if we can shew that this expression, being substituted instead of x in the compound quantity $x^3 - qx$, will make that quantity be equal to the absolute term t of the said equation; or that, if the said expression be cubed, or raised to the third power by multiplying it twice into itself, and also be multiplied into the co-efficient q , the said cube of the said expression will be greater than the said product of its multiplication into q , and that the excess, or difference, will be equal to the absolute term t . This therefore is what I shall now endeavour to demonstrate.

63. The cube of the expression $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum* is $= 8g \times$ the cube of the said series. We must therefore raise the cube of this series by multiplying it twice into itself; which may be done as follows.

*The multiplication of the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum* into itself, in order to obtain its square.*

$$\begin{array}{r}
 1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \\
 1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \\
 \hline
 1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \\
 + \frac{Czz}{gg} + \frac{C^2z^4}{g^4} - \frac{CEz^6}{g^6} + \frac{CGz^8}{g^8} - \frac{CIz^{10}}{g^{10}} + \frac{CLz^{12}}{g^{12}} - \&c \\
 - \frac{Ez^4}{g^4} - \frac{CEz^6}{g^6} + \frac{E^2z^8}{g^8} - \frac{EGz^{10}}{g^{10}} + \frac{EIz^{12}}{g^{12}} - \&c \\
 + \frac{Gz^6}{g^6} + \frac{CGz^8}{g^8} - \frac{EGz^{10}}{g^{10}} + \frac{G^2z^{12}}{g^{12}} - \&c \\
 - \frac{Iz^8}{g^8} - \frac{CIz^{10}}{g^{10}} + \frac{EIz^{12}}{g^{12}} - \&c \\
 + \frac{Lz^{10}}{g^{10}} + \frac{CLz^{12}}{g^{12}} - \&c \\
 - \frac{Nz^{12}}{g^{12}} - \&c \\
 \hline
 1 + \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} + \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} + \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} + \&c \\
 + \frac{C^2z^4}{g^4} - \frac{2CEz^6}{g^6} + \frac{2CGz^8}{g^8} - \frac{2CIz^{10}}{g^{10}} + \frac{2CLz^{12}}{g^{12}} - \&c \\
 + \frac{E^2z^8}{g^8} - \frac{2EGz^{10}}{g^{10}} + \frac{2EIz^{12}}{g^{12}} - \&c \\
 + \frac{G^2z^{12}}{g^{12}} - \&c
 \end{array}$$

This is the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$.

A com-

A comparison between the foregoing compound series, which is equal to the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \text{\&c ad infinitum}$, and the compound series which is equal to the square of the former series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \text{\&c ad infinitum}$, and which is set down above in art. 40.

64. Now, if we compare this square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \text{\&c}$ with the compound series which is equal to the square of the former series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} + \text{\&c}$, and which is set down above in art. 40, we shall find that there are the following resemblances and differences between them.

In the first place, the first terms of both these compound serieses are equal to the same quantity, to wit, 1.

Secondly, the second, and third, and fourth, and other following terms of the latter compound series obtained in the foregoing article 63, involve in them the fractions $\frac{zz}{gg}$, $\frac{z^4}{g^4}$, $\frac{z^6}{g^6}$, $\frac{z^8}{g^8}$, $\frac{z^{10}}{g^{10}}$, $\frac{z^{12}}{g^{12}}$, &c, or the several powers of the fraction $\frac{zz}{gg}$, just as the second, and third, and fourth, and other following terms of the former compound series, obtained in art. 40, involve in them the fractions $\frac{ss}{ee}$, $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, $\frac{s^8}{e^8}$, $\frac{s^{10}}{e^{10}}$, $\frac{s^{12}}{e^{12}}$, &c, or the several powers of the fraction $\frac{ss}{ee}$. And it is evident that this observation will be true of all the following terms of these two compound serieses, because the simple series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \text{\&c}$ contains in its terms the very same powers of the fraction $\frac{zz}{gg}$ as the former simple series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \text{\&c}$ contains of the fraction $\frac{ss}{ee}$ in its several corresponding terms.

Thirdly, the signs of the several members of the third, fifth, and seventh terms of the latter compound series obtained in the preceeding art. 63, are the same with the signs of the corresponding members of the third, fifth, and seventh terms of the former compound series obtained in art. 40. For the third term of the former compound series is $-\frac{2Es^4}{e^4} + \frac{C^2s^4}{e^4}$, the two members of which are marked with the signs $-$ and $+$; and the third term of the latter compound

compound series is $-\frac{2Ez^4}{g^4} + \frac{C^2z^4}{g^4}$, the two members of which are likewise marked with the signs $-$ and $+$. And the fifth term of the former compound series is $-\frac{2Is^8}{e^8} + \frac{2CGs^8}{e^8} + \frac{E^2s^8}{e^8}$, the three members of which are marked with the signs $-$, $+$, and $+$; and the fifth term of the latter compound series is $-\frac{2Iz^8}{g^8} + \frac{2CGz^8}{g^8} + \frac{E^2z^8}{g^8}$, the three members of which are likewise marked with the signs $-$, $+$, and $+$. And the seventh term of the former compound series is $-\frac{2Ns^{12}}{e^{12}} + \frac{2CLs^{12}}{e^{12}} + \frac{2EIs^{12}}{e^{12}} + \frac{G^2s^{12}}{e^{12}}$, the four members of which are marked with the signs $-$, $+$, $+$, and $+$; and the seventh term of the latter compound series is $-\frac{2Nz^{12}}{g^{12}} + \frac{2CLz^{12}}{g^{12}} + \frac{2EIZ^{12}}{g^{12}} + \frac{G^2z^{12}}{g^{12}}$, the four members of which are likewise marked with the signs $-$, $+$, $+$, and $+$. And the same analogy will take place between the signs of the several members of the ninth, eleventh, thirteenth, and other following odd terms of the former compound series obtained in art. 40, and the signs of the corresponding members of the ninth, eleventh, thirteenth, and other following odd terms of the latter compound series obtained in art. 63; as we shall presently endeavour to make appear.

Fourthly, the sign of the second term, $+\frac{2Czz}{gg}$, of the latter compound series, and those of the several members of its fourth term, $+\frac{2Gz^6}{g^6} - \frac{2CEz^6}{g^6}$, and those of the several members of its sixth term, $+\frac{2Lz^{10}}{g^{10}} - \frac{2CIZ^{10}}{g^{10}} - \frac{2EGz^{10}}{g^{10}}$, are respectively contrary to the sign of the second term, $-\frac{2Css}{ee}$, of the former compound series, and to those of the several members of its fourth term, $-\frac{2Gs^6}{e^6} + \frac{2CES^6}{e^6}$, and to those of the several members of its sixth term, $-\frac{2Ls^{10}}{e^{10}} + \frac{2CIS^{10}}{e^{10}} + \frac{2EGS^{10}}{e^{10}}$. And the same contrariety will take place between the signs of the several members of the eighth, tenth, twelfth, and other following even terms of the latter compound series obtained in art. 63, and those of the several corresponding members of the eighth, tenth, twelfth, and other following even terms of the former compound series obtained in art. 40; as we shall now endeavour to make appear.

65. It is evident, from the rules of multiplication in algebra, that, whenever a series of algebraick quantities is multiplied by either the same, or another, series of algebraick quantities, all those horizontal lines of terms in the product, which arise from the multiplication of the first series, or multiplicand, into those terms of the multiplier which are marked with the sign $+$, will have the same signs $+$ and $-$ prefixed to their several terms as are prefixed to the corresponding terms in the multiplicand; and that all those horizontal lines of terms in the product which arise from the multiplication of the first series, or multiplicand, into those terms of the multiplier which are marked with the sign

sign —, will have contrary signs prefixed to their several terms to those which are prefixed to the corresponding terms of the multiplicand. It follows therefore that in the general product of the multiplication of the series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ (of which all the terms after the first term are marked with the sign —) into itself, which is set down above in art. 40 (I mean by the *general* product the first product, before the similar terms in each vertical column of terms are added up together at the bottom, so as to make but one term), the first horizontal line of terms (which arises from the multiplication of the said multiplicand, or series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, into its first term 1, or, in other words, which is the said series itself) must have the sign + prefixed to its first term (or rather, no sign at all; because, being the first term of the whole product, it is that to which all the other terms are to be referred, and to be added to it when they are marked with the sign +, and to be subtracted from it when they are marked with the sign —) and must have the sign — prefixed to all the following terms; and the second, and third, and fourth, and fifth, and sixth, and seventh, and every following horizontal line of terms in the said product (which arise from the multiplication of the said multiplicand, or series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, into the several terms $\frac{C s s}{e e}, \frac{E s^4}{e^4}, \frac{G s^6}{e^6}, \frac{I s^8}{e^8}, \frac{L s^{10}}{e^{10}}, \frac{N s^{12}}{e^{12}}, \&c$, which are all marked with the sign —) must have the sign — prefixed to their several first terms, and the sign + prefixed to all their following terms, to whatever number of terms the said horizontal lines may be continued. And it follows likewise, that in the general product of the multiplication of the series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$ (of which the second, and fourth, and sixth, and all the following even terms, are marked with the sign +, and the third, and fifth, and seventh, and all the following odd terms, are marked with the sign —) into itself, which is set down in art. 63, the first horizontal line of terms (which arises by the multiplication of the said multiplicand, or series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$, into its first term 1, or, in other words, which is the said series itself) must have the sign + prefixed to its second, fourth, sixth, and other following even terms, and the sign — prefixed to its third, fifth, seventh, and other following odd terms (to whatever number of terms the said horizontal line may be continued), being the same signs as those of the several corresponding terms of the said multiplicand itself; and, in like manner, the second horizontal row of terms in the said product, and the fourth, and sixth, and eighth, and tenth, and every following even horizontal row of terms in it (which arise from the multiplication of the said multiplicand, or series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$, into the second, fourth,

fourth, and sixth terms, $\frac{Czz}{gg}$, $\frac{Gz^6}{g^6}$, $\frac{Lz^{10}}{g^{10}}$, and the other following even terms, of the said series, to all which the sign + is prefixed) will have the same signs + and — prefixed to their several terms as are prefixed to the corresponding terms of the said multiplicand, or series itself, to wit, the sign + prefixed to their first, and second, and fourth, and sixth, and eighth, and tenth, and other following even terms, and the sign — prefixed to their third, and fifth, and seventh, and ninth, and eleventh, and other following odd terms; and the third horizontal row of terms in the said product, and the fifth, and seventh, and ninth, and eleventh, and every following odd horizontal row of terms in it (which arise from the multiplication of the said multiplicand, or series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, into the third, and fifth, and seventh, terms $\frac{Ez^4}{g^4}$, $\frac{Iz^8}{g^8}$, $\frac{Nz^{12}}{g^{12}}$, and the other following odd terms of the said series, to all which the sign — is prefixed), will have contrary signs prefixed to their several terms to those which are prefixed to the corresponding terms of the said multiplicand, or series itself, and therefore will have the sign — prefixed to their first, and second, and fourth, and sixth, and eighth, and tenth, and other following even terms, and the sign + prefixed to their third, and fifth, and seventh, and ninth, and eleventh, and other following odd terms.

66. It has been shewn, in the last article, that in the general product of the multiplication of the series $1 - \frac{Czz}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ into itself, set down above in art. 40, all the terms in the first, or highest, horizontal row of terms, except the first term 1, will have the sign — prefixed to them, and that in the second, and third, and fourth, and other following horizontal rows of terms in the said product, the first terms of the said rows will have the sign — prefixed to them, but all the following terms in them will be marked with the sign +. Now the first terms of the several horizontal rows of terms in the said general product are the lowest terms of the several vertical columns of terms in the said product, which involve the same powers of the fraction $\frac{ss}{ee}$; and the second, and third, and fourth, and other following terms of the first, or highest, horizontal row of terms in the said product, are the highest terms of the second, and third, and fourth, and other following vertical columns of terms in the said product. Therefore the highest term and the lowest term of the second, and the third, and the fourth, and every following vertical column of terms in the said product, will have the sign — prefixed to them, and all the other terms of the said vertical columns will be marked with the sign +. Now this will likewise be the case with the third, and the fifth, and the seventh, and the other following odd vertical columns, of the general product of the multiplication of the other series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ into itself, set down in art. 63. For the first, or highest, terms

terms of the said odd vertical columns are the third, fifth, seventh, and other following odd terms of the first, or highest, horizontal row of terms, to wit, the terms $\frac{Ez^4}{g^4}$, $\frac{Iz^8}{g^8}$, $\frac{Nz^{12}}{g^{12}}$, &c, which are all marked with the sign $-$; and the lowest terms of the same odd vertical columns are equal to their highest terms, and are marked also with the sign $-$, because they are the first terms of the several odd horizontal rows of terms, which have been shewn to be marked with the said sign $-$. And the intermediate terms of the said odd vertical columns between the highest term and the lowest must all be marked with the sign $+$, because they are, all of them, the products of the multiplication of factors, which are both marked either with the same sign $+$ or with the same sign $-$. Thus, for example, in the seventh vertical column of the said product, of which the highest term is $-\frac{Nz^{12}}{g^{12}}$, the next term $\frac{CLz^{12}}{g^{12}}$ is the product of the multiplication of the factors $\frac{Lz^{10}}{g^{10}}$ and $\frac{Czz}{gg}$, which are both marked with the sign $+$, and therefore it must be likewise marked with the sign $+$; and the third term $\frac{EIz^{12}}{g^{12}}$ is the product of the multiplication of the factors $\frac{Iz^8}{g^8}$ and $\frac{Ez^4}{g^4}$, which are both marked with the sign $-$, and therefore it must be marked with the sign $+$; and the fourth term $\frac{G^2z^{12}}{g^{12}}$ is the product of the multiplication of $\frac{Gz^6}{g^6}$ by $\frac{Gz^6}{g^6}$, which are both marked with the sign $+$, and therefore it must be marked also with the sign $+$; and the fifth term $\frac{EIz^{12}}{g^{12}}$ is the product of the multiplication of the factors $\frac{Ez^4}{g^4}$ and $\frac{Iz^8}{g^8}$, which are both marked with the sign $-$, and therefore it must be marked with the sign $+$; and the sixth term, $\frac{CLz^{12}}{g^{12}}$, or the lowest term but one, or the last of the intermediate terms between the highest and lowest terms of the said vertical column, is the product of the factors $\frac{Czz}{gg}$ and $\frac{Lz^{10}}{g^{10}}$, which are both marked with the sign $+$, and therefore it must likewise be marked with the sign $+$; and consequently all the five intermediate terms of the said seventh vertical column between its highest and lowest terms must be marked with the sign $+$. And it is easy to perceive that, for the same reasons, all the intermediate terms between the highest and lowest terms of any other vertical column of which the highest term was marked with the sign $-$, that is, of any other odd vertical column, must be marked with the sign $+$. We may therefore conclude that in all the odd vertical columns of terms in the general product set down in art. 63, the signs $+$ and $-$, which are to be prefixed to the several terms of the said columns, will be the same as those which are to be prefixed to the corresponding terms of the same odd vertical columns of terms in the general product set down above in art. 40.

Q. E. D.

67. Since the signs of the terms in the several odd vertical columns of the general product set down in art. 63, are the same as those of the corresponding

terms of the like odd vertical columns of the general product set down in art. 40, it is evident that, when the similar terms contained in the said odd vertical columns are added up together at the bottoms of the said general products, so as to make but single terms, the same analogy between the signs of the terms of these vertical columns will continue; that is, in the several odd vertical columns of the reduced compound series, which is equal to the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, the signs of the several terms, or members of the said columns, will be the same with the signs of the several corresponding terms, or members of the like odd vertical columns in the reduced compound series, which is equal to the square of the former series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \&c$, to whatever number of terms the said serieses may be continued. Therefore the third observation made above in art. 64, concerning the signs of the members of the odd vertical columns of these two compound serieses, is universally true. Q. E. D.

68. But in the second, and fourth, and sixth, and other following *even* vertical columns of terms in the general product set down in art. 63, the signs $+$ and $-$, that are to be prefixed to the several terms or members of the said columns, will be contrary to those which are to be prefixed to the several corresponding terms of the second, and fourth, and sixth, and other following *even* vertical columns in the general product set down in art. 40. For, in the first place, the first, or highest, term in every even vertical column in the general product set down in art. 63, is marked with the sign $+$; whereas the first, or highest, term of every even vertical column of terms in the general product set down in art. 40 (as well as the first, or highest, term of the third and every following odd vertical column of terms in the same general product), is marked with the sign $-$. And, in the second place, the last, or lowest, term in every even vertical column of terms in the general product set down in art. 63, is marked with the sign $+$; because the said lowest term of such even vertical column is the first term of the even horizontal line of terms in which it lies: and it has been shewn in art. 65 that the first terms of the second, and fourth, and sixth, and every following even horizontal row of terms, are marked with the sign $+$. But the lowest terms of the second, and fourth, and sixth, and every following even vertical column of terms, in the general product set down in art. 40 (as well as of every odd vertical column in the said product), are all marked with the sign $-$, as has been shewn in art. 66. And, thirdly, the several intermediate terms between the highest and the lowest terms in every even vertical column of terms, in the general product set down in art. 63, must all be marked with the sign $-$, because they are the products of factors which are marked with different signs. Thus, for example, in the sixth vertical column of the general product, set down in art. 63, of which the highest term is $+\frac{Lz^{10}}{g^{10}}$, the next term $\frac{CIz^{10}}{g^{10}}$ is the product of the multiplication of the factors $\frac{Iz^8}{g^8}$ and $\frac{Czz}{gg}$, of which the former is marked with the sign $-$, and the latter is marked with the sign $+$, in consequence

consequence of which their product must be marked with the sign —; and the third term $\frac{EGz^{10}}{g^{10}}$ is the product of the multiplication of the factors $\frac{Gz^6}{g^6}$ and $\frac{Ez^4}{g^4}$, of which the former is marked with the sign +, and the latter is marked with the sign —, and consequently their product must be marked with the sign —; and the fourth term $\frac{EGz^{10}}{g^{10}}$ is the product of the multiplication of the factors $\frac{Ez^4}{g^4}$ and $\frac{Gz^6}{g^6}$, of which the former is marked with the sign —, and the latter is marked with the sign +, and consequently their product must be marked with the sign —; and the fifth term $\frac{CIZ^{10}}{g^{10}}$, or the lowest term but one, or the last of the said intermediate terms between the highest and the lowest terms of the said vertical column, is the product of the multiplication of the factors $\frac{Czz}{gg}$ and $\frac{Iz^8}{g^8}$, of which the former is marked with the sign +, and the latter is marked with the sign —, and consequently their product must be marked with the sign —. And it is easy to perceive that, for the same reasons, all the intermediate terms between the highest and the lowest terms of any other vertical column, of which the highest term was marked with the sign +, that is, of any other even vertical column, must be marked with the sign —. But in the general product set down in art. 40, all the intermediate terms between the highest and lowest terms of all the vertical columns, both odd and even, are marked with the sign +. Therefore all the intermediate terms between the highest and the lowest terms of all the even vertical columns, in the general product set down in art. 63, are marked with contrary signs to those of the corresponding intermediate terms of the even vertical columns of the general product set down in art. 40. And the same thing has been shewn concerning the highest and lowest terms of the said even vertical columns of the said two general products. We may therefore conclude that in all the even vertical columns of terms in the general product set down in art. 63, the signs + and —, which are to be prefixed to the several terms of the said columns, will be every where contrary to those which are to be prefixed to the corresponding terms of the like even vertical columns of terms in the general product set down above in art. 40. Q. E. D.

69. Since the signs of the terms in the several even vertical columns of the general product set down in art. 63, are respectively contrary to those of the corresponding terms of the like even vertical columns of the general product set down in art. 40, it is evident that, when the similar terms contained in the said even vertical columns are added up together at the bottoms of both the said general products, so as to make but single terms, the same analogy, or rather contrariety, between the signs of the terms of these vertical columns will continue; that is, in the several even vertical columns of the reduced compound series, which is equal to the square of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, the signs of the several terms, or members of the said columns, will be every where contrary to the signs of the several corresponding terms, or members

members of the like even vertical columns, of the reduced compound series, which is equal to the square of the series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, to whatever number of terms the said serieses may be continued. Therefore the fourth observation made above in art. 64, concerning the signs of the members of the even vertical columns of these two compound serieses, is universally true. Q. E. D.

70. In the fifth place we may observe concerning these two compound serieses, which are equal to the squares of the two infinite serieses $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, and $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$, that the co-efficients of the several terms, or members of the several vertical columns, of the compound series which is equal to the square of the latter simple series, and which is set down in art. 63, will be equal to the co-efficients of the corresponding terms, or members of the several vertical columns of the former compound series, which is equal to the square of the former simple series, and which is set down above in art. 40; though the signs + and - that are to be prefixed to them will not every where be alike. For, since the two simple serieses $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, and $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$, contain the very same co-efficients C, E, G, I, L, N, &c, combined with the same powers of the two fractions $\frac{s s}{e e}$ and $\frac{z z}{g g}$, though with different signs + and - prefixed to the even terms $\frac{C s s}{e e}$, $\frac{G s^6}{e^6}$, $\frac{L s^{10}}{e^{10}}$, &c, and $\frac{C z z}{g g}$, $\frac{G z^6}{g^6}$, $\frac{L z^{10}}{g^{10}}$, &c, of the two serieses; and since it has been shewn that, in the third, and fifth, and seventh, and other following *odd* vertical columns of terms, in the two compound serieses which are equal to the squares of the said simple serieses, the signs + and -, that are to be prefixed to the corresponding terms of the said *odd* vertical columns, are the same in both serieses; and that in the second, and fourth, and sixth, and other following *even* vertical columns of terms in the said compound serieses, the signs + and -, that are to be prefixed to the corresponding terms of the said *even* vertical columns, are uniformly contrary to each other in the two serieses; it follows that the co-efficients of the several terms, or members of the vertical columns, of one series, must arise from the same combinations of the original co-efficients C, E, G, I, L, N, &c, by multiplication and addition, by which the co-efficients of the corresponding terms, or members of the several vertical columns, of the other series are produced: and consequently the said co-efficients must be the same in both serieses, though the signs + and - that are to be prefixed to them will, in all the terms of the several even vertical columns, be different. Q. E. D.

71. We

71. We now proceed to multiply the compound series obtained above, in art. 63, for the square of the simple series $1 + \frac{Cz}{g} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, into the said simple series itself, in order to obtain the cube of the said simple series. This multiplication is as follows.

The multiplication of the compound series (obtained above in art. 63) which is equal to the square of the simple series $1 + \frac{Cz}{g} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, into the said simple series, in order to obtain the cube of the said simple series.

$$\begin{array}{r}
 1 + \frac{2Cz}{g} - \frac{2Ez^4}{g^4} + \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} + \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} + \&c \\
 + \frac{C^2z^2}{g^2} - \frac{2CEz^6}{g^6} + \frac{2CGz^8}{g^8} - \frac{2CIz^{10}}{g^{10}} + \frac{2CLz^{12}}{g^{12}} - \&c \\
 + \frac{E^2z^8}{g^8} - \frac{2EGz^{10}}{g^{10}} + \frac{2EIz^{12}}{g^{12}} - \&c \\
 + \frac{G^2z^{12}}{g^{12}} - \&c \\
 \hline
 1 + \frac{Cz}{g} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \\
 \hline
 1 + \frac{2Cz}{g} - \frac{2Ez^4}{g^4} + \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} + \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} + \&c \\
 + \frac{C^2z^2}{g^2} - \frac{2CEz^6}{g^6} + \frac{2CGz^8}{g^8} - \frac{2CIz^{10}}{g^{10}} + \frac{2CLz^{12}}{g^{12}} - \&c \\
 + \frac{E^2z^8}{g^8} - \frac{2EGz^{10}}{g^{10}} + \frac{2EIz^{12}}{g^{12}} - \&c \\
 + \frac{G^2z^{12}}{g^{12}} - \&c \\
 + \frac{Cz}{g} + \frac{2C^2z^2}{g^2} - \frac{2CEz^6}{g^6} + \frac{2CGz^8}{g^8} - \frac{2CIz^{10}}{g^{10}} + \frac{2CLz^{12}}{g^{12}} - \&c \\
 + \frac{C^3z^3}{g^3} - \frac{2C^2Ez^7}{g^7} + \frac{2C^2Gz^9}{g^9} - \frac{2C^2Iz^{11}}{g^{11}} + \frac{2C^2Lz^{13}}{g^{13}} - \&c \\
 + \frac{CE^2z^{10}}{g^{10}} - \frac{2CEGz^{12}}{g^{12}} + \&c \\
 - \frac{Ez^4}{g^4} - \frac{2CEz^6}{g^6} + \frac{2E^2z^8}{g^8} - \frac{2EGz^{10}}{g^{10}} + \frac{2EIz^{12}}{g^{12}} - \&c \\
 - \frac{C^2Ez^8}{g^8} + \frac{2CE^2z^{10}}{g^{10}} - \frac{2CEGz^{12}}{g^{12}} + \&c \\
 - \frac{E^3z^{12}}{g^{12}} + \&c \\
 + \frac{Gz^6}{g^6} + \frac{2CGz^8}{g^8} - \frac{2EGz^{10}}{g^{10}} + \frac{2G^2z^{12}}{g^{12}} - \&c \\
 + \frac{C^2Gz^{10}}{g^{10}} - \frac{2CEGz^{12}}{g^{12}} + \&c \\
 - \frac{Iz^8}{g^8}
 \end{array}$$

$$\begin{aligned}
& - \frac{1z^3}{g^3} - \frac{2CIz^{10}}{g^{10}} + \frac{2EIz^{12}}{g^{12}} - \&c \\
& - \frac{C^2Iz^{12}}{g^{12}} + \&c \\
& + \frac{Lz^{10}}{g^{10}} + \frac{2CLz^{12}}{g^{12}} - \&c \\
& - \frac{Nz^{12}}{g^{12}} - \&c \\
\hline
& 1 + \frac{3Czz}{gg} - \frac{3Ez^4}{g^4} + \frac{3Gz^6}{g^6} - \frac{3Iz^8}{g^8} + \frac{3Lz^{10}}{g^{10}} - \frac{3Nz^{12}}{g^{12}} + \&c \\
& + \frac{3C^2z^4}{g^4} - \frac{6CEz^6}{g^6} + \frac{6CGz^8}{g^8} - \frac{6CIz^{10}}{g^{10}} + \frac{6CLz^{12}}{g^{12}} - \&c \\
& + \frac{C^3z^6}{g^6} + \frac{3E^2z^8}{g^8} - \frac{6EGz^{10}}{g^{10}} + \frac{6EIz^{12}}{g^{12}} - \&c \\
& - \frac{3C^2Ez^8}{g^8} + \frac{3C^2Gz^{10}}{g^{10}} + \frac{3G^2z^{12}}{g^{12}} - \&c \\
& + \frac{3CE^2z^{10}}{g^{10}} - \frac{3C^2Iz^{12}}{g^{12}} + \&c \\
& - \frac{6CEGz^{12}}{g^{12}} + \&c \\
& - \frac{E^3z^{12}}{g^{12}} - \&c.
\end{aligned}$$

This last compound series (which, for the sake of brevity, we will denote by the small Greek letter γ) is the cube of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum*. Therefore $8g \times$ the cube of the said series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum* will be $= 8g \times$ the compound series γ ; and consequently the cube of the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ will be $= 8g \times$ the compound series γ .

72. If the foregoing compound series γ be actually multiplied into $8g$, the product will be the following compound series, to wit,

$$\begin{aligned}
8g + \frac{24Czz}{g} - \frac{24Ez^4}{g^3} + \frac{24Gz^6}{g^5} - \frac{24Iz^8}{g^7} + \frac{24Lz^{10}}{g^9} - \frac{24Nz^{12}}{g^{11}} + \&c \\
+ \frac{24C^2z^4}{g^3} - \frac{48CEz^6}{g^5} + \frac{48CGz^8}{g^7} - \frac{48CIz^{10}}{g^9} + \frac{48CLz^{12}}{g^{11}} - \&c \\
+ \frac{8C^3z^6}{g^5} + \frac{24E^2z^8}{g^7} - \frac{48EGz^{10}}{g^9} + \frac{48EIz^{12}}{g^{11}} - \&c \\
- \frac{24C^2Ez^8}{g^7} + \frac{24C^2Gz^{10}}{g^9} + \frac{24G^2z^{12}}{g^{11}} - \&c \\
+ \frac{24CE^2z^{10}}{g^9} - \frac{24C^2Iz^{12}}{g^{11}} + \&c \\
- \frac{48CEGz^{12}}{g^{11}} + \&c \\
- \frac{8E^3z^{12}}{g^{11}} - \&c;
\end{aligned}$$

which,

which, for the sake of brevity, we shall denote by the small Greek letter δ . And then the cube of the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ will be equal to the said compound series δ .

A comparison between the compound series δ , obtained in the foregoing art. 72, and the compound series Δ obtained above in art. 41.

73. If we compare the last-mentioned compound series δ with the compound series Δ obtained above in art. 41, we shall find that the co-efficients of the corresponding terms are the same in both serieses, and likewise that the signs prefixed to the corresponding terms, or members, of the third, and fifth, and seventh, vertical columns of terms in both serieses are the same, but that the signs prefixed to the second terms of the said two serieses, to wit, the terms $\frac{24Czz}{g}$ and $\frac{24Czz}{g}$, and to the corresponding terms, or members, of their fourth and sixth vertical columns of terms, are contrary to each other. And the same agreement of the signs of the terms of these two serieses will take place in all the following *odd* vertical columns of terms after the seventh; and the same contrariety between the signs of the terms of the said serieses will take place in all the following *even* vertical columns of terms in them after the sixth terms, to whatever number of terms the said serieses may be continued; as might be shewn by reasonings similar to those contained in art. 65, 66, 67, 68, and 69.

74. Since the co-efficients of the several terms of the compound series δ are equal to the co-efficients of the correspondent terms of the compound series Δ ; and the signs prefixed to the corresponding terms of the said two serieses are exactly the same in the third, and fifth, and seventh, and other following *odd* vertical columns of terms in the said serieses, but are contrary to each other in the *second* terms of the said serieses, and in the corresponding terms of the fourth, and sixth, and other following *even* vertical columns of the said serieses; it follows that, if we reduce the several columns of terms in the compound series Δ into single terms by making the necessary multiplications, additions, and subtractions (as is done above in art. 46, 47, 48, 49, 50, and 51), and denote the co-efficients of the single terms thereby obtained by the capital letters P, Q, R, S, T, V, &c, and, if we likewise reduce the several columns of terms in the compound series δ into single terms by making the necessary multiplications, additions, and subtractions, so as to convert the said compound series δ into a simple series; the co-efficients of the terms of this latter simple series, which will be equal to the compound series δ , will be equal to the co-efficients of the corresponding terms of the former simple series which is equal to the compound series

series Δ ; and the signs $+$ and $-$ that are to be prefixed to the third, and fifth, and seventh, and other following odd terms of the latter simple series, will be the same as those which are to be prefixed to the third, and fifth, and seventh, and other following odd terms of the former simple series, which is equal to the compound series Δ ; but the signs to be prefixed to the second, and fourth, and sixth, and other following even terms of this second simple series will be contrary to those which are to be prefixed to the second, and fourth, and sixth, and other following even terms of the former simple series, which is equal to the compound series Δ .

75. Now it has been shewn in art. 60 that y^3 is equal to the simple series $8e - \frac{Pz^3}{e} - \frac{Qz^4}{e^3} - \frac{Rz^6}{e^5} - \frac{Sz^8}{e^7} - \frac{Tz^{10}}{e^9} - \frac{Vz^{12}}{e^{11}} - \&c \text{ ad infinitum}$. Therefore the compound series Δ (which is equal to y^3) must likewise be equal to the simple series $8e - \frac{Pz^3}{e} - \frac{Qz^4}{e^3} - \frac{Rz^6}{e^5} - \frac{Sz^8}{e^7} - \frac{Tz^{10}}{e^9} - \frac{Vz^{12}}{e^{11}} - \&c \text{ ad infinitum}$. It follows therefore, from the foregoing article 74, that the compound series δ will be equal to the simple series $8g + \frac{Pz^2}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$. Therefore the cube of the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^2}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ (which is equal to the compound series δ) will also be equal to the simple series $8g + \frac{Pz^2}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$.

The value of the co-efficient q of the simple power of x in the cubick equation $x^3 - qx = t$ expressed by an infinite series involving the powers of z and g .

76. In the next place we will find a transcendental expression, involving the powers of z and g , for q , the co-efficient of x in the proposed cubick equation $x^3 - qx = t$; after which we shall multiply the said value of q into the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^2}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, which we have asserted to be equal to the value of x in the equation $x^3 - qx = t$. And, if it can be shewn that the said product of the multiplication of the expression which is equal to q into the said expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^2}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is less than the cube of the said last-mentioned expression, or than the simple series $8g + \frac{Pz^2}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$ (which has been shewn in the last article 75 to be equal to the said cube), and that the difference by which it falls

falls short of the said cube, or of the said series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$, is equal to t (which is the absolute term of the equation $x^3 - qx = t$) or to $2 \times \frac{t}{2}$, or to $2g$, or that the said product is equal to the simple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$, which is less than the series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$ by the difference $2g$, or t , we shall be able justly to conclude that the said expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is equal to the root x of the proposed equation $x^3 - qx = t$. This therefore is what we must now endeavour to prove: and, for this purpose, we must, in the first place, find a transcendental expression for the co-efficient q , which shall involve in it only the powers of g and z . Now this may be done in the manner following.

Since zz is $= \frac{q^3}{27} - \frac{tt}{4}$, and gg is $= \frac{tt}{4}$, we shall have $zz = \frac{q^3}{27} - gg$, and consequently $gg + zz = \frac{q^3}{27}$, and $q^3 = 27 \times gg + zz = 27 \times gg \times \sqrt{1 + \frac{zz}{gg}}$,

and $q = 3 \times \sqrt[3]{gg} \times \sqrt[3]{1 + \frac{zz}{gg}} = 3 \times \sqrt[3]{g} \times \sqrt[3]{1 + \frac{zz}{gg}}$. But, by the

binomial theorem in the case of roots, $\sqrt[3]{1 + \frac{zz}{gg}}$ is $=$ the infinite series $1 + \frac{1}{3} \times A \times \frac{zz}{gg} - \frac{2}{6} B \times \frac{z^4}{g^4} + \frac{5}{9} C \times \frac{z^6}{g^6} - \frac{8}{12} D \times \frac{z^8}{g^8} + \frac{11}{15} E \times \frac{z^{10}}{g^{10}} - \frac{14}{18} F \times \frac{z^{12}}{g^{12}} + \&c$ *ad infinitum*, or $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ *ad infinitum*.

Therefore q will be $= 3 \times \sqrt[3]{g} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ *ad infinitum*. Q. E. I.

77. Therefore the product of the multiplication of q into the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ will be $= 3 \times \sqrt[3]{g} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \times 2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c = 3 \times \sqrt[3]{g} \times 2 \times g^{\frac{1}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c = 6g \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$

$\frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ $\times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$. We must therefore now proceed to multiply the series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ into the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$; which may be done in the manner following.

The multiplication of the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ ad infinitum into the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ ad infinitum.

$$\begin{array}{r}
 1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \\
 1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \\
 \hline
 1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \\
 + \frac{Czz}{gg} + \frac{BCz^4}{g^4} - \frac{C^2z^6}{g^6} + \frac{CDz^8}{g^8} - \frac{CEz^{10}}{g^{10}} + \frac{CFz^{12}}{g^{12}} - \&c \\
 - \frac{Ez^4}{g^4} - \frac{BEz^6}{g^6} + \frac{CEz^8}{g^8} - \frac{DEz^{10}}{g^{10}} + \frac{E^2z^{12}}{g^{12}} - \&c \\
 + \frac{Gz^6}{g^6} + \frac{BGz^8}{g^8} - \frac{CGz^{10}}{g^{10}} + \frac{DGz^{12}}{g^{12}} - \&c \\
 - \frac{Iz^8}{g^8} - \frac{BIz^{10}}{g^{10}} + \frac{CIz^{12}}{g^{12}} - \&c \\
 + \frac{Lz^{10}}{g^{10}} + \frac{BLz^{12}}{g^{12}} - \&c \\
 - \frac{Nz^{12}}{g^{12}} - \&c.
 \end{array}$$

Therefore the product of the multiplication of the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ into the coefficient q will be equal to $6g \times$ the compound series just now obtained, to wit, the series

$1 +$

$$\begin{aligned}
& 1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \\
& + \frac{Czz}{gg} + \frac{BCz^4}{g^4} - \frac{C^2z^6}{g^6} + \frac{CDz^8}{g^8} - \frac{CEz^{10}}{g^{10}} + \frac{CFz^{12}}{g^{12}} - \&c \\
& - \frac{Ez^4}{g^4} - \frac{BEz^6}{g^6} + \frac{CEz^8}{g^8} - \frac{DEz^{10}}{g^{10}} + \frac{E^2z^{12}}{g^{12}} - \&c \\
& + \frac{Gz^6}{g^6} + \frac{BGz^8}{g^8} - \frac{CGz^{10}}{g^{10}} + \frac{DGz^{12}}{g^{12}} - \&c \\
& - \frac{Iz^8}{g^8} - \frac{BIz^{10}}{g^{10}} + \frac{CIz^{12}}{g^{12}} - \&c \\
& + \frac{Lz^{10}}{g^{10}} + \frac{BLz^{12}}{g^{12}} - \&c \\
& - \frac{Nz^{12}}{g^{12}} - \&c.
\end{aligned}$$

or, if, for the sake of brevity, we denote this compound series by the small Greek letter λ , the said product will be equal to $6g \times$ the said compound series λ .

Q. E. I.

A comparison between the foregoing compound series λ and the compound series Λ obtained above in art. 42.

78. If we compare this compound series λ with the compound series Λ obtained above in art. 42, we shall find that there are the following resemblances and differences between them.

In the first place, the first terms of both these compound serieses are equal to the same quantity, to wit, 1.

Secondly, the second, and third, and fourth, and fifth, and all the following vertical columns of terms of the latter compound series λ , obtained in the foregoing art. 77, involve in them the several fractions $\frac{zz}{gg}$, $\frac{z^4}{g^4}$, $\frac{z^6}{g^6}$, $\frac{z^8}{g^8}$, $\frac{z^{10}}{g^{10}}$, $\frac{z^{12}}{g^{12}}$, &c, or the several successive powers of the fraction $\frac{zz}{gg}$, just as the second, and third, and fourth, and fifth, and other following vertical columns of terms of the former compound series Λ , obtained in art. 42, involve in them the several fractions $\frac{ss}{ee}$, $\frac{s^4}{e^4}$, $\frac{s^6}{e^6}$, $\frac{s^8}{e^8}$, $\frac{s^{10}}{e^{10}}$, $\frac{s^{12}}{e^{12}}$, or the several successive powers of the fraction $\frac{ss}{ee}$. And it is evident that this observation will be true of all the following

terms of these two compound serieses, to whatever number of terms the said serieses may be continued, as well as of the terms set down in art. 42 and 77, because the two simple serieses $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ (by the multiplication of which the compound series λ is produced) contain in their terms the very same powers of the fraction $\frac{zz}{gg}$ as the two former simple serieses $1 - \frac{Bss}{ee}$

$-\frac{Cs^4}{e^4}$

$$-\frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$$
 and $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$
 (by the multiplication of which the former compound series Λ was produced) contain in their terms of the fraction $\frac{s s}{e e}$.

Thirdly, the co-efficients of the several terms in the compound series λ will be equal to the co-efficients of the corresponding terms, or terms placed in the same situations in the several horizontal lines, of the compound series Λ , though they will not every where have the same signs + and — prefixed to them. For these co-efficients are the products of the co-efficients of the terms of the simple serieses, by the multiplication of which into each other the said compound serieses Λ and λ are produced : and therefore, as the co-efficients of the powers of the fraction $\frac{z z}{g g}$ in the terms of the two simple serieses $1 + \frac{B z z}{g g} - \frac{C z^4}{g^4} + \frac{D z^6}{g^6} - \frac{E z^8}{g^8} + \frac{F z^{10}}{g^{10}} - \frac{G z^{12}}{g^{12}} + \&c$ and $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$ (by the multiplication of which the compound series λ is produced) are the very same with the co-efficients of the powers of the fraction $\frac{s s}{e e}$ in the terms of the two simple serieses $1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$ and $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ (by the multiplication of which the compound series Λ is produced) it follows that the several co-efficients of the terms of the compound series λ must be the same combinations of the several original co-efficients B, C, D, E, F, G, I, L, N, &c, as the co-efficients of the corresponding terms of the compound series Λ are of the same original co-efficients, and consequently that they must be equal to the said co-efficients of the corresponding terms of the compound series Λ .

Q. E. D.

Fourthly, the signs + and — that are prefixed to the terms contained in the third, and fifth, and seventh, vertical columns of the compound series λ , are the same as those which are prefixed to the corresponding terms contained in the third, and fifth, and seventh vertical columns of the compound series Λ , as will be evident upon the inspection of the said two compound serieses. And the same similitude will take place between the signs prefixed to the terms contained in the following odd vertical columns, after the seventh, of the compound series λ , and those prefixed to the corresponding terms of the following odd vertical columns, after the seventh, of the compound series Λ ; as we shall presently endeavour to make appear.

And, fifthly, the signs + and — that are prefixed to the terms contained in the second, and fourth, and sixth vertical columns of the compound series λ are every where contrary to those which are prefixed to the corresponding terms of the second, and fourth, and sixth vertical columns of the compound series Λ ; as will be evident upon the inspection of the said two compound serieses. And the same contrariety will take place between the signs to be prefixed to the terms contained in the following even vertical columns, after the sixth, of the compound

compound series Λ and the signs to be prefixed to the corresponding terms of the following even vertical columns, after the sixth, of the compound series Λ ; as we shall now endeavour to make appear.

79. In the compound series Λ , or the product of the multiplication of the series $1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$ (of which all the terms, after the first term 1, are marked with the sign $-$) into the series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ (of which all the terms, after the first term 1, are likewise marked with the sign $-$) set down above in art. 42, it is evident that all the terms, after the first term 1, of the first, or highest, horizontal row of terms must be marked with the sign $-$, to whatever number of terms the said horizontal row of terms may be continued; and it is evident likewise, that in all the following horizontal rows of terms in the said product the first terms of the said horizontal rows (being the products of the multiplication of 1 by $-\frac{C s s}{e e}$, $-\frac{E s^4}{e^4}$, $-\frac{G s^6}{e^6}$, $-\frac{I s^8}{e^8}$, $-\frac{L s^{10}}{e^{10}}$, $-\frac{N s^{12}}{e^{12}}$, &c) must be marked with the sign $-$, but that all the following terms of the said horizontal rows, to whatever number of terms the said rows may be continued, will be marked with the sign $+$, because they are the products of the multiplication of the several terms $-\frac{B s s}{e e}$, $-\frac{C s^4}{e^4}$, $-\frac{D s^6}{e^6}$, $-\frac{E s^8}{e^8}$, $-\frac{F s^{10}}{e^{10}}$, $-\frac{G s^{12}}{e^{12}}$, &c, as multiplicands, into the several terms $-\frac{C s s}{e e}$, $-\frac{E s^4}{e^4}$, $-\frac{G s^6}{e^6}$, $-\frac{I s^8}{e^8}$, $-\frac{L s^{10}}{e^{10}}$, $-\frac{N s^{12}}{e^{12}}$, &c, as multipliers, and, wherever the sign $-$ is prefixed to both the factors of any multiplication, the sign $+$ is to be prefixed to the product. But the second, and third, and other following terms of the first, or highest, horizontal row of terms in the said product, are the highest terms of the second, and third, and other following vertical columns of terms in it; and the first terms of the second, and third, and other following horizontal rows of terms of the said product are the lowest terms of the second, and third, and other following vertical columns of terms in the same. Therefore, the highest and the lowest terms in the second, and the third, and every following vertical column of the said product will be marked with the sign $-$, and all the intermediate terms in the said vertical columns between the highest and the lowest terms will be marked with the sign $+$.

These will be the signs to be prefixed to the several terms of the compound series Λ , or the product, set down above in art. 42, of the multiplication of the series $1 - \frac{B s s}{e e} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$ *ad infinitum* into the series $1 - \frac{C s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum*. We must now enquire what signs are to be prefixed to the several corresponding terms of the compound series λ , or the product, set down in art. 77, of the multiplication of the series $1 + \frac{B z z}{g g} - \frac{C z^4}{g^4} + \frac{D z^6}{g^6} - \frac{E z^8}{g^8} + \frac{F z^{10}}{g^{10}} - \frac{G z^{12}}{g^{12}} + \&c$ *ad infinitum* into

into the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum*.

80. Now the first, or highest, horizontal row of terms in this product, to wit, $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$, consists of the very same terms as the multiplicand, being the product of the multiplication of the said multiplicand into 1, which will make no change either in the terms of the multiplicand themselves, or in the signs to be prefixed to them. Therefore in this first, or highest, horizontal row of terms, the second, and fourth, and sixth, and eighth, and tenth, terms, and all the following even terms (to whatever number of terms the said horizontal row of terms may be continued) will be marked with the sign +, and the third, and fifth, and seventh, and ninth, and eleventh, terms, and all the following odd terms (whatever be their number) will be marked with the sign —. But the second, and fourth, and sixth, and other following even terms of the first, or highest, horizontal row of terms in the said product, are the first, or highest, terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the said product; and the third, and fifth, and seventh, and other following odd terms of the said first, or highest, horizontal row of terms in the said product are the first, or highest, terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the said product. Therefore the first, or highest, terms of the second, and fourth, and sixth, and other following even vertical columns of the said product will be marked with the sign +; and the first, or highest, terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the said product will be marked with the sign —.

81. We will next enquire what signs must be prefixed to the last, or lowest, terms of the several vertical columns of the said product.

Now, because the first term of the series which is the multiplicand in the multiplication set down above in art. 77, is 1, and the second, and fourth, and sixth, and other following even terms of the series which is the multiplier in that multiplication, are marked with the sign +, and the third, and fifth, and seventh, and other following odd terms of the said multiplier are marked with the sign —, it follows that in the second, and fourth, and sixth, and other following even horizontal rows of the said product, the signs + and — that are to be prefixed to the several terms in the said rows will be the same as those of the corresponding terms of the multiplicand, and consequently the two first terms of each of the said even horizontal rows will be marked with the sign +, and the third, and fourth, and fifth, and sixth, and other following terms of the said even horizontal rows will be marked with the sign — and the sign + alternately; and it follows also, that in the third, and fifth, and seventh, and other following odd horizontal rows of terms, the signs of the terms will be contrary to those of the corresponding terms of the multiplicand, and consequently the two first terms of each of the said odd horizontal rows of terms will be marked with the

the sign —, and the third, and fourth, and fifth, and sixth, and other following terms of the said odd horizontal rows of terms will be marked with the sign + and the sign — alternately. But the first terms of the second, and third, and fourth, and other following horizontal rows of terms are the lowest terms of the second, and third, and fourth, and other following vertical columns of terms. Therefore the lowest terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the said product will be marked with the sign +; and the lowest terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the said product will be marked with the sign —.

82. It appears therefore that in the third, and fifth, and seventh, and other following odd vertical columns of the said compound series λ , or product, set down in art. 77, both the highest and the lowest terms will have the sign — prefixed to them, as is the case in the compound series Λ , or the product set down above in art. 42; and that in the second, and fourth, and sixth, and other following even vertical columns of the said compound series λ , or product set down in art. 77, both the highest and lowest terms will be marked with the sign +, which is contrary to the sign which is prefixed to the highest and lowest terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Λ , or product set down above in art. 42. It remains that we enquire into the signs that are to be prefixed to the several intermediate terms between the highest and the lowest terms of the several vertical columns of terms in the said compound series λ , or product set down in art. 77.

83. Now, wherever the highest term of one of the vertical columns of the product set down in art. 77 is marked with the sign — (which has been shewn to be the case in the third, and fifth, and seventh, and all the following odd vertical columns), all the intermediate terms of the said column between the highest term and the lowest will be marked with the sign +, because they will be the product of two factors which are both marked with the same sign, to wit, first with the sign +, then with the sign —, then with the sign +, and then with the sign — again, and so on alternately. Thus, for example, in the seventh vertical column of the said product, the highest term is $-\frac{Gz^{12}}{g^{12}}$, which is the product of the multiplication of $-\frac{Gz^{12}}{g^{12}}$, the seventh term of the multiplicand, by 1, the first term of the multiplier; and the second term of the said vertical column is $+\frac{CFz^{12}}{g^{12}}$, which is marked with the sign +, because it is the product of the multiplication of $\frac{Fz^{10}}{g^{10}}$, the sixth term of the multiplicand, by $\frac{Cz^2}{g^2}$, the second term of the multiplier, which terms are both marked with the sign +; and the third term of the said vertical column is $+\frac{E^2z^{12}}{g^{12}}$, which is

marked with the sign $+$, because it is the product of the multiplication of $\frac{Ez^8}{g^8}$, the fifth term of the multiplicand, by $\frac{Ez^4}{g^4}$, the third term of the multiplier, which terms are both marked with the sign $-$; and the fourth term of the said vertical column is $+\frac{DGz^{12}}{g^{12}}$, which is marked with the sign $+$, because it is the product of the multiplication of $\frac{Dz^6}{g^6}$, the fourth term of the multiplicand, by $\frac{Gz^6}{g^6}$, the fourth term of the multiplier, which terms are both marked with the sign $+$; and the fifth term of the said vertical column is $+\frac{CIz^{12}}{g^{12}}$, which is in like manner marked with the sign $+$, because it is the product of the multiplication of $\frac{Cz^4}{g^4}$, the third term of the multiplicand, by $\frac{Iz^8}{g^8}$, the fifth term of the multiplier, which terms are both marked with the sign $-$; and the sixth term of the said vertical column, or the lowest term but one, of the said column, or the last of the intermediate terms of the said column between its highest and its lowest terms, is $+\frac{BLz^{12}}{g^{12}}$, which is also marked with the sign $+$, because it is the product of the multiplication of $\frac{Bzz}{gg}$, the second term of the multiplicand, by $\frac{Lz^{10}}{g^{10}}$, the sixth term of the multiplier, which terms are both marked with the same sign $+$. And thus it appears, that all the intermediate terms between the highest and lowest terms of the said seventh vertical column must be marked with the sign $+$, because the two factors, by the multiplication of which they are produced, are always marked with the same sign, whether $+$ or $-$. And it is easy to see, from the alternate succession of the signs $+$ and $-$ to each other, in both the multiplicand and multiplier, that the same thing must take place with respect to the intermediate terms of any other vertical column of terms of which the highest term is marked with the sign $-$, that is, of any other odd vertical column. We may therefore conclude that in all the odd vertical columns of the compound series λ , or the product set down in art. 77 (to whatever number of terms the said product may be continued), the highest and lowest terms of the said vertical columns will be marked with the sign $-$, and all the intermediate terms will be marked with the sign $+$, and consequently (by what is shewn in art. 79) that the signs of the several terms of all the said odd vertical columns of this product, or of the compound series λ , will be the same with those of the corresponding terms of the like odd vertical columns of terms in the product set down above in art. 42, or in the compound series Λ ; or that the fourth observation made above in art. 78 is universally true of the terms contained in all the odd vertical columns of the said two products, or compound series Λ and λ , set down in art. 42 and 77 after the seventh vertical columns (to whatever number of columns the said serieses may be continued), as well as of the terms contained in their third, and fifth, and seventh vertical columns.

Q. E. D.

84. In

84. In the several even vertical columns of the compound series λ , set down in art. 77, the signs $+$ and $-$, that are to be prefixed to the terms of the said columns, will be every where contrary to those which are to be prefixed to the corresponding terms of the like even vertical columns of the compound series Λ , set down above in art. 42. For it has been already shewn in art. 80 and 81, that the highest and lowest terms of the second, and fourth, and sixth, and all the following even vertical columns of terms in the said compound series λ (to whatever number they may be continued) will be marked with the sign $+$, which is contrary to that with which the highest and lowest terms of the second, and fourth, and sixth, and other following even vertical columns of terms, in the compound series Λ , are marked. And all the intermediate terms of the second, and fourth, and sixth, and other following even vertical columns of the compound series λ , between the highest and the lowest terms of the said columns, must be marked with the sign $-$, because they will be the products of two factors which are marked with different signs $+$ and $-$. Thus, for example, in the sixth vertical column of the said product, the highest term is $+\frac{Fz^{10}}{g^{10}}$, which is the product of the multiplication of $+\frac{Fz^{10}}{g^{10}}$, the sixth term of the multiplicand, by 1, the first term of the multiplier; and the second term of the said vertical column is $-\frac{CEz^{10}}{g^{10}}$, which is marked with the sign $-$, because it is the product of the multiplication of $\frac{Ez^8}{g^8}$, the fifth term of the multiplicand, which is marked with the sign $-$, by $\frac{Czz}{gg}$, the second term of the multiplier, which is marked with the sign $+$; and the third term of the said vertical column is $-\frac{DEz^{10}}{g^{10}}$, which is also marked with the sign $-$, because it is the product of the multiplication of $\frac{Dz^6}{g^6}$, the fourth term of the multiplicand, which is marked with the sign $+$, by $\frac{Ez^4}{g^4}$, the third term of the multiplier, which is marked with the sign $-$; and the fourth term of the said vertical column is $-\frac{CGz^{10}}{g^{10}}$, which is also marked with the sign $-$, because it is the product of the multiplication of $\frac{Cz^4}{g^4}$, the third term of the multiplicand, which is marked with the sign $-$, by $\frac{Gz^6}{g^6}$, the fourth term of the multiplier, which is marked with the sign $+$; and the fifth term of the said vertical column, or the lowest term but one of the said column, or the last of the intermediate terms of the said column between the highest term and the lowest, is $-\frac{BIz^{12}}{g^{12}}$, which is also marked with the sign $-$, because it is the product of the multiplication of $\frac{Bzz}{gg}$, the second term of the multiplicand, which is marked with the sign $+$, by $\frac{Iz^8}{g^8}$, the fifth term of the multiplier, which is marked with the sign $-$. And thus it appears that all the intermediate terms between the highest and the lowest terms

of the sixth vertical column must be marked with the sign —, because the two factors, by the multiplication of which they are produced, are always marked with contrary signs. And it is easy to see, from the alternate succession of the signs + and — to each other in both the multiplicand and the multiplier, that the same thing must take place with respect to the intermediate terms of every other vertical column of which the highest term is marked with the sign +, that is, of every other even vertical column. We may therefore conclude, that in all the even vertical columns of the compound series λ , or the product set down in art. 77 (to whatever number of terms the said product may be continued) the highest and lowest terms will be marked with the sign +, and all the intermediate terms will be marked with the sign —, and consequently (by what is shewn in art. 79) that the signs of the several terms of all the said even vertical columns of terms in this product, or compound series λ , will be every where contrary to the signs of the corresponding terms of the like even vertical columns of terms in the compound series Λ , or the product set down above in art. 42; or that the fifth observation made above in art. 78 is universally true of the terms contained in all the even vertical columns of terms of the said two products, or compound serieses Λ and λ , set down above in art. 42 and 77 (to whatever number of columns of terms the said serieses may be continued), as well as of the terms contained in their second, and fourth, and sixth vertical columns.

Q. E. D.

85. If the foregoing compound series λ , set down in art. 77, be actually multiplied into $6g$, the product thence arising will be the following compound series, to wit,

$$\begin{aligned}
 6g &+ \frac{6Bzz}{g} - \frac{6Cz^4}{g^3} + \frac{6Dz^6}{g^5} - \frac{6Ez^8}{g^7} + \frac{6Fz^{10}}{g^9} - \frac{6Gz^{12}}{g^{11}} + \&c \\
 &+ \frac{6Czz}{g} + \frac{6BCz^4}{g^3} - \frac{6C^2z^6}{g^5} + \frac{6CDz^8}{g^7} - \frac{6CEz^{10}}{g^9} + \frac{6CFz^{12}}{g^{11}} - \&c \\
 &\quad - \frac{6Ez^4}{g^3} - \frac{6BEz^6}{g^5} + \frac{6CEz^8}{g^7} - \frac{6DEz^{10}}{g^9} + \frac{6E^2z^{12}}{g^{11}} - \&c \\
 &\quad + \frac{6Gz^6}{g^5} + \frac{6BGz^8}{g^7} - \frac{6CGz^{10}}{g^9} + \frac{6DGz^{12}}{g^{11}} - \&c \\
 &\quad - \frac{6Iz^8}{g^7} - \frac{6BIz^{10}}{g^9} + \frac{6CIz^{12}}{g^{11}} - \&c \\
 &\quad + \frac{6Lz^{10}}{g^9} + \frac{6BLz^{12}}{g^{11}} - \&c \\
 &\quad - \frac{6Nz^{12}}{g^{11}} - \&c;
 \end{aligned}$$

which, for the sake of brevity, we will denote by the small Greek letter π . Then will the product of the multiplication of the expression $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum* into the coefficient q be equal to the compound series π .

We must now compare this compound series π with the compound series Π obtained above in art. 43.

A Com-

A Comparison between the foregoing compound series π and the compound series Π obtained above in art. 43.

86. The compound series Π is equal to $6e \times$ the compound series Λ ; and the compound series π is equal to $6g \times$ the compound series λ . Therefore the signs $+$ and $-$ which are to be prefixed to the terms of the compound series Π will be the same with the signs which are prefixed to the corresponding terms of the compound series Λ ; and the signs $+$ and $-$, which are to be prefixed to the terms of the compound series π will be the same with the signs which are to be prefixed to the correspondent terms of the compound series λ ; because the multiplications of the terms of the two compound serieses Λ and λ into $6e$ and $6g$ cannot make any change in the signs of the terms which are multiplied. It follows therefore that there will be the same similitude and the same differences between the signs $+$ and $-$ that are to be prefixed to the several terms of the two compound serieses Π and π (which are equal to $6e \times \Lambda$ and $6g \times \lambda$) as there are between the signs which are to be prefixed to the terms of the two former compound serieses Λ and λ . But it has been shewn in art. 78, 79, 80, 81, 82, 83, 84, that the signs $+$ and $-$ that are to be prefixed to the several terms of the third, fifth, seventh, and other following odd vertical columns of terms in the compound series λ are the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Λ ; and that the signs $+$ and $-$ which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series λ are uniformly contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Λ . It therefore follows that the signs $+$ and $-$ which are to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series π will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Π ; and that the signs $+$ and $-$, which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of the compound series π will be uniformly contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of the compound series Π .

87. It has been shewn in the third observation of art. 78, that the co-efficients of the several terms of the compound series λ are equal to the co-efficients of the corresponding terms of the compound series Λ . Therefore, if the co-efficients of all the terms of both these serieses Λ and λ be multiplied by the same number, the new numbers, or co-efficients, thereby produced in the one series will be equal to the corresponding new numbers, or co-efficients, thereby produced in the other series. But the co-efficients of the terms of the compound series Π are produced by the multiplication of the co-efficients of the terms

terms of the compound series Λ by 6, because the series Π is $= 6e \times$ the series Λ ; and the co-efficients of the terms of the compound series π are produced by the multiplication of the co-efficients of the terms of the compound series λ by the same number 6, because the series π is $= 6g \times$ the series λ . Therefore the co-efficients of the terms of the compound series π will be equal to the co-efficients of the corresponding terms of the compound series Π .

88. It appears therefore from the two last articles, that the co-efficients of the several terms of the compound series π are equal to the co-efficients of the corresponding terms of the compound series Π ; and that in the third and fifth, and seventh, and other following odd vertical columns of terms in the compound series π , the signs $+$ and $-$ that are to be prefixed to the several terms of the said columns will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Π ; and, lastly, that in the second, and fourth, and sixth, and other following even vertical columns of terms in the said compound series π , the signs $+$ and $-$, that are to be prefixed to the several terms of the said columns, will be respectively contrary to those which are to be prefixed to the corresponding terms of the second and fourth, and sixth, and other following even vertical columns of terms in the compound series Π . Now from hence it follows that, if all the vertical columns of terms in the compound series Π be reduced to single terms, by the addition, or subtraction, of their several members according to the marks $+$ or $-$ which are prefixed to them, and the compound series Π be thereby converted into a simple series, or series consisting of single terms; and, if all the vertical columns of terms in the compound series π be in like manner reduced to single terms, by the addition and subtraction of their several members according to the signs $+$ and $-$ that are prefixed to them, and the said compound series π be thereby converted into a simple series, or series consisting of single terms;—I say, it follows that, if these two compound series Π and π be so converted into simple series, the co-efficients of the second, and third, and fourth, and all the following terms of the simple series which is equal to the compound series π , will be respectively equal to the co-efficients of the second, and third, and fourth, and other following corresponding terms of the simple series which is equal to the compound series Π ; and the signs $+$ and $-$ that are to be prefixed to the third, and fifth, and seventh, and other following odd terms of the simple series which is equal to the compound series π will be the same with those which are to be prefixed to the third, and fifth, and seventh, and other following odd terms of the simple series which is equal to the compound series Π ; but the signs $+$ and $-$ that are to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series which is equal to the compound series π will be respectively contrary to those which are to be prefixed to the corresponding terms of the simple series which is equal to the compound series Π .

89. It has been shewn above in art. 60, that the compound series Π (which is equal to y^3 in the equation $y^3 - qy = r$) is equal to the simple series $6e -$

$\frac{Pz^3}{e} - \frac{Qz^4}{e^3} - \frac{Rz^6}{e^5} - \frac{Sz^8}{e^7} - \frac{Tz^{10}}{e^9} - \frac{Vz^{12}}{e^{11}} - \&c \text{ ad infinitum.}$ It follows therefore from the foregoing art. 88, that the compound series π will be equal to the simple series $6g + \frac{Pz^3}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum.}$

90. But it has been shewn in art. 85, that the product of the multiplication of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^3}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ into q (the co-efficient of x in the equation $x^3 - qx = t$) is equal to the compound series π . Therefore the product arising from the said multiplication of the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^3}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ into the co-efficient q will be equal to the said simple series $6g + \frac{Pz^3}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum.}$

91. We have just now seen that, if the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^3}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ be multiplied into q (the co-efficient of x in the equation $x^3 - qx = t$) the product arising from the said multiplication will be equal to the simple series $6g + \frac{Pz^3}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum.}$ And it was shewn above in art. 75, that the cube of the same transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^3}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ is equal to the simple series $8g + \frac{Pz^3}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum.}$ But the simple series $8g + \frac{Pz^3}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$ is greater than the simple series $6g + \frac{Pz^3}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$, and the difference is $8g - 6g$, or $2g$, or $2 \times \frac{t}{2}$, or t . Therefore the cube of the said transcendental expression will be greater than the product of its multiplication into the co-efficient q , and the excess or difference will be equal to t , or the absolute term of the equation $x^3 - qx = t$. Therefore the said transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz^3}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ must be equal to the root x of the said equation, agreeably to the assertion made above in art. 36. Q. E. D.

*End of the Demonstration of the Proposition, or Theorem,
laid down in art. 36.*

A Re-

A Recapitulation of the Contents of the foregoing Demonstration of the Proposition laid down above in art. 36.

92. The foregoing demonstration of the proposition laid down above in art. 36 is in itself very subtle and intricate, and therefore has been set forth in the foregoing articles at very great length. But now, after it has, I hope, been rendered intelligible and satisfactory to the attentive reader, by the full exposition that has been given of it, he will probably be glad to see the matter of it compressed into a smaller compass, by which the connection of the several steps of which it is composed will be more clearly and easily perceived. This therefore I shall now proceed to do in the following articles.

93. It has been shewn in art. 25 that y , the root of the cubick equation $y^3 - qy = r$ (in which the absolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) is equal to the transcendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum*, in which all the terms, after the first term 1, are marked with the sign $-$. Let this infinite series be denoted by v ; and then we shall have $y = 2\sqrt[3]{e} \times$ the series v .

94. In this expression $2\sqrt[3]{e} \times$ the series v the letter e is put for $\frac{r}{2}$, or half the absolute term r of the equation $y^3 - qy = r$, and ss is $= \frac{rr}{4} - \frac{q^3}{27}$, or $ee - \frac{q^3}{27}$; and consequently $ss + \frac{q^3}{27}$ is $= ee$, and $\frac{q^3}{27}$ is $= ee - ss = ee \times \sqrt{1 - \frac{ss}{ee}}$, and q^3 is $= 27 \times ee \times \sqrt{1 - \frac{ss}{ee}}$, and q is $= 3 \times e^{\frac{2}{3}} \times \sqrt{1 - \frac{ss}{ee}}^{\frac{1}{3}} =$ (by the residual theorem in the case of roots) $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{B ss}{ee} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$ *ad infinitum*.

95. Since y is $= 2\sqrt[3]{e} \times$ the series v , it follows that y^3 will be $= 8e \times$ the series v^3 , which will be a compound series, or series consisting of several different horizontal lines, or rows, of terms placed one under another. And the first term of this series v^3 must evidently be 1, because the first term of the series v is 1. Call this compound series Γ . And we shall have y^3 , or $8e \times v^3$, $= 8e \times$ the compound series Γ .

96. Let all the terms of the compound series Γ be actually multiplied into $8e$; and we shall thereby obtain another compound series, of which the first term will be $8e$ (because the first term of the compound series Γ is 1), and which will be equal to y^3 . Let this second compound series be called Δ . And we shall then have $y^3 =$ the compound series Δ .

97. Now

97. Now let the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{CJ^5}{ee} - \frac{EJ^4}{e^4} - \frac{GJ^6}{e^6} - \frac{IJ^8}{e^8} - \frac{LJ^{10}}{e^{10}} - \frac{NJ^{12}}{e^{12}} - \&c$ (which is equal to y) be multiplied into the expression $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{BJ^5}{ee} - \frac{CJ^4}{e^4} - \frac{DJ^6}{e^6} - \frac{EJ^8}{e^8} - \frac{FJ^{10}}{e^{10}} - \frac{GJ^{12}}{e^{12}} - \&c$ *ad infinitum*, which is equal to the co-efficient q . And we shall have

$qy = 2\sqrt[3]{e} \times 3 \times e^{\frac{2}{3}} \times$ the product of the said two infinite serieses, which product will be a certain compound series, of which the first term will be 1. Let

this compound series be called Λ . Then will qy be $(= 2\sqrt[3]{e} \times 3 \times e^{\frac{2}{3}} \times$ the compound series $\Lambda = 2 \times e^{\frac{1}{3}} \times 3 \times e^{\frac{2}{3}} \times$ the compound series $\Lambda) = 6e \times$ the compound series Λ .

98. Let all the terms of the compound series Λ be actually multiplied into $6e$; and we shall thereby obtain another compound series, of which the first term will be $6e$ (because the first term of the compound series Λ is 1), and which will be equal to qy . Let this new compound series be called Π . And we shall then have $qy =$ the compound series Π .

99. Since $y^3 - qy$ is $= r$, and y^3 is $=$ the compound series Δ , and qy is $=$ the compound series Π , it follows that the excess of the compound series Δ above the compound series Π will also be equal to r , that is, $\Delta - \Pi$ will be $= r = 2 \times \frac{r}{2} = 2e$. But the excess of the first term of the compound series Δ (which is $8e$) above the first term of the compound series Π (which is $6e$) is also $= 2e$. Therefore the excess of the whole compound series Δ above the whole compound series Π is equal to the excess of the first term of the former compound series above the first term of the latter compound series; that is, $\Delta - \Pi$ is $= 8e - 6e$. Consequently (adding $6e$ to both sides) we shall have $\Delta + 6e - \Pi = 8e$. But the compound series Δ is less than its first term $8e$, because the second term of it is marked with the sign $-$. Therefore in the last equation $\Delta + 6e - \Pi = 8e$ it is possible to subtract Δ from both sides. Let it be so subtracted; and we shall then have $6e - \Pi = 8e - \Delta$.

100. Now let the compound series $8e - \Delta$ be reduced to a simple series, or series consisting of single terms, by making the several multiplications, additions, and subtractions that are necessary for that purpose; and let the simple series, to which it is thereby reduced, be $\frac{P J^5}{e} + \frac{Q J^4}{e^3} + \frac{R J^6}{e^5} + \frac{S J^8}{e^7} + \frac{T J^{10}}{e^9} + \frac{V J^{12}}{e^{11}} + \&c$ *ad infinitum*. Then will the said simple series $\frac{P J^5}{e} + \frac{Q J^4}{e^3} + \frac{R J^6}{e^5} + \frac{S J^8}{e^7} + \frac{T J^{10}}{e^9} + \frac{V J^{12}}{e^{11}} + \&c$ *ad infinitum* be equal to the compound series $6e - \Pi$. And, because

this equality between the simple series $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c \text{ ad infinitum}$, and the compound series $6e - \Pi$ takes place in all the different relative magnitudes of ss and ee , or of $\frac{rr}{4} - \frac{q^3}{27}$ and $\frac{rr}{4}$, and consequently when $\frac{rr}{4}$ approaches as near as we please to an equality with $\frac{q^3}{27}$, and $\frac{rr}{4} - \frac{q^3}{27}$ approaches as near as we please to $\frac{q^3}{27} - \frac{q^3}{27}$, or to 0, and $\frac{ss}{ee}$ approaches as near as we please to $\frac{0}{ee}$, or to 0, it follows that every term in the simple series $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c \text{ ad infinitum}$ will be equal to the corresponding vertical column of terms, or column of terms involving the same powers of ss and e in their numerators and denominators, in the compound series $6e - \Pi$. And therefore, if the compound terms, or vertical columns of terms of the said compound series $6e - \Pi$ be reduced to single terms by making the several multiplications, additions, and subtractions, that are necessary for that purpose; the simple series thence arising will be the foregoing simple series $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c \text{ ad infinitum}$, which arose from a similar reduction of the terms of the compound series $8e - \Delta$.

101. Since $8e - \Delta$ is equal to the simple series $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c$, we shall have $8e = \Delta +$ the said series, and $\Delta = 8e -$ the said series $=$ the series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c$. Therefore y^3 (which is equal to the compound series Δ) will also be equal to the series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c$.

102. And, since the compound series $6e - \Pi$ is equal to the said simple series $\frac{Pss}{e} + \frac{Qs^4}{e^3} + \frac{Rs^6}{e^5} + \frac{Ss^8}{e^7} + \frac{Ts^{10}}{e^9} + \frac{Vs^{12}}{e^{11}} + \&c \text{ ad infinitum}$, we shall have $6e = \Pi +$ the said simple series, and $\Pi = 6e -$ the said simple series, or $\Pi =$ the series $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c \text{ ad infinitum}$. Therefore the product qy (which is $= \Pi$) will also be equal to the series $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c \text{ ad infinitum}$.

103. It appears therefore that y^3 is equal to the series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c \text{ ad infinitum}$, and that qy is equal to the series $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c \text{ ad infinitum}$.

$-\frac{P s^5}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$, which is less than the former series $8e - \frac{P s^5}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$, by the quantity, or difference, $8e - 6e$, or $2e$, or $2 \times \frac{r}{2}$, or r , as it ought to be, agreeably to the original equation $y^3 - qy = r$.

104. We then endeavoured to prove that the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ (in which g is equal to $\frac{t}{2}$, or half the absolute term of the equation $x^3 - qx = t$, and zz is equal to $\frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$) was equal to the root x of the same equation. And the method we took to prove the said equality, was to shew that, if the said transcendental expression was substituted instead of x in the compound quantity $x^3 - qx$ (which forms the left-hand side of the said equation) it would make the said compound quantity equal to the absolute term t , or that, if the said expression were to be cubed by multiplying it twice into itself, and were likewise to be multiplied into the co-efficient q , the said cube would be greater than the said product, and the difference would be equal to the absolute term t . These substitutions, and the reasoning grounded on them, are as follows.

105. Let us, for the sake of brevity, denote the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$, by the letter w , and the product of the multiplication of $2\sqrt[3]{g}$ into the said series by the letter m . Then it will be necessary for us to prove that the transcendental quantity m will be equal to the root x of the equation $x^3 - qx = t$, or that, if we raise the transcendental quantity m to its cube, so as to obtain the quantity m^3 , and also multiply m into the co-efficient q , or its value expressed in powers of g and z , so as to obtain the value of the product $q \times m$, the said cube of m will be greater than the said product qm by the quantity, or difference, $2g$, or $2 \times \frac{t}{2}$, or t , the absolute term of the equation $x^3 - qx = t$; from which it will follow that the said transcendental quantity m is equal to x .

106. Now, since m is $= 2\sqrt[3]{g} \times$ the series w , it follows that m^3 will be $= 8g \times$ the series w^3 , which will be a compound series, or series consisting of several different horizontal lines, or rows, of terms placed one under another. Call this compound series γ . And we shall have $m^3 = 8g \times$ the compound series γ .

107. Since the terms of the series w , or $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$, have the same co-efficients $C, E, G, I, L, N, \&c$,

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as

as the corresponding terms of the series v , or $1 - \frac{C s^5}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^3}{e^3} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ *ad infinitum* (though differently connected together by the signs $+$ and $-$), it follows that the co-efficients of the several terms of the compound series w^3 , or γ , will be equal to the co-efficients of the corresponding terms of the compound series v^3 , or Γ , both sets of terms being similar combinations, or products, of the same original factors, or co-efficients, $C, E, G, I, L, N, \&c$. And from the continued repetition of the sign $-$ in the second and other following terms of the series v , or $1 - \frac{C s^5}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^3}{e^3} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, and the alternate succession of the sign $+$ and the sign $-$ in the second and other following terms of the series w , or $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^3}{g^3} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \&c$, it follows that, in the third, and fifth, and seventh, and other following odd vertical columns of terms of the compound series w^3 , or γ , the signs $+$ and $-$ that are to be prefixed to the terms of the said columns will be exactly the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series v^3 , or Γ ; and that in the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series w^3 , or γ , the signs $+$ and $-$, that are to be prefixed to the terms of the said columns, will be every where contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series v^3 , or Γ .

And, as to the first terms of these two compound serieses v^3 and w^3 , or Γ and γ , they will be equal to each other, because each of them will be equal to 1 .

108. Let all the terms of the compound series w^3 , or γ , be multiplied by $8g$. And we shall thereby obtain another compound series of which the first term will be $8g$ (because the first term of the compound series w^3 , or γ , is 1) and which will be equal to m^3 . Let this new compound series be called δ . And we shall then have $m^3 =$ the compound series δ .

109. Now there will evidently be the same analogies and the same differences between the compound series Δ and δ , or $8e \times \Gamma$ and $8g \times \gamma$, with respect to the co-efficients of their several corresponding terms and the signs $+$ and $-$, which are to be prefixed to them, as between the compound serieses Γ and γ themselves; because the said multiplications by $8e$ and $8g$ can make no difference at all in the signs of the terms multiplied, and will increase the co-efficients of all the said terms in the same proportion of 8 to 1 , which will not affect their antecedent equality. For as to the quantities e and g (which are involved in the multipliers $8e$ and $8g$) the multiplications of the terms by them will not affect the numeral co-efficients of the terms arising from the combinations

tions of the original numeral co-efficients C, E, G, I, L, N, &c, but will only affect the literal parts of the said terms by removing one power of e and g from their several denominators. It therefore follows from art. 107, and 108, in the first place, that the first term of the compound series δ , or $8g \times \gamma$, will be ($8g \times 1$, or) $8g$; and, secondly, that the co-efficients of the second term, and all the following terms of the compound series δ , or $8g \times \gamma$, will be respectively equal to the co-efficients of the second term, and all the following corresponding terms of the compound series Δ , or $8e \times \Gamma$; and, thirdly, that the signs $+$ and $-$, which are to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series δ , or $8g \times \gamma$, will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Δ , or $8e \times \Gamma$; and, fourthly, that the signs $+$ and $-$, which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series δ , or $8g \times \gamma$, will be every where contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Δ , or $8e \times \Gamma$.

110. It follows from the last article, that, if the compound series δ , or $8g \times \gamma$, be reduced into a simple series, or series of single terms, by performing all the multiplications, additions, and subtractions that are necessary for that purpose, the co-efficients of the second and other following terms of the simple series thereby obtained will be respectively equal to the co-efficients of the second and other following terms of the simple series which is equal to the compound series Δ , or $8e \times \Gamma$, that is (by art. 101) to the co-efficients P, Q, R, S, T, V, &c; and the signs to be prefixed to the third, and fifth, and seventh, and other following odd terms of the said simple series, which is equal to the compound series δ , or $8g \times \gamma$, will be the same with those which are prefixed to the third, and fifth, and seventh, and other following odd terms of the simple series which is equal to the compound series Δ , or $8e \times \Gamma$, or to the odd terms of the simple series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c$; and the signs to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series which is equal to the said compound series δ , or $8g \times \gamma$, will be respectively contrary to those which are to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series which is equal to the compound series Δ , or $8e \times \Gamma$, or to the even terms of the simple series $8e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} - \&c$. And consequently the simple series that will be equal to the compound series δ , or $8g \times \gamma$, will be $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$ *ad infinitum*. Therefore m^3 , or the cube of the transcendental expression m , or $2\sqrt[3]{g} \times$ the series w (which, by art. 108, is equal to the compound series δ) will be equal to the simple series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c$ *ad infinitum*.

111. In

111. In the expression $2\sqrt[3]{g} \times$ the series w , or $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum*, the letter g is put for $\frac{t}{2}$, or half the absolute term t of the equation $x^3 - qx = t$, and zz is $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$. Therefore $\frac{q^3}{27}$ is $= gg + zz = gg \times \left[1 + \frac{zz}{gg}\right]$, and q^3 is $= 27 \times gg \times \left[1 + \frac{zz}{gg}\right]$, and q is $= 3 \times g^{\frac{2}{3}} \times \left[1 + \frac{zz}{gg}\right]^{\frac{1}{3}} =$ (by the binomial theorem in the case of roots) $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ *ad infinitum*.

112. Now let the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ be multiplied into the expression $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$, which is equal to q ; and the product will be $(= 2\sqrt[3]{g} \times 3 \times g^{\frac{2}{3}} \times$ the product of the said two infinite series $= 2 \times g^{\frac{1}{3}} \times 3 \times g^{\frac{2}{3}} \times$ the product of the said two infinite series) $= 6g \times$ the product of the said two infinite series. Therefore $q \times$ the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, or $q \times 2\sqrt[3]{g} \times w$, or $q \times m$, will be $= 6g \times$ the product of the said two infinite series, which product will be a certain compound series, or series consisting of several different horizontal rows of terms, placed one under another. Let this compound series be denoted by the small Greek letter λ . And then $q \times 2\sqrt[3]{g} \times$ the simple series w , or $q \times m$, will be $= 6g \times$ the compound series λ .

113. Now let the several terms of the compound series λ be multiplied into $6g$, and let the new compound series thence arising be denoted by the small Greek letter π . Then will $q \times 2\sqrt[3]{g} \times$ the simple series w , or $q \times m$, be $=$ the new compound series π .

And in this compound series π it is evident that the first term must be $6g$, because the first term of the compound series λ (which is the product of the multiplication of the series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \&c$ into the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$) must be 1. And the following terms of the said compound series π will involve the successive even powers of z , to wit,

$zz,$

$zz, z^4, z^6, z^8, z^{10}, z^{12}, \&c$, in their numerators, and the successive odd powers of g , to wit, $g, g^3, g^5, g^7, g^9, g^{11}, \&c$, in their denominators.

114. Since the co-efficients of the terms of the two infinite serieses $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ (by the multiplication of which the compound series λ is produced) are the same with those of the corresponding terms of the two infinite serieses $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \&c$, and $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ (by the multiplication of which the compound series Λ is produced in art. 97), though the terms of the two former serieses are not connected with each other by the signs $+$ and $-$ in the same manner as the terms of the two latter serieses; it follows that the co-efficients of the several terms contained in the compound series λ will be equal to the co-efficients of the corresponding terms contained in the compound series Λ ; because the co-efficients of the terms of both the said compound serieses are similar combinations, or products, of the same original factors or co-efficients, $B, C, D, E, F, G, H, I, K, L, M, N, \&c$, of the terms of the simple serieses by the multiplication of which into each other the said compound serieses are produced. And from the continued repetition of the sign $-$ in the second and other following terms of the two serieses $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \&c$, and $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ (by the multiplication of which the compound series Λ is produced), and the alternate succession of the sign $+$ and the sign $-$ in the second and other following terms of the two serieses $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$, and $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ (by the multiplication of which into each other the compound series λ is produced), it follows that in the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series λ the signs $+$ and $-$ that are to be prefixed to the several terms of the said columns, will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Λ ; and that, in the second, and fourth, and sixth, and other following even vertical columns of terms in the said compound series λ the signs $+$ and $-$, that are to be prefixed to the several terms of the said columns, will be respectively contrary to those which are to be prefixed to the corresponding terms of the second and fourth, and sixth, and other following even vertical columns of terms in the said compound series Λ .

115. Since the compound series Π is $= 6e \times$ the compound series Λ , and the compound series π is $= 6g \times$ the compound series λ , it is evident that there will

will be the same analogies and the same differences between the two compound serieses Π and π , with respect to the co-efficients of their several corresponding terms, and the signs $+$ and $-$ which are to be prefixed to them, as between the two compound serieses Λ and λ themselves; because the said multiplications by $6e$ and $6g$ can make no difference at all in the signs of the terms multiplied, and will increase the co-efficients of all the said terms in the same proportion of 6 to 1, which will not change their antecedent equality. For, as to the quantities e and g (which are involved in the multipliers $6e$ and $6g$) the multiplications of the terms by them will not affect the numeral co-efficients of the terms arising from the combinations of the original numeral co-efficients $B, C, D, E, F, G, H, I, K, L, M, N$, &c, but will only affect the literal parts of the said terms by removing one power of e and g out of their several denominators. It therefore follows from the two last articles 113 and 114, in the first place, that the first term of the compound series π , or $6g \times \lambda$, will be $(6g \times 1, \text{ or } 6g)$; and, secondly, that the co-efficients of all the terms of the compound series π , or $6g \times \lambda$, will be, respectively, equal to the co-efficients of the corresponding terms of the compound series Π , or $6e \times \Lambda$; and, thirdly, that the signs $+$ and $-$, which are to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series π , or $6g \times \lambda$, will be the same with those which are to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Π , or $6e \times \Lambda$; and, 4thly, that the signs $+$ and $-$, which are to be prefixed to the several terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series π , or $6g \times \lambda$, will be, respectively, contrary to those which are to be prefixed to the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Π , or $6e \times \Lambda$.

116. It follows, from the last article, that, if the compound series π , or $6g \times \lambda$, be reduced into a simple series, or series consisting of single terms, by making all the multiplications, additions, and subtractions, that are necessary for that purpose, the co-efficients of the second and other following terms of the simple series thereby obtained will be respectively equal to the co-efficients of the second and other following terms of the simple series which is equal to the compound series Π , or $6e \times \Lambda$; and it follows in the second place, that the signs $+$ or $-$, that are to be prefixed to the third, and fifth, and seventh, and other following odd terms of the simple series which is equal to the compound series π , or $6g \times \lambda$, will be the same with those which are to be prefixed to the third, and fifth, and seventh, and other following odd terms of the simple series which is equal to the compound series Π , or $6e \times \Lambda$; and it follows, in the third place, that the signs $+$ or $-$, that are to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series which is equal to the compound series π , or $6g \times \lambda$, will be contrary to those which are to be prefixed to the second, and fourth, and sixth, and other following even terms of the simple series that is equal to the compound series Π , or $6e \times \Lambda$. But it has been shewn in art. 100, 101, 102, that the simple series that is equal to the compound

pound series Π , or $6e \times \Lambda$, is $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} -$
 &c *ad infinitum*. Therefore the simple series that is equal to the compound se-
 ries π , or $6g \times \lambda$, will be $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$
 &c *ad infinitum*. But by art. 112, $q \times m$, or $q \times$ the transcendental expression
 $2\sqrt[3]{g} \times$ the series w , or $q \times$ the transcendental expression $2\sqrt[3]{g} \times$ the in-
 finite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c, is equal to the
 compound series π , or $6g \times \lambda$. Therefore $q \times m$ is equal to the simple series
 $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c, *ad infinitum*.

117. But it was shewn in art. 110, that m^3 , or the cube of the transcendental
 expression m , or $2\sqrt[3]{g} \times$ the series w , or $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4}$
 $+ \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c is equal to the simple series $8g + \frac{Pzz}{g} -$
 $\frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*, which is greater than the
 simple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} +$ &c *ad infinitum*,
 by the difference $(8g - 6g)$, or $2g$. Therefore m^3 is greater than $q \times m$ by
 the same difference $2g$, or the residual quantity $m^3 - q \times m$ is $= 2g$, or $2 \times$
 $\frac{t}{2}$, or t , the absolute term of the equation $x^3 - qx = t$; and consequently m , or
 the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} +$
 $\frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} +$ &c *ad infinitum*, must be equal to the root x of the
 said equation.

Q. E. D.

*End of the Recapitulation of the foregoing Demonstration of the Proposition laid down
 above in art. 36.*

*Another Recapitulation of the Substance of the foregoing
 Demonstration, somewhat shorter than the former.*

118. The substance of the foregoing demonstration may be expressed in a
 more concise manner as follows.

Having shewn in art. 25, that, if $\frac{rr}{4}$ be greater than $\frac{q^3}{27}$, or r be greater than
 $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the root y of the cubick equation $y^3 - qy = r$ will be equal to the tranf-
 cendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8}$

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$$- \frac{Ls^{10}}{e^{10}}$$

$-\frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, *ad infinitum*, in which the letter e denotes $\frac{r}{2}$, or half the absolute term r of the equation $y^3 - qy = r$, and ss is $= \frac{rr}{4} - \frac{q^3}{27}$, we raised the said value of y to its cube, or third-power, by multiplying it twice into itself, whereby we obtained a certain compound series, of which the first term was $8e$, and the following terms involved the several fractions $\frac{ss}{e}, \frac{s^4}{e^3}, \frac{s^6}{e^5}, \frac{s^8}{e^7}, \frac{s^{10}}{e^9}, \frac{s^{12}}{e^{11}}$, &c, or the several successive powers of ss divided by the successive odd powers of e . This compound series we called Δ , and so we had $y^3 =$ the compound series Δ .

119. We then found an expression of the value of q (the co-efficient of y in the equation $y^3 - qy = r$) in terms involving the quantities e and s , to the end that we might multiply the said value of q into the foregoing transcendental expression which is equal to y , and thereby obtain a quantity expressed in powers of e and s , that should be equal to the product qy , which is subtracted from y^3

in the equation $y^3 - qy = r$. This value of q we found to be $3 \times g \frac{2}{3} \times$ the infinite series $1 - \frac{B ss}{ee} - \frac{C s^4}{e^4} - \frac{D s^6}{e^6} - \frac{E s^8}{e^8} - \frac{F s^{10}}{e^{10}} - \frac{G s^{12}}{e^{12}} - \&c$, which, being multiplied into the foregoing transcendental expression of the value of y , produced a certain compound series, of which the first term was $6e$, and the following terms involved the several fractions $\frac{ss}{e}, \frac{s^4}{e^3}, \frac{s^6}{e^5}, \frac{s^8}{e^7}, \frac{s^{10}}{e^9}, \frac{s^{12}}{e^{11}}$, &c, or the several successive even powers of s divided by the successive odd powers of e .

This compound series we called Π , and so had the product qy equal to the compound series Π .

120. We then observed that, since the compound series Δ was equal to y^3 , and the compound series Π was equal to qy , the excess of Δ above Π would be equal to $y^3 - qy$, and consequently to the absolute term r , or $2 \times \frac{r}{2}$, or $2e$.

But this, we observed, was the excess of $8e$, the first term of the compound series Δ above $6e$, the first term of the compound series Π . And hence it followed that $8e - \Delta$, or the excess of the first term of the series Δ above the whole of the said series, would be equal to $6e - \Pi$, or the excess of the first term of the series Π above the whole of the said series. And thus we obtained two compound serieses, $8e - \Delta$ and $6e - \Pi$, which were equal to each other.

121. We then observed that, since this equality between the compound serieses $8e - \Delta$ and $6e - \Pi$ took place in all the possible magnitudes of ss and ee , it followed that each separate compound term, or vertical column of single terms, involving any powers of ss and e , in the compound series Π , must be equal to the corresponding compound term, or vertical column of single terms, involving the same powers of ss and e , in the compound series Δ ; and consequently that, if these two compound serieses Δ and Π were to be reduced to simple serieses,

or

or serieses consisting of single terms, by making all the multiplications, additions, and subtractions, that were necessary for that purpose, the terms of the two simple serieses that would be thereby obtained, would be exactly equal to each other, and marked with the same signs + or —; and consequently that, if the simple series that was equal to the compound series $8e - \Delta$ was $\frac{P s^5}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, the compound series $6e - \Pi$ would likewise be equal to the same simple series $\frac{P s^5}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$.

122. And from the equality of the compound series $8e - \Delta$ to the simple series $\frac{P s^5}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, we inferred that the compound series Δ would be equal to the simple series $8e - \frac{P s^5}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$; and, in like manner, from the equality of the compound series $6e - \Pi$ to the same simple series $\frac{P s^5}{e} + \frac{Q s^4}{e^3} + \frac{R s^6}{e^5} + \frac{S s^8}{e^7} + \frac{T s^{10}}{e^9} + \frac{V s^{12}}{e^{11}} + \&c \text{ ad infinitum}$, we inferred that the compound series Π would be equal to the simple series $6e - \frac{P s^5}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$.

123. And then we concluded that y^3 (which is equal to the compound series Δ) must be equal to the simple series $8e - \frac{P s^5}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$, and that qy (which is equal to the compound series Π) must be equal to the simple series $6e - \frac{P s^5}{e} - \frac{Q s^4}{e^3} - \frac{R s^6}{e^5} - \frac{S s^8}{e^7} - \frac{T s^{10}}{e^9} - \frac{V s^{12}}{e^{11}} - \&c \text{ ad infinitum}$, which is less than the former simple series (which is equal to y^3) by the difference $8e - 6e$, or $2e$, or $2 \times \frac{r}{2}$, or r , as it ought to be.

124. We then turned our attention to the other transcendental expression, which had been declared above in art. 36 to be equal to the root x of the cubick equation $x^3 - qx = t$, in which t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, to wit, the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, in which expression g is $= \frac{t}{2}$, or half the absolute term t of the equation $x^3 - qx = t$, and zz is $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$.

125. This expression we multiplied twice into itself, so as to obtain its cube, which was a compound series, or series consisting of several horizontal lines, or rows, of terms, placed one under another. This compound series we called δ , and made the four following observations concerning it; to wit, 1st, That its first term would be $8g$, and all the following terms of it would involve the several quantities zz , z^4 , z^6 , z^8 , z^{10} , z^{12} , &c, or the several successive even powers of z , in their numerators, and the several quantities g , g^3 , g^5 , g^7 , g^9 , g^{11} , &c, or the several successive odd powers of g , in their denominators; and, secondly, That, from the equality of the co-efficients of the several terms of the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ to the co-efficients of the corresponding terms of the infinite series $1 - \frac{Cz^3}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^5}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} - \&c$, it followed that the co-efficients of the second and other following terms of the compound series δ would be equal to the co-efficients of the second and other following terms of the compound series Δ ; and, thirdly, that from the alternate succession of the sign $+$ and the sign $-$ in the terms of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, it followed that, in the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series δ , the signs $+$ and $-$ that were to be prefixed to the several terms of the said columns, would be the same with those which were to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns in the compound series Δ ; and fourthly, that in the second, and fourth, and sixth, and other following even vertical columns of terms in the said compound series δ , the signs $+$ and $-$, that are to be prefixed to the several terms of the said columns, would be, respectively, contrary to those of the corresponding terms of the second, and fourth, and sixth, and other following even vertical columns of terms in the compound series Δ .

126. And, from the foregoing observations we concluded, that, since the simple series to which the compound series Δ was reduced by making all the multiplications, additions, and subtractions that were necessary for that purpose, was $8e - \frac{Pz^3}{e} - \frac{Qz^4}{e^3} - \frac{Rz^5}{e^5} - \frac{Sz^8}{e^7} - \frac{Tz^{10}}{e^9} - \frac{Vz^{12}}{e^{11}} - \&c$, the simple series to which the compound series δ would be reduced in like manner by making all the multiplications, additions, and subtractions necessary for that purpose, would be $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$. And hence it followed that the cube of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ was equal to the simple series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$.

127. We then sought the value of q in terms that involved the powers of g and z , and found that, since zz was $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$, we should have $\frac{q^3}{27} = gg + zz = gg \times \left[1 + \frac{zz}{gg}\right]$, and $q^3 = 27 \times gg \times \left[1 + \frac{zz}{gg}\right]$, and $q = 3 \times g^{\frac{2}{3}} \times \left[1 + \frac{zz}{gg}\right]^{\frac{1}{3}} =$ (by the binomial theorem in the case of roots) $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$ *ad infinitum*.

128. Having thus found this value of q , we multiplied it into the expression $2 \sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ *ad infinitum*, and found the product to be a certain compound series which we called π , of which the first term was $6g$, and the second and other following terms involved the several quantities $zz, z^4, z^6, z^8, z^{10}, z^{12}, \&c$, or the several successive even powers of z , in their numerators, and the several quantities $g, g^3, g^5, g^7, g^9, g^{11}, \&c$, or the several successive odd powers of g , in their denominators.

129. We then compared this compound series π with the compound series Π , which was produced by the multiplication of the transcendental expression $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \&c$ (which was the value of q in the former equation $y^3 - qy = r$) into the transcendental expression $2 \sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ *ad infinitum*, and we observed that, from the equality of the co-efficients of the terms of the two simple serieses, by the multiplication of which into each other and into $3 \times g^{\frac{2}{3}} \times 2 \times g^{\frac{1}{3}}$ the compound series π is produced, to the co-efficients of the corresponding terms of the two simple serieses, by the multiplication of which into each other and into $3 \times e^{\frac{2}{3}} \times 2 \times e^{\frac{1}{3}}$ the compound series Π was produced, it followed that the numeral co-efficients of the several terms of the compound series π would be equal to the numeral co-efficients of the corresponding terms of the compound series Π ; and we likewise observed that, from the continued repetition of the sign $-$ before the second and all the following terms of the series $1 - \frac{Bss}{ee} - \frac{Cs^4}{e^4} - \frac{Ds^6}{e^6} - \frac{Es^8}{e^8} - \frac{Fs^{10}}{e^{10}} - \frac{Gs^{12}}{e^{12}} - \&c$, and likewise before the second and all the following terms of the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$, and the alternate succession of the sign $+$ and the sign $-$ before the second and all the following terms of the series

1 +

$1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$, and likewise before the second and all the following terms of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, it followed that in the compound series π (which was produced by the multiplication of the two latter serieses into each other, and into $3g^{\frac{2}{3}} \times 2g^{\frac{1}{3}}$) the signs $+$ and $-$, that would be to be prefixed to the several terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the said compound series would be the same with those which were to be prefixed to the corresponding terms of the third, and fifth, and seventh, and other following odd vertical columns of terms in the compound series Π , which was produced by the multiplication of the two former simple serieses into each other and into $3e^{\frac{2}{3}} \times 2e^{\frac{1}{3}}$; and that in the second, and fourth, and sixth, and other following even vertical columns of the said two compound serieses Π and π the signs prefixed to the terms of the one series would be every where contrary to the signs prefixed to the corresponding terms of the other series.

130. And from the foregoing observations we concluded, that, if the said two compound serieses Π and π were to be reduced to simple serieses, by making the several multiplications, additions, and subtractions that were necessary for that purpose, the co-efficients of the second, and other following terms of the simple series that was equal to the compound series π , would be equal to the co-efficients of the second and other following terms of the simple series that was equal to the compound series Π , and the signs that would be prefixed to the third, and fifth, and seventh, and other following odd terms of both these simple serieses would be the same, but those which would be prefixed to the second, and fourth, and sixth, and other following even terms of these simple serieses would be different in the two serieses.

131. And hence it followed that, since the compound series Π was found before to be equal to the simple series $6e - \frac{Pss}{e} - \frac{Qs^4}{e^3} - \frac{Rs^6}{e^5} - \frac{Ss^8}{e^7} - \frac{Ts^{10}}{e^9} - \frac{Vs^{12}}{e^{11}} + \&c \text{ ad infinitum}$, the compound series π must be equal to the simple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$.

132. But the compound series π was equal to the product of the multiplication of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ into the expression $3 \times g^{\frac{2}{3}}$ $\times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c \text{ ad infinitum}$, which is equal to q . Therefore the product of the multiplication of q into

into the transcendental expreffion $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$, is equal to the fimple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$.

133. It appears, therefore, that the cube of the transcendental expreffion $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ is equal to the fimple series $8g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$, and that the product of the multiplication of q into the faid transcendental expreffion is equal to the fimple series $6g + \frac{Pzz}{g} - \frac{Qz^4}{g^3} + \frac{Rz^6}{g^5} - \frac{Sz^8}{g^7} + \frac{Tz^{10}}{g^9} - \frac{Vz^{12}}{g^{11}} + \&c \text{ ad infinitum}$, which is lefs than the former fimple series by $8g - 6g$, or by $2g$, or $2 \times \frac{t}{2}$, or t . Therefore the cube of the transcendental expreffion $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$ is greater than the product of the multiplication of the faid transcendental expreffion into q , and their difference is equal to t , the absolute term of the equation $x^3 - qx = t$. Therefore the faid transcendental expreffion muft be equal to the root x of the faid equation.

Q. E. D.

End of the second and shorter Recapitulation of the Substance of the foregoing Demonstration of the Propofition laid down above in art. 36.

134. I have now compleated the demonftration of the propofition, or theorem, laid down above in art. 36, and which is the principal fubject of this difcourfe, to wit, "That, if the absolute term, t , of the cubick equation $x^3 - qx = t$ be lefs than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, but greater than $\sqrt[3]{2} \times \frac{q\sqrt[3]{q}}{3\sqrt[3]{3}}$ (or tt be lefs than $\frac{4q^3}{27}$, but greater than $\frac{2q^3}{27}$), or $\frac{tt}{4}$ be lefs than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$, and g be put $= \frac{t}{2}$, or half the absolute term t of the faid equation, and zz be $= \frac{q^3}{27} - \frac{tt}{4}$, or $\frac{q^3}{27} - gg$, the root x of the faid equation will be equal to the following transcendental quantity, to wit, $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c \text{ ad infinitum}$, in which series the numeral co-efficients $C, E, G, I, L, N, \&c$ of the fecond and other following terms are the co-efficients

co-efficients of the third, and fifth, and seventh, and ninth, and eleventh, and thirteenth, and other following odd terms of the infinite series which is equal to the cube-root of a binomial quantity, as $1 + x$, and the second and third, and fourth, and fifth, and sixth, and seventh, and other following terms are marked with the sign $+$ and with the sign $-$ alternately." This demonstration has been, I confess, exceedingly long and intricate; but I did not know how to make it shorter or easier, without lessening the evidence of the reasonings contained in it. And I hope that the reader will have found it convincing and satisfactory, so as to have no doubt remaining on his mind of the truth of the proposition it is intended to demonstrate. But it may possibly be asked, "How could such an expression be discovered to be equal to the root of the cubick equation $x^3 - qx = r$, if we did not already know that it is so? seeing that, when we have found out that it is equal to the said root, it requires such a long train of subtle and abstruse reasoning to prove that it is so." In answer to this question I will now proceed to state a method of investigating the value of x in this equation $x^3 - qx = t$ by which we might have been led to this discovery.

A method of investigating the value of the root x of the cubick equation $x^3 - qx = t$, in which the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, by means of the transcendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{css}{ee} - \frac{Ess^4}{e^4} - \frac{Gss^6}{e^6} - \frac{Iss^8}{e^8} - \frac{Lss^{10}}{e^{10}} - \frac{Nss^{12}}{e^{12}} - \&c$ ad infinitum, which has been shewn above in art. 25 to be equal to the root y of the cubick equation $y^3 - qy = r$, in which the absolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$.

135. Since the transcendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{css}{ee} - \frac{Ess^4}{e^4} - \frac{Gss^6}{e^6} - \frac{Iss^8}{e^8} - \frac{Lss^{10}}{e^{10}} - \frac{Nss^{12}}{e^{12}} - \&c$ is equal to the root y of the cubick equation $y^3 - qy = r$, it follows that, if the said expression were to be raised to its cube, or third power, by multiplying it twice into itself, and if it were also to be multiplied into the co-efficient q , or into the value of the said co-efficient expressed in powers of e and s , the said cube would be greater than the said product, and their difference would be equal to r . Now, because e is $= \frac{r}{2}$, and ss is $= \frac{rr}{4} - \frac{q^3}{27}$, or $ee - \frac{q^3}{27}$, we shall have $ss + \frac{q^3}{27} = ee$, and $\frac{q^3}{27} = ee - ss = ee \times \sqrt{1 - \frac{ss}{ee}}$, and $q^3 = 27 \times ee \times \sqrt{1 - \frac{ss}{ee}}$, and consequently $q = 3 \times e^{\frac{2}{3}}$

X

$\times \sqrt[3]{1 - \frac{ss}{ee}} =$ (by the residual theorem in the case of roots) $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cj^4}{e^4} - \frac{Dj^6}{e^6} - \frac{Ej^8}{e^8} - \frac{Fj^{10}}{e^{10}} - \frac{Gj^{12}}{e^{12}} - \&c.$ It follows therefore that the product of the multiplication of this last expression into the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Ej^4}{e^4} - \frac{Gj^6}{e^6} - \frac{Ij^8}{e^8} - \frac{Lj^{10}}{e^{10}} - \frac{Nj^{12}}{e^{12}} - \&c$ will be less than the cube of the said expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Ej^4}{e^4} - \frac{Gj^6}{e^6} - \frac{Ij^8}{e^8} - \frac{Lj^{10}}{e^{10}} - \frac{Nj^{12}}{e^{12}} - \&c$, and that their difference will be equal to r , or to $2 \times \frac{r}{2}$, or $2 \times e$, or $2e$.

136. Now from hence it seems natural to conjecture that the root x of the second equation $x^3 - qx = t$, in which the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, will be equal to a transcendental expression that shall bear a great resemblance to the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Ej^4}{e^4} - \frac{Gj^6}{e^6} - \frac{Ij^8}{e^8} - \frac{Lj^{10}}{e^{10}} - \frac{Nj^{12}}{e^{12}} - \&c$ (which is equal to the root y of the former equation $y^3 - qy = r$), and shall differ from it only in such circumstances as shall be the consequences of the absolute term's being less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, instead of being greater than that quantity, as it was in the foregoing equation $y^3 - qy = r$. Therefore, if in the equation $x^3 - qx = t$ we were to put g for $\frac{t}{2}$, or half the absolute term t (as we before put e for $\frac{r}{2}$, or half the absolute term r) and were to put zz for the excess of $\frac{q^3}{27}$ above $\frac{tt}{4}$, or gg (as we before put ss for the excess of $\frac{rr}{4}$, or ee , above $\frac{q^3}{27}$) it seems reasonable to conjecture that the root x of the equation $x^3 - qx = t$ will be equal to a transcendental expression of this kind, to wit, $2\sqrt[3]{g} \times$ the infinite series $1, \frac{Czz}{gg}, \frac{Ez^4}{g^4}, \frac{Gz^6}{g^6}, \frac{Iz^8}{g^8}, \frac{Lz^{10}}{g^{10}}, \frac{Nz^{12}}{g^{12}}, \&c$, in which I have not prefixed any signs $+$ or $-$ to the second and other following terms, because it does not hitherto appear whether they ought all to have the sign $-$ prefixed to them (as is the case with the second and other following terms of the former series $1 - \frac{Css}{ee} - \frac{Ej^4}{e^4} - \frac{Gj^6}{e^6} - \frac{Ij^8}{e^8} - \frac{Lj^{10}}{e^{10}} - \frac{Nj^{12}}{e^{12}} - \&c$), or whether some of them are to be marked with the sign $-$, and others of them to be marked with the sign $+$. And we shall have reason to suppose that these terms are not, all of them, to be marked with the sign $-$ (as the corresponding terms of the former series were) because the signs prefixed to the terms of the infinite series which enters into the expression of the value of q in this case, will not be the same as those which are prefixed to the terms of the infinite series

which enters into the expression of the value of q in the former case. For we have seen in art. 135, that the value of q in the former case is $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cj^4}{e^4} - \frac{Dj^6}{e^6} - \frac{Ej^8}{e^8} - \frac{Fj^{10}}{e^{10}} - \frac{Gj^{12}}{e^{12}} - \&c$, in which all the terms after the first term 1 are marked with the sign $-$. But in the present case we have $zz = \frac{q^3}{27} - \frac{tt}{4} = \frac{q^3}{27} - gg$, and consequently $\frac{q^3}{27} = gg + zz = gg \times \sqrt{1 + \frac{zz}{gg}}$, and $q^3 = 27 \times gg \times \sqrt{1 + \frac{zz}{gg}}$, and $q = 3 \times g^{\frac{2}{3}} \times \sqrt{1 + \frac{zz}{gg}}^{\frac{1}{3}}$
 $=$ (by the binomial theorem in the case of roots) $3 \times g^{\frac{2}{3}} \times$ the infinite series $1 + \frac{Bzz}{gg} - \frac{Cz^4}{g^4} + \frac{Dz^6}{g^6} - \frac{Ez^8}{g^8} + \frac{Fz^{10}}{g^{10}} - \frac{Gz^{12}}{g^{12}} + \&c$, in which the second and other following terms are marked with the sign $+$ and the sign $-$ alternately. Therefore, if this value of q were to be multiplied into the expression $2\sqrt[3]{g} \times$ the infinite series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \&c$, the signs $+$ and $-$, that would be prefixed to the several terms of the product, or compound series, thence arising, would in many instances be different from those which would be prefixed to the corresponding terms of the compound series or product, arising from the multiplication of the former value of q , to wit, $3 \times e^{\frac{2}{3}} \times$ the infinite series $1 - \frac{Bss}{ee} - \frac{Cj^4}{e^4} - \frac{Dj^6}{e^6} - \frac{Ej^8}{e^8} - \frac{Fj^{10}}{e^{10}} - \frac{Gj^{12}}{e^{12}} - \&c$, into the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Ej^4}{e^4} - \frac{Gj^6}{e^6} - \frac{Ij^8}{e^8} - \frac{Lj^{10}}{e^{10}} - \frac{Nj^{12}}{e^{12}} - \&c$. Therefore, since the expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Ej^4}{e^4} - \frac{Gj^6}{e^6} - \frac{Ij^8}{e^8} - \frac{Lj^{10}}{e^{10}} - \frac{Nj^{12}}{e^{12}} - \&c$, being substituted instead of y in compound quantity $y^3 - qy$ makes the said quantity equal to r , or $2e$, it follows that the expression $2\sqrt[3]{g} \times$ the infinite series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \&c$, being substituted instead of x in the compound quantity $x^3 - qx$, cannot (on account of the different signs prefixed to the terms of the value of the product qx from those which are prefixed to the corresponding terms of the value of the product qy) make the said quantity equal to $2g$, or t , and consequently the said expression $2\sqrt[3]{g} \times$ the infinite series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \&c$ cannot be equal to the root x of the equation $x^3 - qx = t$.

137. Having thus found that the expression $2\sqrt[3]{g} \times$ the infinite series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \&c$ is not equal to the root x of the equation $x^3 - qx = t$;—and having observed that the signs of the second and
6 other

other following terms of the infinite series that enters into the expression of the value of q in this case are alternately $+$ and $-$, whereas in the former case the second and all the following terms of the infinite series which entered into the expression of the value of q , were marked with the sign $-$;—it seems reasonable to conjecture, in the next place, that the root x of the equation $x^3 - qx = t$ may be equal to the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, in which the second and all the following terms are marked with the sign $+$ and the sign $-$ alternately. And to this expression we should find that the said root was really equal, if, when we had made this conjecture, we had tried it in a few numerical examples, by computing the values of $2\sqrt[3]{g}$ and of a few of the first terms of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, and multiplying $2\sqrt[3]{g}$ into the result of the said terms, and then substituting the product thence arising instead of x in the compound quantity $x^3 - qx$. For it would always appear, upon such substitutions, that the values of $x^3 - qx$ thence arising would be equal to the absolute terms, t , of the said equations. And thus we might have discovered that the said expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is equal to the root x of the equation $x^3 - qx = t$; after which it would have become necessary to our further satisfaction on the subject, to endeavour to demonstrate the truth of this proposition synthetically; which we have done at great length, and with as much exactness as possible, in the foregoing articles of this discourse.

138. But there is also another method of discovering that the said transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is equal to the root x of the equation $x^3 - qx = t$. And this method is more direct than the former, and carries with it a proof that the said expression must be equal to the said root, instead of affording us only a probable conjecture that it is so, which is afterwards to be confirmed by trials, in particular numerical examples, and by a synthetical demonstration. This method I shall now endeavour to explain.

Another method of discovering that the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ is equal to the root x of the cubick equation $x^3 - qx = t$, in which t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

139. We have seen, that in the equation $y^3 - qy = r$ (in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$) the transcendental expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ is equal to the root y . Now there are two different ways of computing the infinite series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$ contained in this expression, which (though not equally short and convenient in practice) are, nevertheless, equally just and true: and therefore they must both produce the same result for the value of the said series. These different methods of computing the said series are as follows.

140. The first way of computing it is the common one, which consists of the following processes; to wit, first, to compute the quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$; as was done in the foregoing example in art. 30, 31, where $\frac{rr}{4}$ was found to be = 1,110,916, and $\frac{q^3}{27}$ was found to be = 1000,000; and, secondly, to subtract $\frac{q^3}{27}$ from $\frac{rr}{4}$, in order to obtain the quantity ss , which is equal to their difference $\frac{rr}{4} - \frac{q^3}{27}$, and which in the foregoing example was 110,916; and, thirdly, to divide ss by ee , so as to obtain the value of the fraction $\frac{ss}{ee}$; as in the foregoing example we found the fraction $\frac{110,916}{1,110,916}$ to be = 0.099,841,932,2; and, fourthly, to compute the powers of the value found for the fraction $\frac{ss}{ee}$; as in the foregoing example we computed the powers of the decimal fraction 0.099,841,932,2, and found its square to be = 0.009,968,411,4, and its cube to be = 0.000,995,265,4, and its fourth power to be = 0.000,099,369,2, and its fifth, sixth, seventh, and eighth powers to be equal to 0.000,009,921,2, and 0.000,000,990,5, and 0.000,000,098,8, and 0.000,000,009,8, respectively; and, fifthly, to multiply $\frac{ss}{ee}$, and its powers $\frac{s^4}{e^4}, \frac{s^6}{e^6}, \frac{s^8}{e^8}, \frac{s^{10}}{e^{10}}, \frac{s^{12}}{e^{12}}, \frac{s^{14}}{e^{14}}, \frac{s^{16}}{e^{16}}, \frac{s^{18}}{e^{18}}, \&c$, into the coefficients $C, E, G, I, L, N, P, R, T, \&c$, respectively; as in the foregoing example we multiplied 0.099,841,932,2 into the fraction $\frac{1}{9}$ (which is = C), and

0.009,

0.009,968,411,4 into the fraction $\frac{10}{243}$ (which is = E), and 0.000,995,265,4 into the fraction $\frac{154}{6561}$ (which is = G), and 0.000,099,369,2 into the fraction $\frac{935}{59,049}$ (which is = I), and 0.000,009,921,2 into the fraction $\frac{55.913}{4,782,969}$ (which is = L), and 0.000,000,990,5 into the fraction $\frac{1,179.256}{129,140,163}$ (which is = N), and 0.000,000,098,8 into the fraction $\frac{8,617,640}{1,162,261,467}$ (which is = P), and 0.000,000,009,8 into the fraction $\frac{194,327,782}{31,381,059,609}$ (which is = R), and found the products to be 0.011,093,548,0, 0.000,410,222,6, 0.000,023,360,9, 0.000,001,573,4, 0.000,000,115,9, 0.000,000,009,0, 0.000,000,000,7, and 0.000,000,000,0; and, sixthly, to subtract the sum of all the products so obtained from 1, the first term of the series. This is the common, and the proper, way of computing the series $1 - \frac{C ss}{ee} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \frac{P s^{14}}{e^{14}} - \frac{R s^{16}}{e^{16}} - \frac{T s^{18}}{e^{18}} - \&c$, when we want to make use of it in practice. But it may also be computed in another manner, which may be described as follows.

141. Instead of ss let us insert the compound quantity $\frac{rr}{4} - \frac{q^3}{27}$ itself, to which ss is equal, in all the terms of the said series. And it will thereby be converted into the following series, to wit, $1 - \frac{C}{ee} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right] - \frac{E}{e^4} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^2 - \frac{G}{e^6} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^3 - \frac{I}{e^8} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^4 - \frac{L}{e^{10}} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^5 - \frac{N}{e^{12}} \times \left[\frac{rr}{4} - \frac{q^3}{27} \right]^6 - \&c$, or (if, for the sake of brevity, we substitute ee for $\frac{rr}{4}$, and ff for $\frac{q^3}{27}$) into the series $1 - \frac{C}{ee} \times [ee - ff] - \frac{E}{e^4} \times [ee - ff]^2 - \frac{G}{e^6} \times [ee - ff]^3 - \frac{I}{e^8} \times [ee - ff]^4 - \frac{L}{e^{10}} \times [ee - ff]^5 - \frac{N}{e^{12}} \times [ee - ff]^6 - \&c \text{ ad infinitum}$, or (substituting instead of $[ee - ff]^2$, $[ee - ff]^3$, $[ee - ff]^4$, $[ee - ff]^5$, $[ee - ff]^6$, &c their expanded values) into the series $1 - \frac{C}{ee} \times [ee - ff] - \frac{E}{e^4} \times [e^4 - 2e^2ff + f^4] - \frac{G}{e^6} \times [e^6 - 3e^4ff + 3e^2f^4 - f^6] - \frac{I}{e^8} \times [e^8 - 4e^6ff + 6e^4f^4 - 4e^2f^6 + f^8] - \frac{L}{e^{10}} \times [e^{10} - 5e^8ff + 10e^6f^4 - 10e^4f^6 + 5e^2f^8 - f^{10}] - \frac{N}{e^{12}} \times [e^{12} - 6e^{10}ff + 15e^8f^4 - 20e^6f^6 + 15e^4f^8 - 6e^2f^{10} + f^{12}] - \&c \text{ ad infinitum}$, or (multiplying the fractions $\frac{C}{e^2}$, $\frac{E}{e^4}$, $\frac{G}{e^6}$, $\frac{I}{e^8}$, $\frac{L}{e^{10}}$, $\frac{N}{e^{12}}$, &c into the separate terms of the compound quantities, or powers of $ee - ff$, with which they are connected, respectively) into the series

1 - C

$$\begin{aligned}
1 &= C + \frac{c f f}{e e} \\
&- E + \frac{2 E f f}{e e} - \frac{E f^4}{e^4} \\
&- G + \frac{3 G f f}{e e} - \frac{3 G f^4}{e^4} + \frac{G f^6}{e^6} \\
&- I + \frac{4 I f f}{e e} - \frac{6 I f^4}{e^4} + \frac{4 I f^6}{e^6} - \frac{I f^8}{e^8} \\
&- L + \frac{5 L f f}{e e} - \frac{10 L f^4}{e^4} + \frac{10 L f^6}{e^6} - \frac{5 L f^8}{e^8} + \frac{L f^{10}}{e^{10}} \\
&- N + \frac{6 N f f}{e e} - \frac{15 N f^4}{e^4} + \frac{20 N f^6}{e^6} - \frac{15 N f^8}{e^8} + \frac{6 N f^{10}}{e^{10}} - \frac{N f^{12}}{e^{12}}
\end{aligned}$$

— &c *ad infinitum*; which series consists of a much greater number of terms than the corresponding part of the series $1 - \frac{c s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, from which it is derived, and many of its terms are more complicated, and more difficult to compute, than the terms of the said former series. Nevertheless, since the compound quantity $\frac{r r}{4} - \frac{q^3}{27}$, or $e e - f f$, is equal to $s s$, the insertion of it instead of $s s$ in the terms of the said former series cannot alter its value, or magnitude, though it will make it much more difficult to compute. It must therefore be true of the new and complicated series just now obtained, as well as of the former series $1 - \frac{c s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ (from which it was derived), that, if it be multiplied into $2 \sqrt[3]{e}$, or $2 \sqrt[3]{\frac{r}{2}}$, the quantity thereby produced will be equal to the value of y in the equation $y^3 - q y = r$, or that, if the said last-mentioned quantity be raised to its cube, or third power, by multiplying it twice into itself, and also be multiplied into the co-efficient q , the product of the said multiplication will be less than the said cube, and their difference will be equal to the absolute term r .

142. Having shewn that the product of the multiplication of $2 \sqrt[3]{e}$ or $2 \sqrt[3]{\frac{r}{2}}$, into the new and complicated series last obtained in the foregoing article (which series we will, for the sake of brevity, denote by the Greek capital letter Γ) is equal to the root y of the cubick equation $y^3 - q y = r$ in the first case of that equation, or when $\frac{r r}{4}$ is greater than $\frac{q^3}{27}$, or $e e$ is greater than $f f$, as well as the product of the multiplication of $2 \sqrt[3]{e}$ into the series $1 - \frac{c s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$, from which the series Γ was derived;—we may, by means of this equality between these two products and the root of the equation $y^3 - q y = r$ in the said first case of that equation, deduce from the series $1 - \frac{c s s}{e e} - \frac{E s^4}{e^4} - \frac{G s^6}{e^6} - \frac{I s^8}{e^8} - \frac{L s^{10}}{e^{10}} - \frac{N s^{12}}{e^{12}} - \&c$ another series (resembling it in the powers of the literal quantity involved in its second and other following

lowing terms, and likewise in the numeral co-efficients C, E, G, I, L, N, &c, by which the said powers are to be multiplied, but differing from it in the signs which are to be prefixed to some of its terms) which will be of such a magnitude that, if it be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, the product of the said multiplication shall be equal to the root y of the said cubick equation $y^3 - qy = r$ in the second case of the said equation, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, and which cannot be resolved by Cardan's second rule above explained. This may be done in the manner following.

143. If in this second case of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or (if we still denote $\frac{rr}{4}$ by ee , and $\frac{q^3}{27}$ by ff) ee is less than ff , we subtract $\frac{rr}{4}$ from $\frac{q^3}{27}$, or ee from ff , and call the difference, or remainder, zz , and then raise the several powers of zz , to wit, $zz^2, zz^3, zz^4, zz^5, zz^6, zz^7, zz^8, zz^9$, &c, or $z^4, z^6, z^8, z^{10}, z^{12}, z^{14}, z^{16}, z^{18}$, &c, and also the corresponding powers of its value, the residual quantity $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$, to wit, $ff - ee^2, ff - ee^3, ff - ee^4, ff - ee^5, ff - ee^6, ff - ee^7, ff - ee^8, ff - ee^9$, &c, the even powers of the difference $ff - ee$, to wit, $ff - ee^2, ff - ee^4, ff - ee^6, ff - ee^8$, &c, will consist of the very same terms, or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff , and with the same signs $+$ and $-$ prefixed to them respectively, as were before contained in the same even powers of the opposite difference $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$, or ee was greater than ff .

Thus, for example, the square of $ee - ff$ in the former case was $e^4 - 2e^2f^2 + f^4$; and in the present case, the square of $ff - ee$ is $f^4 - 2f^2e^2 + e^4$, which consists of the same terms (or the same powers and products of ee and ff) as were contained in the square of the difference $ee - ff$, to wit, $e^4 - 2e^2f^2 + f^4$, and agrees with the said former square in the signs which are prefixed to its several terms, and differs from it only in the order in which the extreme terms e^4 and f^4 , and the letters e and f in the middle terms $2e^2f^2$ and $2f^2e^2$, are placed. And the same observation is true of the fourth power of the difference $ee - ff$, to wit, $e^8 - 4e^6f^2 + 6e^4f^4 - 4e^2f^6 + f^8$, and the fourth power of the opposite difference $ff - ee$, to wit, $f^8 - 4f^6e^2 + 6f^4e^4 - 4f^2e^6 + e^8$, and of all the following even powers of the said opposite differences $ee - ff$ and $ff - ee$.

144. Also the odd powers of the difference $ff - ee$, to wit, $ff - ee$ itself, and $ff - ee^3, ff - ee^5, ff - ee^7, ff - ee^9$, &c, will consist of the same terms (or of

of the same powers, products, and multiples of the two original quantities ee and ff) as were contained in the same odd powers of the opposite difference $ee - ff$, when ee , or $\frac{rr}{4}$, was greater than ff , or $\frac{q^3}{27}$. But the signs prefixed to the said terms will be contrary to those which were prefixed to them in the former case.

Thus, for example, the cube of the difference $ee - ff$ in the former case was $e^6 - 3e^4f^2 + 3e^2f^4 - f^6$; and the cube of the difference $ff - ee$ in the present case is $f^6 - 3f^4e^2 + 3f^2e^4 - e^6$, which consists of the same terms (or the same powers, and products of ee and ff) as are contained in the cube of $ee - ff$, to wit, $e^6 - 3e^4f^2 + 3e^2f^4 - f^6$; but they are placed in a contrary order to that in which they stood in the former case, and the signs, which are prefixed to them, are contrary in every term to what they were before. And the same thing is true of the fifth power of the difference $ee - ff$, to wit, $e^{10} - 5e^8f^2 + 10e^6f^4 - 10e^4f^6 + 5e^2f^8 - f^{10}$, and the fifth power of the opposite difference $ff - ee$, to wit, $f^{10} - 5f^8e^2 + 10f^6e^4 - 10f^4e^6 + 5f^2e^8 - e^{10}$, and of all the following odd powers of the said opposite differences $ee - ff$ and $ff - ee$.

145. It follows, therefore, that, if zz be put for $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$, in the second case of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or ee is less than ff , the even powers of zz , to wit, zz^2 , zz^4 , zz^6 , zz^8 , &c, or z^4 , z^8 , z^{12} , z^{16} , &c, will represent, or be equal to, the same terms (or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff) in the present case, as were represented by the same even powers of ss , to wit, ss^2 , ss^4 , ss^6 , ss^8 , &c, or s^4 , s^8 , s^{12} , s^{16} , &c, in the former case, when $\frac{rr}{4}$, or ee , was greater than $\frac{q^3}{27}$, or ff , and ss was put for the difference $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$; and the several terms (or powers, products, and multiples, of the original quantities ee and ff) represented by the said even powers of zz in the second case of the equation $y^3 - qy = r$, when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or ee is less than ff , will have the same signs $+$ and $-$ prefixed to them as they had in the former case of the said equation, when $\frac{rr}{4}$ was greater than $\frac{q^3}{27}$, or ee was greater than ff , and the said terms were represented by the even powers of ss ; and the only difference between the terms represented by the even powers of zz in the latter case of the said equation and the terms represented by the even powers of ss in the former case of the said equation will be in the order in which the said terms will be placed in the two cases, and in which the letters e and f in the several middle terms of the values of the said even powers of zz and ss (which will involve both the quantities ee and ff) will follow each other.

146. And

146. And it likewise follows, in the second place, that the odd powers of zz , to wit, zz itself and $\overline{zz}^3, \overline{zz}^5, \overline{zz}^7, \overline{zz}^9, \&c$, or $z^6, z^{10}, z^{14}, z^{18}, \&c$, will also represent, or be equal to, the same terms (or the same powers, products, and multiples of the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff) in the present case as were represented by the same odd powers of ss , to wit, ss itself, and $\overline{ss}^3, \overline{ss}^5, \overline{ss}^7, \overline{ss}^9, \&c$, or $s^6, s^{10}, s^{14}, s^{18}, \&c$, in the former case, or when $\frac{rr}{4}$, or ee , was greater than $\frac{q^3}{27}$, or ff , and ss was put for the difference $\frac{rr}{4} - \frac{q^3}{27}$, or $ee - ff$. But the signs $+$ and $-$, that will be prefixed to the terms that are represented by the said odd powers of zz , to wit, $zz, z^6, z^{10}, z^{14}, z^{18}, \&c$, will be respectively contrary to those which were prefixed to the same terms in the former case, when they were represented by the same odd powers of ss , to wit, $ss, s^6, s^{10}, s^{14}, s^{18}, \&c$: and the terms represented by the said odd powers of zz will likewise differ from the terms represented by the same odd powers of ss in the order in which they will be placed, and in which the letters e and f in the several middle terms of each of the said powers of $ee - ff$ and $ff - ee$ (which middle terms will involve both the quantities ee and ff) will follow each other.

147. If therefore, in the second case of the equation $y^3 - qy = r$, in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or ee is less than ff , we put zz for $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$, the series $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} - \&c$, *ad infinitum*, will represent, or be equal to, a system of terms derived from the two original quantities $\frac{rr}{4}$ and $\frac{q^3}{27}$, or ee and ff , that will be the very same in point of composition, that is, will be the very same powers, products, and multiples of ee and ff , as the terms of the complicated series Γ , which was derived above, in art. 141, from the series $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$, *ad infinitum*, by substituting $ee - ff$ in its terms instead of ss in the former case of the equation $y^3 - qy = r$, or when $\frac{rr}{4}$, or ee , was greater than $\frac{q^3}{27}$, or ff . But the terms of the said two complicated serieses, or systems of terms, so represented by the two serieses $1 - \frac{Css}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \&c$, *ad infinitum*, and $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} - \&c$, *ad infinitum*, will not *all* have the same signs $+$ and $-$ prefixed to them; but those terms only of the latter complicated series, or system of terms, which is represented by the series $1 - \frac{Czz}{ee} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} - \&c$ (which latter system of terms we will, for the sake of brevity, denote by the Greek capital letter Δ), which correspond to, or are represented by, the terms which involve the even powers of zz , to wit, the

terms $\frac{Ez^4}{e^4}$, $\frac{Iz^8}{e^8}$, $\frac{Nz^{12}}{e^{12}}$, &c, will have the same signs + and - prefixed to them as were prefixed to them in the former system of terms, or complicated series, called Γ (which was derived above in art. 141 from the series $1 - \frac{Cz}{ce} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} - \text{\&c}$), when they corresponded to, or were represented by, the terms which involved the even powers of ss , to wit, the terms $\frac{Es^4}{e^4}$, $\frac{Is^8}{e^8}$, $\frac{Ns^{12}}{e^{12}}$, &c; and those terms of the complicated series Δ which correspond to, or are represented by, the terms which involve the odd powers of zz , to wit, the terms $\frac{Czz}{ee}$, $\frac{Gz^6}{e^6}$, $\frac{Lz^{10}}{e^{10}}$, &c, will have contrary signs prefixed to them to those which were prefixed to them in the former complicated series Γ , when they corresponded to, or were represented by, those terms of the series $1 - \frac{Cz}{ce} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \text{\&c}$, which involved the odd powers of ss , to wit, the terms $\frac{Cz}{ce}$, $\frac{Gs^6}{e^6}$, $\frac{Ls^{10}}{e^{10}}$, &c. And, lastly, the order in which the several terms, or powers, products and multiples of the two original quantities ee and ff are placed in the complicated series Δ , will be contrary to that in which the same terms are placed in the complicated series Γ . But this last circumstance will make no change in the magnitude of the terms, nor in that of the whole series that is composed of them, and therefore need not be attended to any further.

148. It follows from the last article that, if the terms of the complicated series Δ be placed in the same order as the terms of the complicated series Γ , the said complicated series Δ will be as follows; to wit,

$$\begin{aligned}
 &1 + C - \frac{cff}{ee} \\
 &- E + \frac{2E ff}{ee} - \frac{Ef^4}{e^4} \\
 &+ G - \frac{3G ff}{ee} + \frac{3G f^4}{e^4} - \frac{Gf^6}{e^6} \\
 &- I + \frac{4I ff}{ee} - \frac{6I f^4}{e^4} + \frac{4I f^6}{e^6} - \frac{If^8}{e^8} \\
 &+ L - \frac{5L ff}{ee} + \frac{10L f^4}{e^4} - \frac{10L f^6}{e^6} + \frac{5L f^8}{e^8} - \frac{Lf^{10}}{e^{10}} \\
 &- N + \frac{6N ff}{ee} - \frac{15N f^4}{e^4} + \frac{20N f^6}{e^6} - \frac{15N f^8}{e^8} + \frac{6N f^{10}}{e^{10}} - \frac{Nf^{12}}{e^{12}} \\
 &+ \text{\&c, ad infinitum.}
 \end{aligned}$$

149. Therefore, if in the series $1 - \frac{Cz}{ce} - \frac{Ez^4}{e^4} - \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} - \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} - \text{\&c}$, we change the signs of those terms which involve the odd powers of zz , to wit, the terms $\frac{Cz}{ce}$, $\frac{Gz^6}{e^6}$, $\frac{Lz^{10}}{e^{10}}$, &c, and in the terms of the new series thereby produced,

produced, to wit, the series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c, we substitute $ff - ee$ instead of zz , and thereby produce a third complicated series (which, for the sake of brevity, we will denote by the Greek capital letter Λ), this third complicated series Λ will consist of the same terms in point of composition (or the same powers, products, and multiples, of the two original quantities ee and ff) as each of the two former complicated serieses denoted by the Greek capital letters Γ and Δ , and its terms will have the same signs $+$ and $-$ prefixed to them as are prefixed to the same terms in the complicated series Γ , which is set down above, in art. 141; and the only difference between the said two complicated serieses Γ and Λ will be in the order in which their terms are ranged, those of the series Γ proceeding according to the powers of the letter e , and those of the series Λ proceeding according to the powers of the letter f .

150. Now, since this last complicated series Λ consists of the same terms in point of composition (or of the same powers, products, and multiples of the original quantities ee and ff) as the first complicated series Γ , which is set down in art. 141, and has every where the same signs $+$ and $-$ prefixed to its terms; and it has been shewn above, in the same art. 141, that if the said series Γ be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, and the series thereby produced, to wit, the complicated series $2\sqrt[3]{e} \times \Gamma$, be cubed, or raised to its third power, by multiplying it twice into itself, and also be multiplied into the co-efficient q , the said product, to wit, $q \times 2\sqrt[3]{e} \times \Gamma$, will be less than the said cube, to wit, $8e \times \Gamma^3$, and their difference, to wit, $8e \times \Gamma^3 - q \times 2\sqrt[3]{e} \times \Gamma$, will be equal to the absolute term r ;—it will follow that, if the third complicated series Λ (which agrees so entirely in all its terms, and the signs which are prefixed to them, with the complicated series Γ) be multiplied into $2\sqrt[3]{e}$, or $2\sqrt[3]{\frac{r}{2}}$, and the complicated series thereby produced, to wit, the complicated series $2\sqrt[3]{e} \times \Lambda$, be cubed, or raised to its third power by multiplying it twice into itself, and also be multiplied into the co-efficient q , the said product, to wit, $q \times 2\sqrt[3]{e} \times \Lambda$, will be less than the said cube, or $8e \times \Lambda^3$, and their difference, to wit, $8e \times \Lambda^3 - q \times 2\sqrt[3]{e} \times \Lambda$ will be equal to the absolute term r .

151. But the complicated series Λ is equal to the series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c, *ad infinitum*, from which it was derived by substituting $ff - ee$ in its terms, instead of zz . Therefore $2\sqrt[3]{e} \times$ the said series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{Iz^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} +$ &c will be equal to $2\sqrt[3]{e} \times$ the complicated series Λ , or to the complicated series $2\sqrt[3]{e} \times \Lambda$. And consequently, if the quantity $2\sqrt[3]{e} \times$ the infinite series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} -$

$\frac{1z^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} + \&c$ (which is equal to the complicated series $2\sqrt[3]{e} \times \Lambda$) be raised to its cube, or third power, by multiplying it twice into itself, and likewise be multiplied into the co-efficient q , the said product will be less than the said cube, and their difference will be equal to the absolute term r . Therefore the said quantity $2\sqrt[3]{e} \times$ the infinite series $1 + \frac{Czz}{ee} - \frac{Ez^4}{e^4} + \frac{Gz^6}{e^6} - \frac{1z^8}{e^8} + \frac{Lz^{10}}{e^{10}} - \frac{Nz^{12}}{e^{12}} + \&c$, *ad infinitum*, is equal to the root y of the cubick equation $y^3 - qy = r$ in the second case of it, or when $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, or ee is less than ff , and zz is put for the difference $\frac{q^3}{27} - \frac{rr}{4}$, or $ff - ee$; or, if in this second case of the equation $y^3 - qy = r$, we substitute t instead of r for the absolute term of the equation; and put g , instead of e , for half the absolute term t ; and denote the root of the equation by the letter x instead of the letter y , the transcendental quantity $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{1z^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, *ad infinitum*, will be equal to the root x of the cubick equation $x^3 - qx = t$. Q. E. I.

End of the Investigation of the Transcendental Expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{1z^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, which is equal to the root x of the cubick equation $x^3 - qx = t$, in which t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

A Remark on the foregoing Investigation.

152. The foregoing investigation is very abstruse and difficult, and therefore has been set forth at great length, in order to make the reasonings used in it as clear as possible: and I hope the attentive reader will have found them, in general, intelligible and satisfactory. There is, however, one part of the deduction which is more subtle than the rest, and may therefore require some further elucidation; I mean that part of it which is contained in art. 150. In that article the reasoning is as follows: "Since the complicated series Λ consists of the very same terms in point of composition (or the very same powers, products, and multiples of the two original quantities ee and ff) as the complicated series Γ , and with the same signs $+$ and $-$ prefixed to them; and since it has been shewn in art. 141, that, if the expression $2\sqrt[3]{e} \times$ the complicated series Γ be cubed, or raised to its third power by multiplying it twice into itself, and also be multiplied into the co-efficient q , the said product $q \times 2\sqrt[3]{e} \times \Gamma$ will be less than the said cube, or $8e \times \Gamma^3$, and their difference will be equal to the absolute

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lute term r , or $2e$;—it will follow that, if the expression $2\sqrt[3]{e} \times$ the complicated series Λ be cubed, or raised to its third power by multiplying it twice into itself, and also be multiplied into the co-efficient q , the said product $q \times 2\sqrt[3]{e} \times \Lambda$ must in like manner be less than the said cube, or $8e \times \Lambda^3$, and their difference must be equal to the absolute term r , or $2e$."

An Objection to the Conclusion stated in the last Article.

153. Now to this conclusion it may be objected, "That the letters r and e do not denote the same quantities in the two expressions $2\sqrt[3]{e} \times \Gamma$ and $2\sqrt[3]{e} \times \Lambda$, and in the two cases of the equation $y^3 - qy = r$ which correspond to those expressions; but that in the first case of the equation $y^3 - qy = r$ the letter r signifies a quantity greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the corresponding expression $2\sqrt[3]{e} \times \Gamma$ the letter e signifies a quantity greater than f , and that in the second case of the equation $y^3 - qy = r$ the letter r signifies a quantity less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and in the corresponding expression $2\sqrt[3]{e} \times \Lambda$ the letter e signifies a quantity less than f ; and that consequently the supposed resemblance between the expression $2\sqrt[3]{e} \times \Gamma$ and $2\sqrt[3]{e} \times \Lambda$ is only apparent, and not real, and therefore will not warrant the conclusion drawn from it."

An Answer to the said Objection.

154. In answer to this objection we must observe that it never has been asserted, that the expression $2\sqrt[3]{e} \times$ the series Λ was equal to the expression $2\sqrt[3]{e} \times$ the series Γ . For that would not be true; because $2\sqrt[3]{e} \times \Lambda$ is always less than $2\sqrt[3]{e} \times \Gamma$. But it was only said that these two expressions consisted of terms composed in the same manner of the two original quantities ee and ff , and that therefore, since the cube of $2\sqrt[3]{e} \times \Gamma$ was greater than the product of the multiplication of $2\sqrt[3]{e} \times \Gamma$ into the co-efficient q , and their difference was equal to r , or $2e$, that is, to the greater value of r or $2e$, which belongs to the first case of the equation $y^3 - qy = r$, it followed that the cube of $2\sqrt[3]{e} \times \Lambda$ must be greater than the product of the multiplication of $2\sqrt[3]{e} \times \Lambda$ into the co-efficient q , and that their difference must be equal to the corresponding, or lesser, value of r , or $2e$, which belongs to the second case of the said equation $y^3 - qy = r$. And this conclusion is most certainly just and true, notwithstanding the inequality of the two values of r in the two different cases of the equation $y^3 - qy = r$, and of the two values of e in the two expressions $2\sqrt[3]{e} \times \Gamma$ and $2\sqrt[3]{e} \times \Lambda$.

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A further Proof of the Conclusion stated in Art. 152.

155. But that the truth of this conclusion may be made as plain as possible, let us substitute the letter g instead of e in the expression $2\sqrt[3]{e} \times$ the series Λ . Then will this expression become $2\sqrt[3]{g} \times$ the series Λ , or $2\sqrt[3]{g} \times$ the following complicated series, to wit,

$$\begin{aligned} 1 &= C + \frac{c\sqrt[3]{g}}{gg} \\ &= E + \frac{2E\sqrt[3]{g}}{gg} - \frac{Ef^4}{g^4} \\ &= G + \frac{3G\sqrt[3]{g}}{gg} - \frac{3Gf^4}{g^4} + \frac{Gf^6}{g^6} \\ &= I + \frac{4I\sqrt[3]{g}}{gg} - \frac{6If^4}{g^4} + \frac{4If^6}{g^6} - \frac{If^8}{g^8} \\ &= L + \frac{5L\sqrt[3]{g}}{gg} - \frac{10Lf^4}{g^4} + \frac{10Lf^6}{g^6} - \frac{5Lf^8}{g^8} + \frac{Lf^{10}}{g^{10}} \\ &= N + \frac{6N\sqrt[3]{g}}{gg} - \frac{15Nf^4}{g^4} + \frac{20Nf^6}{g^6} - \frac{15Nf^8}{g^8} + \frac{6Nf^{10}}{g^{10}} - \frac{Nf^{12}}{g^{12}} \\ &= \&c, \text{ ad infinitum.} \end{aligned}$$

We must therefore endeavour to prove that, since the cube of the expression $2\sqrt[3]{e} \times$ the complicated series Γ (which is set down above in art. 141) is greater than the product of the multiplication of the said expression $2\sqrt[3]{e} \times \Gamma$ into the co-efficient q , and their difference is equal to r , or $2e$, it must follow that the cube of the other expression $2\sqrt[3]{g} \times$ the complicated series Λ (which has been just now set down) will be greater than the product of the multiplication of the said expression $2\sqrt[3]{g} \times \Lambda$ into the co-efficient q , and that their difference will be equal to t , or $2g$.

156. Since $\sqrt[3]{g}$ is $= \frac{q^{\frac{2}{3}}}{27}$, we shall have $q^3 = 27 \times \sqrt[3]{g}$, and $q = 3 \times \sqrt[3]{g}$. Therefore the product of the multiplication of the expression $2\sqrt[3]{e} \times \Gamma$ into the co-efficient q will be equal to $2\sqrt[3]{e} \times \Gamma \times 3\sqrt[3]{g}$, or to $6e\frac{1}{3}\sqrt[3]{g}$ $\times$ the complicated series Γ ; and the product of the multiplication of the expression $2\sqrt[3]{g} \times \Lambda$ into the co-efficient q will be equal to $2\sqrt[3]{g} \times \Lambda \times 3\sqrt[3]{g}$, or to $6g\frac{1}{3}\sqrt[3]{g}$ $\times$ the complicated series Λ . And consequently we must now endeavour to prove that, since the cube of the expression $2\sqrt[3]{e} \times$ the complicated series Γ , or of $2e\frac{1}{3}\sqrt[3]{g}$ $\times$ the complicated series Γ , is greater than the product of the multiplication of $6e\frac{1}{3}\sqrt[3]{g}$ into the complicated series Λ , and their difference is equal to $2e$, or r , it must follow that the cube of the other expression $2\sqrt[3]{g} \times$ the complicated series Λ , or of $2g\frac{1}{3}\sqrt[3]{g}$ $\times$ the complicated series Λ , will be greater than the product of the multiplication of $6g\frac{1}{3}\sqrt[3]{g}$ into the com-

complicated series Λ , and that their difference will be equal to $2g$, or t . Now this may be shewn in the manner following.

157. The complicated series Γ (which is set down above in art. 141) consists of 1 together with the several numeral co-efficients C, E, G, I, L, N, &c, *ad infinitum* (which are all subtracted from 1), and of several following vertical columns of terms involving the fraction $\frac{ff}{ee}$ and its powers $\frac{f^4}{e^4}$, $\frac{f^6}{e^6}$, $\frac{f^8}{e^8}$, $\frac{f^{10}}{e^{10}}$, $\frac{f^{12}}{e^{12}}$, &c, *ad infinitum*. Therefore, if we multiply this complicated series Γ into

$2\sqrt[3]{e}$, or $2e^{\frac{1}{3}}$, the product of the said multiplication will be another complicated series which will involve either the quantity e or the quantity f , or some of their powers, products, or multiples, in all its terms, and consequently $8e \times \Gamma^3$, or the cube of the said product will be another and still more complicated series, which will also involve the quantities e and f , or some of their powers, products, or multiples, in all its terms. And, for the same reason, the quantity $6e^{\frac{1}{3}}f^{\frac{2}{3}} \times \Gamma$, or the product of the multiplication of $6e^{\frac{1}{3}}f^{\frac{2}{3}}$ into the complicated series Γ , will be a complicated series which will involve the quantities e and f , or some of their powers, products, and multiples, in all its terms.

And in like manner the complicated series Λ (which is set down in art. 155) consists of 1, together with the several numeral co-efficients C, E, G, I, L, N, &c, *ad infinitum* (which are all subtracted from 1, as in the series Γ), and of several following vertical columns of terms involving the fraction $\frac{ff}{gg}$, and its powers $\frac{f^4}{g^4}$, $\frac{f^6}{g^6}$, $\frac{f^8}{g^8}$, $\frac{f^{10}}{g^{10}}$, $\frac{f^{12}}{g^{12}}$, &c, *ad infinitum*. Therefore, if we multiply this com-

plicated series Λ into $2\sqrt[3]{g}$, or $2g^{\frac{1}{3}}$, the product of the said multiplication will be another complicated series which will involve either the quantity g or the quantity f , or some of their powers, products, or multiples, in all its terms; and consequently $8g \times \Lambda^3$, or the cube of the said product, will be another and still more complicated series, which will also involve the quantities g and f , or some of their powers, products, or multiples, in all its terms. And, for the same reason, the quantity $6g^{\frac{1}{3}}f^{\frac{2}{3}} \times \Lambda$, or the product of the multiplication of $6g^{\frac{1}{3}}f^{\frac{2}{3}}$ into the complicated series Λ , will be another complicated series which will involve the quantities g and f , or some of their powers, products, or multiples, in all its terms.

Further, since the terms of the complicated series Λ agree exactly with those of the complicated series Γ in their co-efficients, and in the signs + and — that are prefixed to them, and in every thing except the quantities which form the denominators of the literal parts of them, which in the series Γ consist of the powers of ee , and in the series Λ consist of the powers of gg , it is evident that the same analogy, or resemblance, must take place between all the complicated serieses derived in any manner from the series Γ and the complicated serieses derived in the same manner from the series Λ ; and consequently the complicated series

series $8g \times \Lambda^3$ will consist of terms that will be exactly similar to those of the complicated series $8e \times \Gamma^3$, in their co-efficients, and in the signs + and — that are prefixed to them, and in every particular whatsoever, except that, wherever the letter e occurs in the terms of the series $8e \times \Gamma^3$, the letter g will occur in the corresponding terms of the series $8g \times \Lambda^3$; and, in like manner, the complicated series $6g \frac{1}{3} f \frac{2}{3} \times \Lambda$ will consist of terms that will be exactly similar to those of the complicated series $6e \frac{1}{3} f \frac{2}{3} \times \Gamma$, in their co-efficients, and in the signs + and — which are prefixed to them, and in every particular whatsoever, except that wherever the letter e occurs in the terms of the series $6e \frac{1}{3} f \frac{2}{3} \times \Gamma$, the letter g will occur in the corresponding terms of the series $6g \frac{1}{3} f \frac{2}{3} \times \Lambda$.

And hence it follows, that, since, when we subtract the complicated series $6e \frac{1}{3} f \frac{2}{3} \times \Gamma$ from the complicated series $8e \times \Gamma^3$, the several terms of the one series so balance and destroy the corresponding terms of the other series, as to leave a remainder that is equal to $2e$ or r , the like effect must result from the subtraction of the complicated series $6g \frac{1}{3} f \frac{2}{3} \times \Lambda$ from the complicated series $8g \times \Lambda^3$, or that the several terms of the one series will so balance and destroy the corresponding terms of the other series as to leave a remainder that shall be equal to $2g$, or t . Q. E. D.

158. This demonstration will, I hope, be thought sufficient to refute the objection stated in art. 153 to the reasoning used in art. 150, and consequently to render the whole of the investigation given above in art. 139, 140, 141 151, of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{czx}{gg} - \frac{Ex^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, *ad infinitum*, for the value of x in the equation $x^3 - qx = t$, perfectly satisfactory.

Of the Method of resolving the Cubick Equation $x^3 - qx = t$, when t is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{54}$.

159. The foregoing infinite series $1 + \frac{czx}{gg} - \frac{Ex^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ will converge only when zx is less than gg , that is, when $\frac{q^3}{27} - \frac{tt}{4}$ is less than $\frac{tt}{4}$, or $\frac{q^3}{27}$ is less than $\frac{2tt}{4}$, or $\frac{q^3}{2 \times 27}$ is less than $\frac{tt}{4}$, or $\frac{q^3}{54}$ is less than $\frac{tt}{4}$. Therefore it is only when $\frac{tt}{4}$ (though less than $\frac{q^3}{27}$) is greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than

than $\frac{q^3}{54}$, that the foregoing expression $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ will be equal to the root of the equation $x^3 - qx = t$. But, when $\frac{tt}{4}$ is less than $\frac{q^3}{54}$, we may find the lesser root of the opposite equation $qx - x^3 = t$ by the method set forth in the preceeding tract contained in pages 379, 380, 381, 382 440 of the present volume of tracts; and, calling this lesser root v , we shall have $qv - v^3 (= t) = x^3 - qx$, and $qv = x^3 + v^3 - qx$, and $qx + qv = x^3 + v^3$, and $q = \frac{x^3 + v^3}{x + v} = xx - xv + vv$, and $xx - vx = q - vv$, and $xx - vx + \frac{vv}{4} (= q - vv + \frac{vv}{4} = q - \frac{3vv}{4}) = \frac{4q - 3vv}{4}$, and consequently $x - \frac{v}{2} = \frac{\sqrt{4q - 3vv}}{2}$, and $x = \frac{v + \sqrt{4q - 3vv}}{2}$. So that, of whatever magnitude less than $\frac{q^3}{27}$ the quantity $\frac{tt}{4}$ may happen to be, or of whatever magnitude less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$ the absolute term t of the equation $x^3 - qx = t$ may be supposed to be taken, the value of the root x may always be found either by the method explained in the present discourse, or by the method explained in the preceeding tract.

160. The last investigation of this transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, *ad infinitum* (which has been explained in art. 139, 140, 141, &c 157), was that by which I discovered this expression to be equal to the root x of the equation $x^3 - qx = t$, after having seen it asserted to be so by Monsieur *Clairaut* in his *Eléments d'Algèbre*. But, as the proposition appeared to me a very curious one, and worthy to be established by more than one method of proof, I afterwards sought for, and discovered, the long synthetical demonstration of it which has taken up so great a part of this discourse, and which, I apprehend, will have confirmed the truth of it beyond any possibility of doubt.

161. I will now proceed to give a few examples of the resolution of cubick equations of the aforesaid form $x^3 - qx = t$, in which t is supposed to be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or tt is supposed to be less than $\frac{4q^3}{27}$, but greater than $\frac{2q^3}{27}$, or $\frac{tt}{4}$ is supposed to be less than $\frac{q^3}{27}$, but greater than $\frac{2q^3}{4 \times 27}$, or $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by means of the foregoing transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, *ad infinitum*, in order to confirm the truth of the reasonings by which the said series has been obtained.

Examples of the resolution of cubick equations of the foregoing form, $x^3 - qx = t$, when the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{2 \times 27}$, or $\frac{q^3}{54}$, by means of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, ad infinitum.

E X A M P L E . I.

162. Let it be required to resolve the equation $x^3 - 50x = 120$ by means of the said transcendental expression.

Here q is $= 50$; t is $= 120$; $\frac{t}{2}$, or g , is $= 60$; $\frac{tt}{4}$, or gg , is $= 3600$; q^3 is $= 125,000$; and $\frac{q^3}{27}$ is $= \frac{125,000}{27} = 4629.629,629,629, \&c$, which is greater than 3600 , or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Car-

dan's rule, but may by the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, provided that the said series converges.

Now, since $\frac{q^3}{27}$ is $= 4629.629,629,629, \&c$, and $\frac{tt}{4}$ is 3600 , we shall have $zz (= \frac{q^3}{27} - \frac{tt}{4} = 4629.629,629,629, \&c - 3600) = 1029.629,629,629, \&c$, which is much less than 3600 , or gg ; and consequently the series will converge.

163. We shall therefore have $\frac{zz}{gg} = \frac{1029.629,629,629,6}{3600} = 0.286,008,230,4$,
and consequently $\frac{z^4}{g^4} (= \overline{0.286,008,230,4}^2) = 0.081,800,707,8$;
and $\frac{z^6}{g^6} (= \frac{z^4}{g^4} \times \frac{zz}{gg} = 0.081,800,707,8 \times 0.286,008,230,4) = 0.023,395,675,6$;
and $\frac{z^8}{g^8} (= \frac{z^6}{g^6} \times \frac{zz}{gg} = 0.023,395,675,6 \times 0.286,008,230,4) = 0.006,691,355,7$;
and $\frac{z^{10}}{g^{10}} (= \frac{z^8}{g^8} \times \frac{zz}{gg} = 0.006,691,355,7 \times 0.286,008,230,4) = 0.001,913,782,8$;
and

$$\text{and } \frac{z^{12}}{g^{12}} (= \frac{z^{10}}{g^{10}} \times \frac{zz}{gg} = 0.001,913,782,8 \times 0.286,008,230,4) = 0.000,547,357,6;$$

$$\text{and } \frac{z^{14}}{g^{14}} (= \frac{z^{12}}{g^{12}} \times \frac{zz}{gg} = 0.000,547,357,6 \times 0.286,008,230,4) = 0.000,156,548,7;$$

$$\text{and } \frac{z^{16}}{g^{16}} (= \frac{z^{14}}{g^{14}} \times \frac{zz}{gg} = 0.000,156,548,7 \times 0.286,008,230,4) = 0.000,044,774,2;$$

$$\text{and } \frac{z^{18}}{g^{18}} (= \frac{z^{16}}{g^{16}} \times \frac{zz}{gg} = 0.000,044,774,2 \times 0.286,008,230,4) = 0.000,012,805,7.$$

$$\text{And consequently } \frac{Czz}{gg} \text{ will be } (= C \times 0.286,008,230,4 = \frac{1}{9} \times 0.286,008,230,4 = \frac{0.286,008,230,4}{9}) = 0.031,778,692,2;$$

$$\text{and } \frac{Ez^4}{g^4} \text{ will be } (= E \times 0.081,800,707,8 = \frac{10}{243} \times 0.081,800,707,8 = \frac{10 \times 0.081,800,707,8}{243} = \frac{0.818,007,078,0}{243}) = 0.003,366,284,2;$$

$$\text{and } \frac{Gz^6}{g^6} \text{ will be } (= G \times 0.023,395,675,6 = \frac{154}{6561} \times 0.023,395,675,6 = \frac{154 \times 0.023,395,675,6}{6561} = \frac{3.602,934,042,4}{6561}) = 0.000,549,144,0;$$

$$\text{and } \frac{Iz^8}{g^8} \text{ will be } (= I \times 0.006,691,355,7 = \frac{935}{59,049} \times 0.006,691,355,7 = \frac{935 \times 0.006,691,355,7}{59,049} = \frac{6.256,417,579,5}{59,049}) = 0.000,105,952,9;$$

$$\text{and } \frac{Lz^{10}}{g^{10}} \text{ will be } (= L \times 0.001,913,782,8 = \frac{55,913}{4,782,969} \times 0.001,913,782,8 = \frac{55,913 \times 0.001,913,782,8}{4,782,969} = \frac{107,005,337,696,4}{4,782,969}) = 0.000,022,372,1;$$

$$\text{and } \frac{Nz^{12}}{g^{12}} \text{ will be } (= N \times 0.000,547,357,6 = \frac{1,179,256}{129,140,163} \times 0.000,547,357,6 = \frac{1,179,256 \times 0.000,547,357,6}{129,140,163} = \frac{645,474,733,945,6}{129,140,163}) = 0.000,004,998,2;$$

$$\text{and } \frac{Pz^{14}}{g^{14}} \text{ will be } (= P \times 0.000,156,548,7 = \frac{8,617,640}{1,162,261,467} \times 0.000,156,548,7 = \frac{8,617,640 \times 0.000,156,548,7}{1,162,261,467} = \frac{1,349,080,339,068,0}{1,162,261,467}) = 0.000,001,160,7;$$

$$\text{and } \frac{Rz^{16}}{g^{16}} \text{ will be } (= R \times 0.000,044,774,2 = \frac{194,327,782}{31,381,059,609} \times 0.000,044,774,2 = \frac{194,327,782 \times 0.000,044,774,2}{31,381,059,609} = \frac{8,700,870,976,824,4}{31,381,059,609}) = 0.000,000,277,2;$$

$$\text{and } \frac{Tz^{18}}{g^{18}} \text{ will be } (= T \times 0.000,012,805,7 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,012,805,7 = \frac{13,431,479,050 \times 0.000,012,805,7}{2,541,865,828,329} = \frac{171,999,491,270,585,0}{2,541,865,828,329}) = 0.000,000,067,6;$$

4 A 2

And

And consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ will be

$$\begin{aligned} &= 1.000,000,000,0 \\ &+ 0.031,778,692,2 - 0.003,366,284,2 \\ &+ 0.000,549,144,0 - 0.000,105,952,9 \\ &+ 0.000,022,372,1 - 0.000,004,998,2 \\ &+ 0.000,001,160,7 - 0.000,000,277,2 \\ &+ 0.000,000,067,6 - \&c \end{aligned}$$

$$\begin{aligned} &= 1.032,351,436,6 - 0.003,477,512,5 \\ &= 1.028,873,924,1. \end{aligned}$$

Further, since g is $= 60$, we shall have $\sqrt[3]{g} = \sqrt[3]{60} = 3.914,867,641,1$ and consequently $2\sqrt[3]{g} (= 2 \times 3.914,867,641,1) = 7.829,735,282,2$. Therefore the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ is $= 7.829,735,282,2 \times 1.028,873,924,1 = 8.055,810,464,4$. Therefore the root x of the proposed equation $x^3 - 50x = 120$ is $= 8.055,810,464,4$. Q. E. I.

164. This value of the root x in the equation $x^3 - 50x = 120$ is exact in the first seven figures 8.055,810, its more accurate value being 8.055,810,345,702, as may easily be found by Mr. Raphson's method of approximation.

N. B. This equation $x^3 - 50x = 120$ expresses the relation between the diameter of a circle and three chords in it that lie contiguous to each other, and together take up the arch of a semicircle, and form a trapezium of which the diameter of the circle is the fourth side. For, if the three chords are called b , k , and t , and the diameter of the circle is called x , the relation between them will be expressed by the cubick equation $x^3 - \left. \begin{matrix} bb \\ - kk \\ - tt \end{matrix} \right\} \times x = 2bkt$; which, if the

numbers 3, 4, and 5 are substituted instead of the letters b , k , and t , will become $x^3 - 50x = 120$. See Sir Isaac Newton's *Arithmetica Universalis*, edit. 2d, A. D. 1722, page 101.

EXAMPLE II.

165. Let it be required to find by means of the same transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, the root of the equation $x^3 - x = \frac{1}{3}$.

In

In this equation q is $= 1$, and t is $= \frac{1}{3}$; and consequently $\frac{t}{2}$, or g , is $= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, and $\frac{tt}{4}$, or gg , is $= \frac{1}{36}$; and $\frac{q}{3}$ is $= \frac{1}{3}$, and $\frac{q^3}{27}$ is $= \frac{1}{27}$, which is greater than $\frac{1}{36}$, or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Cardan's rule, but may by the transcendental expression $2 \sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$, in case that series is a converging one.

Now, since $\frac{q^3}{27}$ is $= \frac{1}{27}$, and $\frac{tt}{4}$ is $= \frac{1}{36}$, we shall have zz , or $\frac{q^3}{27} - \frac{tt}{4} = \frac{1}{27} - \frac{1}{36} = \frac{36}{27 \times 36} - \frac{27}{27 \times 36} = \frac{9}{27 \times 36} = \frac{1}{3 \times 36} = \frac{1}{108}$, which is less than $\frac{1}{36}$, or $\frac{tt}{4}$, or gg , in the proportion of 1 to 3. Consequently the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$ will be a converging series, and the expression $2 \sqrt[3]{g} \times$ the said series will be equal to the root of the equation $x^3 - x = \frac{1}{3}$.

166. Since zz is $= \frac{1}{3 \times 36}$, and $\frac{tt}{4}$, or gg , is $= \frac{1}{36}$, we shall have $\frac{zz}{gg} = \frac{1}{3} =$

0.333,333,333,3;

and consequently $\frac{z^4}{g^4} (= \frac{zz}{gg} \times \frac{zz}{gg} = 0.333,333,333,3 \times \frac{1}{3} = \frac{0.333,333,333,3}{3}$

$= 0.111,111,111,1$;

and $\frac{z^6}{g^6} (= \frac{z^4}{g^4} \times \frac{zz}{gg} = 0.111,111,111,1 \times \frac{1}{3} = \frac{0.111,111,111,1}{3} = 0.037,037,$

037,0;

and $\frac{z^8}{g^8} (= \frac{z^6}{g^6} \times \frac{zz}{gg} = 0.037,037,037,0 \times \frac{1}{3} = \frac{0.037,037,037,0}{3} = 0.012,$

345,679,0;

and $\frac{z^{10}}{g^{10}} (= \frac{z^8}{g^8} \times \frac{zz}{gg} = 0.012,345,679,0 \times \frac{1}{3} = \frac{0.012,345,679,0}{3} = 0.004,$

115,226,3;

and $\frac{z^{12}}{g^{12}} (= \frac{z^{10}}{g^{10}} \times \frac{zz}{gg} = 0.004,115,226,3 \times \frac{1}{3} = \frac{0.004,115,226,3}{3} = 0.001,$

371,742,1;

and $\frac{z^{14}}{g^{14}} (= \frac{z^{12}}{g^{12}} \times \frac{zz}{gg} = 0.001,371,742,1 \times \frac{1}{3} = \frac{0.001,371,742,1}{3} = 0.000,$

457,247,3;

and $\frac{z^{16}}{g^{16}} (= \frac{z^{14}}{g^{14}} \times \frac{zz}{gg} = 0.000,457,247,3 \times \frac{1}{3} = \frac{0.000,457,247,3}{3} = 0.000,$

152,415,7;

and $\frac{z^{18}}{g^{18}} (= \frac{z^{16}}{g^{16}} \times \frac{zz}{gg} = 0.000,152,415,7 \times \frac{1}{3} = \frac{0.000,152,415,7}{3} = 0.000,$

050,805,2.

Therefore $\frac{Czz}{gg}$ will be $(= C \times 0.333,333,333,3 = \frac{1}{9} \times 0.333,333,333,3$

$\frac{0.333,333,333,3}{9} = 0.037,037,037,0$;

and

$$\text{and } \frac{E z^4}{g^4} \text{ will be } (= E \times 0.111,111,111,1 = \frac{10}{243} \times 0.111,111,111,1 = \frac{10 \times 0.111,111,111,1}{243} = \frac{1.111,111,111,1}{243}) = 0.004,572,473,7;$$

$$\text{and } \frac{G z^6}{g^6} \text{ will be } (= G \times 0.037,037,037,0 = \frac{154}{6561} \times 0.037,037,037,0 = \frac{154 \times 0.037,037,037,0}{6561} = \frac{5.703,703,698,0}{6561}) = 0.000,869,344,5;$$

$$\text{and } \frac{I z^8}{g^8} \text{ will be } (= I \times 0.012,345,679,0 = \frac{935}{59,049} \times 0.012,345,679,0 = \frac{935 \times 0.012,345,679,0}{59,049} = \frac{11.543,209,865,0}{59,049}) = 0.000,195,485,2;$$

$$\text{and } \frac{L z^{10}}{g^{10}} \text{ will be } (= L \times 0.004,115,226,3 = \frac{55,913}{4,782,969} \times 0.004,115,226,3 = \frac{55,913 \times 0.004,115,226,3}{4,782,969} = \frac{230.094,648,111,9}{4,782,969}) = 0.000,048,107,0;$$

$$\text{and } \frac{N z^{12}}{g^{12}} \text{ will be } (= N \times 0.001,371,742,1 = \frac{1,179,256}{129,140,163} \times 0.001,371,742,1 = \frac{1,179,256 \times 0.001,371,742,1}{129,140,163} = \frac{1617.635,101,877,6}{129,140,163}) = 0.000,012,526,1;$$

$$\text{and } \frac{P z^{14}}{g^{14}} \text{ will be } (= P \times 0.000,457,247,3 = \frac{8,617,640}{1,162,261,467} \times 0.000,457,247,3 = \frac{8,617,640 \times 0.000,457,247,3}{1,162,261,467} = \frac{3940.392,622,372,0}{1,162,261,467}) = 0.000,003,390,2;$$

$$\text{and } \frac{R z^{16}}{g^{16}} \text{ will be } (= R \times 0.000,152,415,7 = \frac{194,327,782}{31,381,059,609} \times 0.000,152,415,7 = \frac{194,327,782 \times 0.000,152,415,7}{31,381,059,609} = \frac{29618.604,922,977,4}{31,381,059,609}) = 0.000,000,943,8;$$

$$\text{and } \frac{T z^{18}}{g^{18}} \text{ will be } (= T \times 0.000,050,805,2 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,050,805,2 = \frac{13,431,479,050 \times 0.000,050,805,2}{2,541,865,828,329} = \frac{682,388.979,431,060,0}{2,541,865,828,329}) = 0.000,000,268,4;$$

$$\text{and consequently the series } 1 + \frac{C z z}{g^2} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c \text{ will be}$$

$$\begin{aligned} &= 1.000,000,000,0 \\ &+ 0.037,037,037,0 - 0.004,572,473,7 \\ &+ 0.000,869,334,5 - 0.000,195,485,2 \\ &+ 0.000,048,107,0 - 0.000,012,526,1 \\ &+ 0.000,003,390,2 - 0.000,000,943,8 \\ &+ 0.000,000,268,4 - \&c \end{aligned}$$

$$\begin{aligned} &= 1.037,958,137,1 - 0.004,781,428,8 \\ &= 1.033,176,708,3, \end{aligned}$$

$$\text{Further, since } g \text{ is } = \frac{1}{6}, \text{ we shall have } \sqrt[3]{g} = \sqrt[3]{\frac{1}{6}} = \frac{1}{\sqrt[3]{6}} = \frac{1}{1.817,121,9} \text{ and}$$

and consequently $2\sqrt[3]{g} = \frac{2}{1.817,121}$. Therefore the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ will be $= \frac{2}{1.817,121} \times 1.033,176,708,3 = \frac{2.066,353,416,6}{1.817,121} = 1.137,157,853,8, \&c$. Therefore the root x of the proposed equation $x^3 - x = \frac{1}{3}$ will be equal to $1.137,157,853,8, \&c$. Q. E. I.

167. This value of the root x of the equation $x^3 - x = \frac{1}{3}$ is exact in the first six figures of it, to wit, $1.137,15$, the more accurate value of the said root being $1.137,158,164$, as may easily be found by Mr. Raphson's method of approximation.

E X A M P L E III.

168. Let it be required to find the root of the equation $x^3 - 5x = 4$ by means of the same transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$.

Here q is $= 5$; t is $= 4$; $\frac{t}{2}$, or g , is $= 2$; $\frac{tt}{4}$, or gg , is $= 4$; q^3 is $= 125$, and $\frac{q^3}{27}$ is $= \frac{125}{27} = 4.629,629,629,629, \&c$, which is greater than 4 , or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Cardan's rule, but may by means of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, in case the said series is a converging one.

Now, since $\frac{q^3}{27}$ is $= 4.629,629,629,629, \&c$, and $\frac{tt}{4}$, or gg , is $= 4$, we shall have $\frac{q^3}{27} - \frac{tt}{4}$, or zz , $= 0.629,629,629,629, \&c$, which is less than 4 , or gg , in the proportion of about 6 to 40 , which is a pretty large proportion of minority, and much larger than the proportion of zz to gg in either of the former examples; and consequently the said series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$ will converge with a greater degree of swiftness than in either of those examples. Therefore the said transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$ will be equal to the root x of the equation $x^3 - 5x = 4$. This expression may be computed as follows.

169. Since

169. Since gg is $= 4$, and $\frac{q^3}{27} - \frac{tt}{4}$, or zz , is $= 0.629,629,629,629$, &c, we

$$\text{shall have } \frac{zz}{gg} = \frac{0.629,629,629,629}{4} = 0.157,407,407,4;$$

$$\text{and consequently } \frac{z^4}{g^4} (= \frac{zz}{gg} \times \frac{zz}{gg} = 0.157,407,407,4 \times 0.157,407,407,4) \\ = 0.024,777,091,9;$$

$$\text{and } \frac{z^6}{g^6} (= \frac{z^4}{g^4} \times \frac{zz}{gg} = 0.024,777,091,9 \times 0.157,407,407,4) = 0.003,900, \\ 097,7;$$

$$\text{and } \frac{z^8}{g^8} (= \frac{z^6}{g^6} \times \frac{zz}{gg} = 0.003,900,097,7 \times 0.157,407,407,4) = 0.000,613, \\ 904,2;$$

$$\text{and } \frac{z^{10}}{g^{10}} (= \frac{z^8}{g^8} \times \frac{zz}{gg} = 0.000,613,904,2 \times 0.157,407,407,4) = 0.000,096, \\ 633,0;$$

$$\text{and } \frac{z^{12}}{g^{12}} (= \frac{z^{10}}{g^{10}} \times \frac{zz}{gg} = 0.000,096,633,0 \times 0.157,407,407,4) = 0.000, \\ 015,210,7;$$

$$\text{and } \frac{z^{14}}{g^{14}} (= \frac{z^{12}}{g^{12}} \times \frac{zz}{gg} = 0.000,015,210,7 \times 0.157,407,407,4) = 0.000, \\ 002,394,2;$$

$$\text{and } \frac{z^{16}}{g^{16}} (= \frac{z^{14}}{g^{14}} \times \frac{zz}{gg} = 0.000,002,394,2 \times 0.157,407,407,4) = 0.000, \\ 000,376,8;$$

$$\text{and } \frac{z^{18}}{g^{18}} (= \frac{z^{16}}{g^{16}} \times \frac{zz}{gg} = 0.000,000,376,8 \times 0.157,407,407,4) = 0.000, \\ 000,059,3.$$

$$\text{Therefore } \frac{Czz}{gg} \text{ will be } (= C \times 0.157,407,407,4 = \frac{1}{9} \times 0.157,407,407,4 \\ = \frac{0.157,407,407,4}{9}) = 0.017,489,711,9;$$

$$\text{and } \frac{Ez^4}{g^4} \text{ will be } (= E \times 0.024,777,091,9 = \frac{10}{243} \times 0.024,777,091,9 = \\ \frac{10 \times 0.024,777,091,9}{243} = \frac{0.247,770,919,0}{243}) = 0.001,019,633,4;$$

$$\text{and } \frac{Gz^6}{g^6} \text{ will be } (= G \times 0.003,900,097,7 = \frac{154}{6561} \times 0.003,900,097,7 = \\ \frac{154 \times 0.003,900,097,7}{6561} = \frac{0.600,615,045,8}{6561}) = 0.000,091,543,2;$$

$$\text{and } \frac{Iz^8}{g^8} \text{ will be } (= I \times 0.000,613,904,2 = \frac{935}{59,049} \times 0.000,613,904,2 = \\ \frac{935 \times 0.000,613,904,2}{59,049} = \frac{0.574,000,427,0}{59,049}) = 0.000,009,720,7;$$

$$\text{and } \frac{Lz^{10}}{g^{10}} \text{ will be } (= L \times 0.000,096,633,0 = \frac{55,913}{4,782,969} \times 0.000,096,633,0 = \\ \frac{5.403,040,929,0}{4,782,969}) = 0.000,001,129,6;$$

and

and $\frac{N z^{12}}{g^{12}}$ will be ($= N \times 0.000,015,210,7 = \frac{1,179,256}{129,140,163} \times 0.000,015,210,7$
 $= \frac{1,179,256 \times 0.000,015,210,7}{129,140,163} = \frac{17.937,309,239,2}{129,140,163}$) $= 0.000,000,138,8$;

and $\frac{P z^{14}}{g^{14}}$ will be ($= P \times 0.000,002,394,2 = \frac{8,617,640}{1,162,261,467} \times 0.000,002,394,2$
 $= \frac{8,617,640 \times 0.000,002,394,2}{1,162,261,467} = \frac{20.632,353,688,0}{1,162,261,467}$) $= 0.000,000,017,7$;

and $\frac{R z^{16}}{g^{16}}$ will be ($= R \times 0.000,000,376,8 = \frac{194,327,782}{31,381,059,609} \times 0.000,000,376,8$
 $= \frac{194,327,782 \times 0.000,000,376,8}{31,381,059,609} = \frac{73.222,708,257,6}{31,381,059,609}$) $= 0.000,000,002,3$;

and $\frac{T z^{18}}{g^{18}}$ will be ($= T \times 0.000,000,059,3 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,000,$
 $059,3 = \frac{13,431,479,050 \times 0.000,000,059,3}{2,541,865,828,329} = \frac{796.486,707,665,0}{2,541,865,828,329}$) $= 0.000,000,000,3$;

and consequently the series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} +$
 $\frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c$ will be

$$\begin{aligned}
 &= 1.000,000,000,0 \\
 &\quad + 0.017,489,711,9 - 0.001,019,633,4 \\
 &\quad + 0.000,091,543,2 - 0.000,009,720,7 \\
 &\quad + 0.000,001,129,6 - 0.000,000,138,8 \\
 &\quad + 0.000,000,017,7 - 0.000,000,002,3 \\
 &\quad + 0.000,000,000,3 - 0.000,000,000,0 \\
 &= 1.017,582,402,7 - 0.001,029,495,2 \\
 &= 1.016,552,907,5.
 \end{aligned}$$

Further, since g is $= 2$, we shall have $\sqrt[3]{g} = \sqrt[3]{2} = 1.259,921,049,8$,
and consequently $2 \sqrt[3]{g} = 2 \times 1.259,921,049,8 = 2.519,842,099,6$. There-
fore the transcendental expression $2 \sqrt[3]{g} \times$ the infinite series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} +$
 $\frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c$ will be equal to $2.519,$
 $842,099,6 \times 1.016,552,907,5 = 2.561,552,812,7$. Therefore the root of the
proposed equation $x^3 - 5x = 4$ is $= 2.561,552,812,7$. Q. E. I.

170. This number $2.561,552,812,7$ is true to ten places of figures, and errs
only in the last figure, which ought to be an 8 instead of a 7. For the accurate
value of x in this equation is $\frac{1+\sqrt{17}}{2}$, or $\frac{1+4.123,105,625,6}{2}$, or $\frac{5.123,105,625,6}{2}$, or
 $2.561,552,812,8$. For, if we substitute $\frac{1+\sqrt{17}}{2}$ instead of x in the compound
quantity $x^3 - 5x$, we shall find that the said quantity will be equal to 4, which is
the absolute term of the equation $x^3 - 5x = 4$. For, if x is $= \frac{1+\sqrt{17}}{2}$, we

shall have $x^3 (= \frac{1+3 \times 1^2 \times \sqrt{17} + 3 \times 1 \times 17 + 17\sqrt{17}}{8} = \frac{1+3\sqrt{17}+3 \times 17+17\sqrt{17}}{8}$
 $= \frac{1+3\sqrt{17}+51+17\sqrt{17}}{8} = \frac{52+20\sqrt{17}}{8} = \frac{13 \times 4 + 5 \times 4\sqrt{17}}{4 \times 2} = \frac{13+5\sqrt{17}}{2}$, and $5x$
 $(= 5 \times \sqrt{\frac{1+\sqrt{17}}{2}}) = \frac{5+5\sqrt{17}}{2}$, and consequently $x^3 - 5x (= \frac{13+5\sqrt{17}}{2}$
 $\frac{-5-5\sqrt{17}}{2} = \frac{13-5}{2} = \frac{8}{2}) = 4$. Q. E. D.

171. These examples sufficiently prove that the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ (which we derived from the former expression $2\sqrt[3]{e} \times$ the infinite series $1 - \frac{Cs}{ee} - \frac{Es^4}{e^4} - \frac{Gs^6}{e^6} - \frac{Is^8}{e^8} - \frac{Ls^{10}}{e^{10}} - \frac{Ns^{12}}{e^{12}} - \frac{Ps^{14}}{e^{14}} - \frac{Rs^{16}}{e^{16}} - \frac{Tz^{18}}{e^{18}} - \&c$, by the peculiar train of reasoning used in the investigation set forth in art. 139, 140, 141, &c. 158) gives the true root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, or when the said equation cannot be resolved by Cardan's rule; provided that t (though less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$) is greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or that $\frac{tt}{4}$ (though less than $\frac{q^3}{27}$) is greater than $\frac{q^3}{54}$.

I will, however, subjoin one more example to the same purpose; which shall be that of the equation $x^3 - 63x = 162$, which both Dr. Wallis and Mr. De Moivre have resolved by extracting what they call the impossible cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$. Now this equation may be resolved by the foregoing expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, in the manner following.

EXAMPLE IV.

172. Let it be required to find the root of the equation $x^3 - 63x = 162$ by means of the transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$, *ad infinitum*.

Here q is $= 63$; t is $= 162$; $\frac{t}{2}$, or g , is $= 81$; $\frac{tt}{4}$, or gg , is $= 6561$; $\frac{q}{3}$ is $= 21$; and $\frac{q^3}{27}$ is $= 9261$, which is greater than 6561 , or $\frac{tt}{4}$. Therefore this equation cannot be resolved by Cardan's rule, but may by the expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, in case that series is a converging one.

Now, since $\frac{q^3}{27}$ is = 9261, and $\frac{tt}{4}$, or gg , is = 6561, we shall have $\frac{q^3}{27} - \frac{tt}{4}$ (= 9261 - 6561) = 2700, that is, zz will be = 2700, which is less than 6561, or gg , in the proportion of 100 to ($\frac{6561}{27}$, or) 243. Therefore the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$ will be a converging series, and the expression $2\sqrt[3]{g} \times$ the said series will be equal to the root of the equation $x^3 - 63x = 162$.

173. Since zz is = 2700, and gg is = 6561, we shall have $\frac{zz}{gg}$ (= $\frac{2700}{6561}$ = $\frac{27 \times 100}{27 \times 243}$ = $\frac{100}{243}$) = 0.411,522,633,7;

and consequently $\frac{z^4}{g^4}$ (= $\frac{zz}{gg} \times \frac{zz}{gg}$ = 0.411,522,633,7 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.411,522,633,7}{243}$ = $\frac{41.152,263,370,0}{243}$) = 0.169,350,878,0;

and $\frac{z^6}{g^6}$ (= $\frac{z^4}{g^4} \times \frac{zz}{gg}$ = 0.169,350,878,0 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.169,350,878,0}{243}$ = $\frac{16.935,087,800,0}{243}$) = 0.069,691,719,3;

and $\frac{z^8}{g^8}$ (= $\frac{z^6}{g^6} \times \frac{zz}{gg}$ = 0.069,691,719,3 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.069,691,719,3}{243}$ = $\frac{6.969,171,930,0}{243}$) = 0.028,679,719,8;

and $\frac{z^{10}}{g^{10}}$ (= $\frac{z^8}{g^8} \times \frac{zz}{gg}$ = 0.028,679,719,8 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.028,679,719,8}{243}$ = $\frac{2.867,971,980,0}{243}$) = 0.011,802,353,8;

and $\frac{z^{12}}{g^{12}}$ (= $\frac{z^{10}}{g^{10}} \times \frac{zz}{gg}$ = 0.011,802,353,8 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.011,802,353,8}{243}$ = $\frac{1.180,235,380,0}{243}$) = 0.004,445,413,0;

and $\frac{z^{14}}{g^{14}}$ (= $\frac{z^{12}}{g^{12}} \times \frac{zz}{gg}$ = 0.004,445,413,0 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.004,445,413,0}{243}$ = $\frac{0.444,541,300,0}{243}$) = 0.001,829,388,0;

and $\frac{z^{16}}{g^{16}}$ (= $\frac{z^{14}}{g^{14}} \times \frac{zz}{gg}$ = 0.001,829,388,0 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.001,829,388,0}{243}$ = $\frac{0.182,938,800,0}{243}$) = 0.000,752,834,5;

and $\frac{z^{18}}{g^{18}}$ (= $\frac{z^{16}}{g^{16}} \times \frac{zz}{gg}$ = 0.000,752,834,5 $\times$ $\frac{100}{243}$ = $\frac{100 \times 0.000,752,834,5}{243}$ = $\frac{0.075,283,450,0}{243}$) = 0.000,309,808,4.

Therefore $\frac{Czz}{gg}$ will be (= $C \times 0.411,522,633,7$ = $\frac{1}{9} \times 0.411,522,633,7$ = $\frac{0.411,522,633,7}{9}$) = 0.045,724,737,0;

$$\text{and } \frac{E z^4}{g^4} \text{ will be } (= E \times 0.169,350,878,0 = \frac{10}{243} \times 0.169,350,878,0 = \frac{10 \times 0.169,350,878,0}{243} = \frac{1.693,508,780,0}{243}) = 0.006,969,171,9;$$

$$\text{and } \frac{G z^6}{g^6} \text{ will be } (= G \times 0.069,691,719,3 = \frac{154}{6561} \times 0.069,691,719,3 = \frac{10.732,524,772,2}{6561}) = 0.001,635,806,2;$$

$$\text{and } \frac{I z^8}{g^8} \text{ will be } (= I \times 0.028,679,719,8 = \frac{935}{59,049} \times 0.028,679,719,8 = \frac{935 \times 0.028,679,719,8}{59,049} = \frac{26.815,538,013,0}{59,049}) = 0.000,454,123,4;$$

$$\text{and } \frac{L z^{10}}{g^{10}} \text{ will be } (= L \times 0.011,802,353,8 = \frac{55,913}{4,782,969} \times 0.011,802,353,8 = \frac{55,913 \times 0.011,802,353,8}{4,782,969} = \frac{659.905,008,019,4}{4,782,969}) = 0.000,137,969,7;$$

$$\text{and } \frac{N z^{12}}{g^{12}} \text{ will be } (= N \times 0.004,445,413,0 = \frac{1,179,256}{129,140,163} \times 0.004,445,413,0 = \frac{1,179,256 \times 0.004,445,413,0}{129,140,163} = \frac{5,242,279,952,728,0}{129,140,163}) = 0.000,040,593,7;$$

$$\text{and } \frac{P z^{14}}{g^{14}} \text{ will be } (= P \times 0.001,829,388,0 = \frac{8,617,640}{1,162,261,467} \times 0.001,829,388,0 = \frac{8,617,640 \times 0.001,829,388,0}{1,162,261,467} = \frac{15765.007,204,320,0}{1,162,261,467}) = 0.000,013,564,0;$$

$$\text{and } \frac{R z^{16}}{g^{16}} \text{ will be } (= R \times 0.000,752,834,5 = \frac{194,327,782}{31,381,059,609} \times 0.000,752,834,5 = \frac{194,327,782 \times 0.000,752,834,5}{31,381,059,609} = \frac{146,296.658,598,079,0}{31,381,059,609}) = 0.000,004,661,9;$$

$$\text{and } \frac{T z^{18}}{g^{18}} \text{ will be } (= T \times 0.000,309,808,4 = \frac{13,431,479,050}{2,541,865,828,329} \times 0.000,309,808,4 = \frac{13,431,479,050 \times 0.000,309,808,4}{2,541,865,828,329} = \frac{4,161,185.034,114,020,0}{2,541,865,828,329}) = 0.000,001,637,0;$$

$$\text{and consequently the series } 1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c \text{ will be}$$

$$\begin{aligned} & 1.000,000,000,0 \\ & + 0.045,724,737,0 - 0.006,969,171,9 \\ & + 0.001,635,806,2 - 0.000,454,123,4 \\ & + 0.000,137,969,7 - 0.000,040,593,7 \\ & + 0.000,013,564,0 - 0.000,004,661,9 \\ & + 0.000,001,637,0 - \&c \end{aligned}$$

$$\begin{aligned} & = 1.047,513,713,9 - 0.007,468,550,9 \\ & = 1.040,045,163,0. \end{aligned}$$

Further, since g is $= 81$, we shall have $\sqrt[3]{g} = \sqrt[3]{81} = 4.326,749$, and consequently $2\sqrt[3]{g} (= 2 \times 4.326,749) = 8.653,498$. Therefore the transcendental

cendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^9}{g^9} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$ will be $= 8.653,498 \times 1.040,045,163,0 = 9.000,028,737,9$, and consequently the root of the proposed equation $x^3 - 63x = 162$ is equal to $9.000,028,737,9$. Q. E. I.

174. This number $9.000,028,737,9$ is true to five places of figures, the true value of x in the equation $x^3 - 63x = 162$ being the whole number 9, as will appear by substituting 9 instead of x in the compound quantity $x^3 - 63x$. For, if we suppose x to be equal to 9, we shall have $x^3 = 729$, and $63x (= 63 \times 9) = 567$, and consequently $x^3 - 63x (= 729 - 567) = 162$; and consequently 9 is equal to the value of x in the equation $x^3 - 63x = 162$.

Q. E. D.

A S C H O L I U M.

175. This resolution of the equation $x^3 - 63x = 162$ answers to Dr. Wallis's resolution of it by extracting the cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$, in as much as both resolutions are originally derived from Cardan's rule. But the difference between them is, that the method here delivered is intelligible in every step of it, whereas Dr. Wallis's method treats of impossible quantities, or quantities of which no clear idea can be formed, in the whole course of the process by which the value of x is investigated, though it concludes with a result that is intelligible, to wit, that x is equal to the sum of the two impossible quantities $\frac{9}{2} + \frac{1}{2} \times \sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2} \times \sqrt{-3}$, of which quantities the impossible members $+\frac{1}{2} \times \sqrt{-3}$ and $-\frac{1}{2} \times \sqrt{-3}$ are equal to each other, and are marked with the contrary signs $+$ and $-$, and therefore (when added together in order to obtain the sum of the said two impossible quantities $\frac{9}{2} + \frac{1}{2} \sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2} \sqrt{-3}$) will destroy each other, and leave us only the two possible members of the said two quantities, to wit, $\frac{9}{2}$ and $\frac{9}{2}$, of which the sum is the whole number 9. Doctor Wallis's method of finding $\frac{9}{2} + \frac{1}{2} \sqrt{-3}$ and $\frac{9}{2} - \frac{1}{2} \sqrt{-3}$ to be the cube-roots of the impossible binomial quantities $81 + \sqrt{-2700}$ and $81 - \sqrt{-2700}$, is considered by both Professor Saunderson and Mr. De Moivre as only *tentative*, and not likely to succeed in equations of which the roots are incommensurable to unity, which is the case with ninety-nine equations out of a hundred, when the equations are taken at random, and not framed on purpose with rational numbers for their roots. But Mr. De Moivre has supplied this defect, and given a *certain* method of finding the cube-roots of such impossible binomial quantities: but not without the trisection of an angle, or finding (by the help of a table

a table of sines, or otherwise) the cosine of the third part of a circular arc of which the cosine is given; by means of which trisection it is well known (independently both of Cardan's rule, and of Mr. De Moivre's process) that the second case of the cubick equation $y^3 - qy = r$ (in which $\frac{rr}{4}$ is less than $\frac{q^3}{27}$) may be resolved. So that Mr. De Moivre's method of doing this business, though more perfect than Dr. Wallis's, does not seem to be of much use in the resolution of these equations. And both methods are equally liable to the objection above-mentioned, of exhibiting to our eyes during the whole course of the processes, a parcel of algebraick quantities, of which our understandings cannot form any idea; though, by means of the ultimate exclusion of those quantities, the results become intelligible and are true. It is by the introduction of such needless difficulties and mysteries into algebra (which, for the most part, take their rise from the supposition of the existence of negative quantities, or quantities less than nothing, or of the possibility of subtracting a greater quantity from a lesser) that the otherwise clear and elegant science of algebra has been clouded and obscured, and rendered disgusting to numbers of men of learning, who are possessed of a just taste for reasoning, and could therefore, if they pleased, make great advances in the mathematical sciences, but who are apt to complain of this branch of them, and despise it on that account. And, doubtless, they have too much reason to do so; and to say, in the words of the famous French mathematician and philosopher, Monsieur Des Cartes, in his dissertation *De Methodo*, page 11—*Algebra verò, ut solet doceri, animadverti certis regulis et numerandi formulis ita esse contentam, ut videatur potius ars quædam confusa, cujus usu ingenium quodam modo turbatur et obscuratur, quam scientia, quæ excolatur et perspicacius reddatur*. If this complaint was just in Des Cartes's time, there is certainly much more reason for it now.

176. The passage above alluded to in Dr. Wallis's Algebra, is in the 48th chapter, pages 179, 180, of the folio edition printed at London in 1685. And Mr. De Moivre's method of extracting the cube-root of an impossible binomial quantity, as $81 + \sqrt{-2700}$, or $a + \sqrt{-b}$, is published in the appendix to the second volume of Professor Saunderson's Algebra, pages 744, 745, 746, 747. It is very ingenious, and shews that author's great skill in the use and management of algebraick quantities. See also on this subject Monsieur Clairaut's *Elémens d'Algèbre*, and a paper of Mr. Nicole in the Memoirs of the French Academy of Sciences for the year 1738, pages 99 and 100. See also Mac Laurin's Algebra, part 1, the supplement to the 14th chapter, pages 127, 128, 129, 130; and the Philosophical Transactions, No. 451.

Another expression of the value of the root of the equation

$x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$, derived from the foregoing expression of it.

177. But there is another expression for the value of the root x of the equation $x^3 - qx = t$ in the case here supposed, which, as it may be derived from the foregoing expression of it, to wit, $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$, ought not, I think, to be omitted. This expression does not consist entirely of an infinite series (as the foregoing expression does), but partly of a finite algebraick expression, and partly of an infinite series; and fewer terms of the infinite series are necessary to be computed and added together in order to obtain the value of the series to any proposed degree of exactness, than of the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \&c$, contained in the foregoing expression. It is as follows, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$, *ad infinitum*; of which expression the first part, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, is algebraick; and the latter part, to wit, $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ is transcendental.

The terms of the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$, are taken from the series that is equal to $\sqrt[3]{1 + \frac{z}{g}}$, or the cube root of the binomial quantity $1 + \frac{z}{g}$, to wit, the series $1 + \frac{Bz}{g} - \frac{Czz}{gg} + \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} + \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} + \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} + \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} + \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} + \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} + \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} + \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} + \&c$, *ad infinitum*, by beginning with the third term, $\frac{Czz}{gg}$, and taking every fourth term reckoned from it. This mixt expression $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$, may be derived from the foregoing transcendental expression $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$, in the manner following.

The

The derivation of the mixt expression of the value of the root of the cubick equation $x^3 - qx = t$, given in the preceeding article 177, from the former transcendental expression of it.

178. By the binomial theorem in the case of roots we have $\sqrt[3]{1 + \frac{z}{g}} =$ the infinite series $1 + \frac{Bz}{g} - \frac{Czz}{gg} + \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} + \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} + \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} + \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} + \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} + \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} + \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} + \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} + \&c$; and, by the residual theorem in the case of roots, we have $\sqrt[3]{1 - \frac{z}{g}} =$ the infinite series $1 - \frac{Bz}{g} - \frac{Czz}{gg} - \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} - \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} - \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} - \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} - \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} - \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} - \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} - \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} - \&c$. Therefore, if we add $\sqrt[3]{1 - \frac{z}{g}}$ to $\sqrt[3]{1 + \frac{z}{g}}$, and add the latter of the two foregoing serieses (which is equal to $\sqrt[3]{1 - \frac{z}{g}}$) to the former series (which is equal to $\sqrt[3]{1 + \frac{z}{g}}$), the sums thereby obtained will be equal to each other; that is, $\sqrt[3]{1 + \frac{z}{g}} + \sqrt[3]{1 - \frac{z}{g}}$ will be equal to the infinite series $2 - \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \&c$, and consequently $\frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}} + \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}}$ will be equal to the infinite series $1 - \frac{Czz}{gg} - \frac{Ez^4}{g^4} - \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} - \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} - \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} - \frac{Tz^{18}}{g^{18}} - \&c$. Let the infinite series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}} + \&c$ be added to both sides of the last equation; and we shall have $\frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}} + \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}} +$ the infinite series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}} + \&c =$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$. Therefore (multiplying both sides by $2\sqrt[3]{g}$) we shall have $2\sqrt[3]{g} \times \frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}} + 2\sqrt[3]{g} \times \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}} + 2\sqrt[3]{g} \times$ the series $\frac{2Czz}{gg} + \frac{2Gz^6}{g^6} + \frac{2Lz^{10}}{g^{10}} + \frac{2Pz^{14}}{g^{14}} + \frac{2Tz^{18}}{g^{18}} + \&c$ *ad infinitum* $= 2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c$

$-\frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c \text{ ad infinitum.}$ But $2 \sqrt[3]{g} \times \frac{1}{2} \times \sqrt[3]{1 + \frac{z}{g}}$ is $(= \sqrt[3]{g} \times \sqrt[3]{1 + \frac{z}{g}} = \sqrt[3]{g \times (1 + \frac{z}{g})} = \sqrt[3]{g + z};$
 and $2 \sqrt[3]{g} \times \frac{1}{2} \times \sqrt[3]{1 - \frac{z}{g}}$ is $(= \sqrt[3]{g} \times \sqrt[3]{1 - \frac{z}{g}} = \sqrt[3]{g \times (1 - \frac{z}{g})} = \sqrt[3]{g - z};$ and $2 \sqrt[3]{g} \times$ the series $\frac{2 C z z}{g g} + \frac{2 G z^6}{g^6} + \frac{2 L z^{10}}{g^{10}} + \frac{2 P z^{14}}{g^{14}} + \frac{2 T z^{18}}{g^{18}} + \&c$ is equal to $4 \sqrt[3]{g} \times$ the infinite series $\frac{C z z}{g g} + \frac{G z^6}{g^6} + \frac{L z^{10}}{g^{10}} + \frac{P z^{14}}{g^{14}} + \frac{T z^{18}}{g^{18}} + \&c$ will be
 $= 2 \sqrt[3]{g} \times$ the infinite series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c,$ and consequently will be equal to the root of the
 equation $x^3 - q x = t$, when t is less than $\frac{2 q \sqrt[3]{q}}{3 \sqrt[3]{3}}$, but greater than $\sqrt[3]{2} \times \frac{q \sqrt[3]{q}}{3 \sqrt[3]{3}}$, or
 when $\frac{t t}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{1}{2} \times \frac{q^3}{27}$, or than $\frac{q^3}{54}$. Q. E. D.

An application of the last expression of the value of the root
 of the equation $x^3 - q x = t$ to the resolution of the
 above-mentioned numeral equations $x^3 - 50 = 120$,
 $x^3 - x = \frac{1}{3}$, $x^3 - 5 x = 4$, and $x^3 - 63 x = 162$.

179. This new expression $\sqrt[3]{g + z} + \sqrt[3]{g - z} + 4 \sqrt[3]{g} \times$ the infinite
 series $\frac{C z z}{g g} + \frac{G z^6}{g^6} + \frac{L z^{10}}{g^{10}} + \frac{P z^{14}}{g^{14}} + \frac{T z^{18}}{g^{18}} + \&c \text{ ad infinitum}$ may be applied to
 the resolution of the four foregoing cubick equations $x^3 - 50 x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5 x = 4$, and $x^3 - 63 x = 162$, in the manner following.

In the first equation $x^3 - 50 x = 120$ we have seen above in art. 162, 163,
 that g is $= 60$, and $z z$ is $= 1029.629,629,629,629, \&c$, and $\frac{C z z}{g g}$ is $= 0.031,$
 $778,692,2$, and $\frac{G z^6}{g^6}$ is $= 0.000,549,144,0$, and $\frac{L z^{10}}{g^{10}}$ is $= 0.000,022,372,1$, and
 $\frac{P z^{14}}{g^{14}}$ is $= 0.000.001,160,7$, and $\frac{T z^{18}}{g^{18}}$ is $= 0.000,000,067,6$, and $\sqrt[3]{g}$ is $=$
 $3.914,867,641,1$. Therefore the series $\frac{C z z}{g g} + \frac{G z^6}{g^6} + \frac{L z^{10}}{g^{10}} + \frac{P z^{14}}{g^{14}} + \frac{T z^{18}}{g^{18}} +$
 $\&c$ will be $(= 0.031,778,692,2 + 0.000,549,144,0 + 0.000,022,372,1$
 $+ 0.000,001,160,7 + 0.000,000,067,6 + \&c) = 0.032,351,436,6 \&c$, and
 VOL. II. 4 C $4 \sqrt[3]{g}$

$4\sqrt[3]{g}$ will be $(= 4 \times 3.914,867,641,1) = 15.659,470,564,4$, and $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be $(= 15.659,470,564,4 \times 0.032,351,436,6) = 0.506,606,369,1$.

And z will be $(= \sqrt{1029,629,629,629, \&c}) = 32.087,842,395,9$; and consequently $g + z$ will be $(= 60 + 32.087,842,395,9) = 92.087,842,395,9$, and $g - z$ will be $(= 60.000,000,000,0 - 32.087,842,395,9) = 27.912,157,604,1$, and $\sqrt[3]{g + z}$ will be $(= \sqrt[3]{92.087,842,395,9}) = 4.515,793,760,9$ and $\sqrt[3]{g - z}$ will be $(= \sqrt[3]{27.912,157,604,1}) = 3.033,410,154,2$. Therefore $\sqrt[3]{g + z} + \sqrt[3]{g - z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be $= 4.515,793,760,9 + 3.033,410,154,2 + 0.506,606,369,1 = 8.055,810,284,2$; and consequently the root of the equation $x^3 - 50x = 120$ will be $= 8.055,810,284,2$. Q. E. I.

This value of x is exact in the first seven figures 8.055,810, its more accurate value being 8.055,810,345,702.

180. In the next equation $x^3 - x = \frac{1}{3}$ we have seen above, in art. 165, 166, that g is $= \frac{1}{6}$, and zz is $= \frac{1}{3 \times 36}$, or $\frac{1}{108}$, and $\frac{zz}{gg}$ is $= \frac{1}{3}$, and $\frac{Czz}{gg}$ is $= 0.037,037,037,0$, and $\frac{Gz^6}{g^6}$ is $= 0.000,869,344,5$ and $\frac{Lz^{10}}{g^{10}}$ is $= 0.000,048,107,0$, and $\frac{Pz^{14}}{g^{14}}$ is $= 0.000,003,390,2$, and $\frac{Tz^{18}}{g^{18}}$ is $= 0.000,000,268,4$, and that $\sqrt[3]{g}$ is $= \frac{1}{1.817,121}$, and $2\sqrt[3]{g}$ is $= \frac{2}{1.817,121}$. Therefore the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ is $(= 0.037,037,037,0 + 0.000,869,344,5 + 0.000,048,107,0 + 0.000,003,390,2 + 0.000,000,268,4 + \&c) = 0.037,958,147,1$ and consequently $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be $(= 4\sqrt[3]{g} \times 0.037,958,147,1 = \sqrt[3]{g} \times 4 \times 0.037,958,147,1 = \sqrt[3]{g} \times 0.151,832,588,4 = \frac{1}{1.817,121} \times 0.151,832,588,4 = \frac{0.151,832,588,4}{1.817,121}) = 0.083,556,674,7$.

And z will be $(= \sqrt{\frac{1}{108}} = \sqrt{0.009,259,259,2}) = 0.096,225,044,8$, and consequently $g + z$ will be $(= \frac{1}{6} + 0.096,225,044,8 = 0.166,666,666,6 + 0.096,225,044,8) = 0.262,891,711,4$, and $g - z$ will be $(= 0.166,666,666,6 - 0.096,225,044,8) = 0.070,441,621,8$, and $\sqrt[3]{g + z}$ will be $(= \sqrt[3]{0.262,891,711,4}) = 0.640,607,911,4$, and $\sqrt[3]{g - z}$ will be $(= \sqrt[3]{0.070,441,621,8}) = 0.412,993,403,9$. Therefore $\sqrt[3]{g + z} + \sqrt[3]{g - z} + 4\sqrt[3]{g} \times$ the

× the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be = 0.640, 607, 911, 4 + 0.412, 993, 403, 9 + 0.083, 556, 674, 7 = 1.137, 157, 989, 0; and consequently the root of the equation $x^3 - x = \frac{1}{3}$ will be = 1.137, 157, 989, 0.

Q. E. I.

This value of x is exact in the first six figures, 1.137, 15, its more accurate value being 1.137, 158, 164.

181. In the third equation $x^3 - 5x = 4$ we have seen above, in art. 168, 169, that g is = 2, and $\sqrt[3]{g}$ is = 1.259, 921, 049, 8, and zz is = 0.629, 629, 629, 629, &c, and $\frac{zz}{gg}$ is = 0.157, 407, 407, 4, and $\frac{Czz}{gg}$ is = 0.017, 489, 711, 9, and $\frac{Gz^6}{g^6}$ is = 0.000, 091, 543, 2, and $\frac{Lz^{10}}{g^{10}}$ is = 0.000, 001, 129, 6, and $\frac{Pz^{14}}{g^{14}}$ is = 0.000, 000, 017, 7, and $\frac{Tz^{18}}{g^{18}}$ is = 0.000, 000, 000, 3; and consequently the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be (= 0.017, 489, 711, 9 + 0.000, 091, 543, 2 + 0.000, 001, 129, 6 + 0.000, 000, 017, 7 + 0.000, 000, 000, 3 + &c) = 0.017, 582, 402, 7 &c, and $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be (= $4 \times 1.259, 921, 049, 8 \times 0.017, 582, 402, 7$ = 5.039, 684, 199, 2 $\times$ 0.017, 582, 402, 7) = 0.088, 609, 757, 0.

Further z will be (= $\sqrt[3]{0.629, 629, 629, 629, 629, 629}$) = 0.793, 492, 047, 6, and consequently $g + z$ will be (= $2 + 0.793, 492, 047, 6$) = 2.793, 492, 047, 6, and $g - z$ will be (= $2.000, 000, 000, 0 - 0.793, 492, 047, 6$) = 1.206, 507, 952, 4, and $\sqrt[3]{g + z}$ will be (= $\sqrt[3]{2.793, 492, 047, 6}$) = 1.408, 366, 911, 5, and $\sqrt[3]{g - z}$ will be (= $\sqrt[3]{1.206, 507, 952, 4}$) = 1.064, 576, 143, 3. Therefore $\sqrt[3]{g + z} + \sqrt[3]{g - z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be (= $1.408, 366, 911, 5 + 1.064, 576, 143, 3 + 0.088, 609, 757, 0$) = 2.561, 552, 811, 8; and consequently the root of the equation $x^3 - 5x = 4$ will be = 2.561, 552, 811, 8. Q. E. I.

This value of the root of the equation $x^3 - 5x = 4$, is exact in the first nine figures 2.561, 552, 81, its more accurate value being 2.561, 552, 812, 8. See above, art. 170.

182. In the fourth equation $x^3 - 63x = 162$ we have seen above in art. 172, 173, that g is = 81, and $\sqrt[3]{g}$ is = 4.326, 749, and zz is = 2700, and $\frac{zz}{gg}$ is = $\frac{100}{243}$, and $\frac{Czz}{gg}$ is = 0.045, 724, 737, 0, and $\frac{Gz^6}{g^6}$ is = 0.001, 635, 806, 2, and $\frac{Lz^{10}}{g^{10}}$ is

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is = 0.000,137,969,7, and $\frac{Pz^{14}}{g^{14}}$ is = 0.000,013,564,0, and $\frac{Tz^{18}}{g^{18}}$ is = 0.000,001,637,0.

Therefore the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be (= 0.045,724,737,0 + 0.001,635,806,2 + 0.000,137,969,7 + 0.000,013,564,0 + 0.000,001,637,0 + $\&c$) = 0.047,513,713,9, and $4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ will be (= $4\sqrt[3]{g} \times 0.047,513,713,9 = 4 \times 4.326,749 \times 0.047,513,713,9 = 17.306,996 \times 0.047,513,713,9$) = 0.822,319,656,4.

Further z will be (= $\sqrt{2700} = \sqrt{900 \times 3} = \sqrt{900} \times \sqrt{3} = 30 \times \sqrt{3} = 30 \times 1.732,050,807,5$) = 51.961,524,225,0; and consequently $g + z$ will be (= $81 + 51.961,524,225,0$) = 132.961,524,225,0, and $g - z$ will be (= $81.000,000,000,0 - 51.961,524,225,0$) = 29.038,475,775,0, and $\sqrt[3]{g + z}$ will be (= $\sqrt[3]{132.961,524,225,0}$) = 5.103,976,447,9, and $\sqrt[3]{g - z}$ will be (= $\sqrt[3]{29.038,475,775,0}$) = 3.073,674,958,2. Therefore $\sqrt[3]{g + z} + \sqrt[3]{g - z} + 4\sqrt[3]{g} \times$ the series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$ *ad infinitum* will be = 5.103,976,447,9 + 3.073,674,958,2 + 0.822,319,656,4 = 8.999,971,062,5; and consequently the root of the equation $x^3 - 63x = 162$ will be = 8.999,971,062,5. Q. E. I.

This number 8.999,971,062,5 is exact in the first five figures, 8.9999, the true value of the root x in this equation being the whole number 9. See above, art. 174.

End of the resolution of the four equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$, by means of the mixt expression $\sqrt[3]{g + z} + \sqrt[3]{g - z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Czz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c$.

A third expression of the value of the root of the equation $x^3 - qx = t$, derived from the expression obtained for it above in art. 139, 140, 141, $\&c$ 157.

183. We may also derive another expression for the value of the root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, from the former transcendental expression of it, to wit, $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \&c$

$\frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c$, which was obtained by the investigation set forth in art. 139, 140, 141, &c 157. This expression is $4 \sqrt[3]{g} \times$ the infinite series $1 - \frac{E z^4}{g^4} - \frac{I z^8}{g^8} - \frac{N z^{12}}{g^{12}} - \frac{R z^{16}}{g^{16}} - \frac{W z^{20}}{g^{20}} - \&c$ *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$; which (like the second expression of the value of x given above in art. 177) is partly algebraïck, and partly transcendental; but in this expression the transcendental part, to wit, $4 \sqrt[3]{g} \times$ the infinite series $1 - \frac{E z^4}{g^4} - \frac{I z^8}{g^8} - \frac{N z^{12}}{g^{12}} - \frac{R z^{16}}{g^{16}} - \frac{W z^{20}}{g^{20}} - \&c$ *ad infinitum*, is greater than the algebraïck part, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, whereas in the second expression before-mentioned, to wit, the expression $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4 \sqrt[3]{g} \times$ the infinite series $\frac{C z z}{g g} + \frac{G z^6}{g^6} + \frac{L z^{10}}{g^{10}} + \frac{P z^{14}}{g^{14}} + \frac{T z^{18}}{g^{18}} + \&c$, the algebraïck part, $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, was greater than the transcendental part $4 \sqrt[3]{g} \times$ the infinite series $\frac{C z z}{g g} + \frac{G z^6}{g^6} + \frac{L z^{10}}{g^{10}} + \frac{P z^{14}}{g^{14}} + \frac{T z^{18}}{g^{18}} + \&c$ *ad infinitum*. And, further, in the last, or third, expression, to wit, $4 \sqrt[3]{g} \times$ the infinite series $1 - \frac{E z^4}{g^4} - \frac{I z^8}{g^8} - \frac{N z^{12}}{g^{12}} - \frac{R z^{16}}{g^{16}} - \frac{W z^{20}}{g^{20}} - \&c - \sqrt[3]{g+z} - \sqrt[3]{g-z}$, it is the difference of the transcendental part and the algebraïck part, or the excess of the former above the latter, that is equal to the root of the equation $x^3 - q x = t$; whereas in the former, or second, expression, to wit, $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4 \sqrt[3]{g} \times$ the infinite series $\frac{C z z}{g g} + \frac{G z^6}{g^6} + \frac{L z^{10}}{g^{10}} + \frac{P z^{14}}{g^{14}} + \frac{T z^{18}}{g^{18}} + \&c$ *ad infinitum*, it is the sum of the algebraïck and the transcendental parts of the expression that is equal to the root of the said equation. In both expressions the indexes of the powers of z and g in the numerators and denominators of the terms of the infinite serieses contained in the transcendental parts of them, increase continually by 4.

This third expression $4 \sqrt[3]{g} \times$ the infinite series $1 - \frac{E z^4}{g^4} - \frac{I z^8}{g^8} - \frac{N z^{12}}{g^{12}} - \frac{R z^{16}}{g^{16}} - \frac{V z^{20}}{g^{20}} - \&c$ *ad infinitum* $- \sqrt[3]{g+z} - \sqrt[3]{g-z}$, may be derived from the foregoing transcendental expression $2 \sqrt[3]{g} \times$ the infinite series $1 + \frac{C z z}{g g} - \frac{E z^4}{g^4} + \frac{G z^6}{g^6} - \frac{I z^8}{g^8} + \frac{L z^{10}}{g^{10}} - \frac{N z^{12}}{g^{12}} + \frac{P z^{14}}{g^{14}} - \frac{R z^{16}}{g^{16}} + \frac{T z^{18}}{g^{18}} - \&c$ *ad infinitum* in the manner following.

The Derivation of the third expression of the value of the root of the equation $x^3 - qx = t$, given in the preceding article 183, from the former transcendental expression of it.

184. Since $\sqrt[3]{g+z}$ is $(= \sqrt[3]{g} \times \sqrt[3]{1 + \frac{z}{g}})$ (by the binomial theorem in the case of roots) $\sqrt[3]{g} \times$ the infinite series $1 + \frac{Bz}{g} - \frac{Cz^2}{gg} + \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} + \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} + \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} + \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} + \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} + \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} + \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} + \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} + \frac{Vz^{19}}{g^{19}} - \frac{Wz^{20}}{g^{20}} + \&c$, and $\sqrt[3]{g-z}$ is $(= \sqrt[3]{g} \times \sqrt[3]{1 - \frac{z}{g}})$ (by the residual theorem in the case of roots) $\sqrt[3]{g} \times$ the infinite series $1 - \frac{Bz}{g} - \frac{Cz^2}{gg} - \frac{Dz^3}{g^3} - \frac{Ez^4}{g^4} - \frac{Fz^5}{g^5} - \frac{Gz^6}{g^6} - \frac{Hz^7}{g^7} - \frac{Iz^8}{g^8} - \frac{Kz^9}{g^9} - \frac{Lz^{10}}{g^{10}} - \frac{Mz^{11}}{g^{11}} - \frac{Nz^{12}}{g^{12}} - \frac{Oz^{13}}{g^{13}} - \frac{Pz^{14}}{g^{14}} - \frac{Qz^{15}}{g^{15}} - \frac{Rz^{16}}{g^{16}} - \frac{Sz^{17}}{g^{17}} - \frac{Tz^{18}}{g^{18}} - \frac{Vz^{19}}{g^{19}} - \frac{Wz^{20}}{g^{20}} - \&c$, it follows that $\sqrt[3]{g+z} + \sqrt[3]{g-z}$ will be $= \sqrt[3]{g} \times$ the infinite series $2 - \frac{2Cz^2}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} - \&c$. Now the series $2 - \frac{2Cz^2}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} - \&c$ *ad infinitum* is evidently less than the series $2 - \frac{2Ez^4}{g^4} - \frac{2Iz^8}{g^8} - \frac{2Nz^{12}}{g^{12}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Wz^{20}}{g^{20}} - \&c$ *ad infinitum*, and therefore, *à fortiori*, is less than twice that series, or than the series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} - \&c$ *ad infinitum*. Therefore $\sqrt[3]{g} \times$ the series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} - \&c$ *ad infinitum* will be greater than $\sqrt[3]{g} \times$ the series $2 - \frac{2Cz^2}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} - \&c$ *ad infinitum*, and consequently will be greater also than $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, which is equal to that quantity. Let $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, and, its equal, the quantity $\sqrt[3]{g} \times$ the infinite series $2 - \frac{2Cz^2}{gg} - \frac{2Ez^4}{g^4} - \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} - \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} - \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} - \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} - \&c$ *ad infinitum*, be subtracted from the quantity $\sqrt[3]{g} \times$ the infinite series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} - \&c$ *ad infinitum*; and the remainders will

will be equal to each other; that is, $\sqrt[3]{g} \times$ the infinite series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} - \&c \text{ ad infinitum} - \sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be equal to $\sqrt[3]{g} \times$ the series $2 + \frac{2Czz}{gg} - \frac{2Ez^4}{g^4} + \frac{2Gz^6}{g^6} - \frac{2Iz^8}{g^8} + \frac{2Lz^{10}}{g^{10}} - \frac{2Nz^{12}}{g^{12}} + \frac{2Pz^{14}}{g^{14}} - \frac{2Rz^{16}}{g^{16}} + \frac{2Tz^{18}}{g^{18}} - \frac{2Wz^{20}}{g^{20}} + \&c \text{ ad infinitum}$, and consequently to $2\sqrt[3]{g} \times$ the series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \frac{Wz^{20}}{g^{20}} + \&c \text{ ad infinitum}$, or to the transcendental expression for the root of the equation $x^3 - qx = t$ discovered by the investigation contained in art. 139, 140, 141, &c. . . . 157. Therefore the expression $\sqrt[3]{g} \times$ the infinite series $4 - \frac{4Ez^4}{g^4} - \frac{4Iz^8}{g^8} - \frac{4Nz^{12}}{g^{12}} - \frac{4Rz^{16}}{g^{16}} - \frac{4Wz^{20}}{g^{20}} - \&c \text{ ad infinitum} - \sqrt[3]{g+z} - \sqrt[3]{g-z}$, or the expression $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}} - \&c \text{ ad infinitum} - \sqrt[3]{g+z} - \sqrt[3]{g-z}$, will be equal to $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Czz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \frac{Wz^{20}}{g^{20}} + \&c \text{ ad infinitum}$, or to the root of the equation $x^3 - qx = t$. Q. E. D.

An application of the last, or third, expression of the value of the root of the equation $x^3 - qx = t$ to the resolution of the above-mentioned numeral equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$.

185. This last expression, $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \frac{Wz^{20}}{g^{20}} - \&c \text{ ad infinitum} - \sqrt[3]{g+z} - \sqrt[3]{g-z}$, may be applied to the resolution of the foregoing equations $x^3 - 50x = 120$, $x^3 - x = \frac{1}{3}$, $x^3 - 5x = 4$, and $x^3 - 63x = 162$ in the manner following.

In the first equation $x^3 - 50x = 120$ we have seen above, in art. 162, 163, that g is $= 60$, and zz is $= 1029.629,629,629,629, \&c$, and $\frac{Ez^4}{g^4}$ is $0.003,366,284,2$, and $\frac{Iz^8}{g^8}$ is $= 0.000,105,952,9$, and $\frac{Nz^{12}}{g^{12}}$ is $= 0.000,004,998,2$, and $\frac{Rz^{16}}{g^{16}}$ is $= 0.000,000,277,2$, and that $\sqrt[3]{g}$ is $= 3.914,867,641,1$. Therefore the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ will be $(= 1.000,000,000,0$
 2. - 0.003,

$- 0.003,366,284,2 - 0.000,105,952,9 - 0.000,004,998,2 - 0.000,000,277,2 - \&c = 1.000,000,000,0 - 0.003,477,512,5) = 0.996,522,487,5$, and $4\sqrt[3]{g}$ will be $(= 4 \times 3.914,867,641,1) = 15.659,470,564,4$, and $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ *ad infinitum* will be $(= 15.659,470,564,4 \times 0.996,522,487,5) = 15.605,014,559,7$.

And we have seen above, in art. 179, that $g + z$ is $= 92.087,842,395,9$, and $g - z$ is $= 27.912,157,604,1$, and $\sqrt[3]{g + z}$ is $= 4.515,793,760,9$, and $\sqrt[3]{g - z}$ is $= 3.033,410,154,2$. Therefore $\sqrt[3]{g + z} + \sqrt[3]{g - z}$ will be $(= 4.515,793,760,9 + 3.033,410,154,2) = 7.549,203,915,1$, and $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ *ad infinitum* $- \sqrt[3]{g + z} - \sqrt[3]{g - z}$ will be $(= 15.605,014,559,7 - 7.549,203,915,1) = 8.055,810,644,6$. Therefore the root of the equation $x^3 - 50x = 120$ will be $= 8.055,810,644,6$. Q. E. I.

This value of x is exact in the first seven figures 8.055,810, its more accurate value being 8.055,810,345,702.

186. In the next equation $x^3 - x = \frac{1}{3}$ we have seen above, in art. 165, 166, that g is $= \frac{1}{6}$, and zz is $= \frac{1}{3 \times 36}$, or $\frac{1}{108}$, and $\frac{zz}{gg}$ is $= \frac{1}{3}$, and $\frac{Ez^4}{g^4}$ is $= 0.004,572,473,7$, and $\frac{Iz^8}{g^8}$ is $= 0.000,195,485,2$, and $\frac{Nz^{12}}{g^{12}}$ is $= 0.000,012,526,1$, and $\frac{Rz^{16}}{g^{16}}$ is $= 0.000,000,943,8$ and $\sqrt[3]{g}$ is $= \frac{1}{1.817,121}$. Therefore the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ will be $(= 1.000,000,000,0 - 0.004,572,473,7 - 0.000,195,485,2 - 0.000,012,526,1 - 0.000,000,943,8 - \&c = 1.000,000,000,0 - 0.004,781,428,8) = 0.995,218,571,2$, and $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ will be $(= 4\sqrt[3]{g} \times 0.995,218,571,2 = 4 \times \frac{1}{1.817,121} \times 0.995,218,571,2 = \frac{4 \times 0.995,218,571,2}{1.817,121} = \frac{3.980,874,284,8}{1.817,121}) = 2.190,759,055,0$.

And we have seen in art. 181 that z is $= 0.096,225,044,8$, and $g + z$ is $= 0.262,891,711,4$, and $g - z$ is $= 0.070,441,621,8$, and $\sqrt[3]{g + z}$ is $= 0.640,607,911,4$, and $\sqrt[3]{g - z}$ is $= 0.412,993,403,9$. Therefore $\sqrt[3]{g + z} + \sqrt[3]{g - z}$ will be $(= 0.640,607,911,4 + 0.412,993,403,9) = 1.053,601,315,3$, and $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ *ad infinitum* $- \sqrt[3]{g + z} - \sqrt[3]{g - z}$ will be $(= 2.190,759,055,0 - 1.053,601,315,3) = 1.137,157,739,1$; and consequently the root of the equation $x^3 - x = \frac{1}{3}$ will be $= 1.137,157,739,1$. Q. E. D.

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This value of x is exact in the first six figures, 1.137,15, its more accurate value being 1.137,158,164.

187. In the third equation $x^3 - 5x = 4$ we have seen above, in art. 168, 169, that g is $= 2$, and $\sqrt[3]{g}$ is $= 1.259,921,049,8$ and zz is $= 0.629,629,629,629$, &c, and $\frac{zz}{gg}$ is $= 0.157,407,407,4$, and $\frac{Ez^4}{g^4}$ is $= 0.001,019,633,4$, and $\frac{Iz^8}{g^8}$ is $= 0.000,009,720,7$, and $\frac{Nz^{12}}{g^{12}}$ is $= 0.000,000,138,8$, and $\frac{Rz^{16}}{g^{16}}$ is $= 0.000,000,002,3$. Therefore the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ will be $(= 1.000,000,000,0 - 0.001,019,633,4 - 0.000,009,720,7 - 0.000,000,138,8 - 0.000,000,002,3 - \&c = 1.000,000,000,0 - 0.001,029,495,2) = 0.998,970,504,8$, and $4\sqrt[3]{g} \times$ the said series will be $(= 4\sqrt[3]{g} \times 0.998,970,504,8 = 4 \times 1.259,921,049,8 \times 0.998,970,504,8 = 5.039,684,199,2 \times 0.998,970,504,8) = 5.034,495,868,5$.

And we have seen in art. 181 that $\sqrt[3]{g+z}$ is $= 1.408,366,911,5$, and that $\sqrt[3]{g-z}$ is $= 1.064,576,143,3$. Therefore $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c - \sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be $= 5.034,495,868,5 - 1.408,366,911,5 - 1.064,576,143,3 = 5.034,495,868,5 - 2.472,943,054,8 = 2.561,552,813,7$; and consequently the root of the equation $x^3 - 5x = 4$ will be $= 2.561,552,813,7$. Q. E. I.

This value of x is exact in the first nine figures 2.561,552,81, the more accurate value of it being 2.561,552,812,8. See above, art. 170.

188. In the 4th and last equation $x^3 - 63x = 162$ we have seen above, in art. 172, 173, that g is $= 81$, and $\sqrt[3]{g}$ is $= 4.326,749$, and zz is $= 2700$, and $\frac{zz}{gg}$ is $= \frac{100}{243}$, and $\frac{Ez^4}{g^4}$ is $= 0.006,969,171,9$, and $\frac{Iz^8}{g^8}$ is $= 0.000,454,123,4$, and $\frac{Nz^{12}}{g^{12}}$ is $= 0.000,040,593,7$, and $\frac{Rz^{16}}{g^{16}}$ is $= 0.000,004,661,9$.

Therefore the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ will be $(= 1.000,000,000,0 - 0.006,969,171,9 - 0.000,454,123,4 - 0.000,040,593,7 - 0.000,004,661,9 - \&c = 1.000,000,000,0 - 0.007,468,550,9) = 0.992,531,449,1$, and $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ *ad infinitum* will be $(= 4\sqrt[3]{g} \times 0.992,531,449,1 = 4 \times 4.326,749 \times 0.992,531,449,1 = 17.306,996 \times 0.992,531,449,1) = 17.177,737,819,4$.

And we have seen in art. 182, that $\sqrt[3]{g+z}$ is $= 5.103,976,447,9$, and that $\sqrt[3]{g-z}$ is $= 3.073,674,958,2$. Therefore $\sqrt[3]{g+z} + \sqrt[3]{g-z}$ will be $(= 5.103,976,447,9 + 3.073,674,958,2) = 8.177,651,406,1$. Therefore $4\sqrt[3]{g} \times$ the series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c$ *ad infinitum* —

$\sqrt[3]{g+z} - \sqrt[3]{g-z}$ will be $= 17.177,737,819,4 - 8.177,651,406,1 = 9.000,086,413,3$; and consequently the root of the equation $x^3 - 63x = 162$ will be $= 9.000,086,413,3$. Q. E. I.

This value of x is exact in the first five places of figures, 9.0000, the true value of x being the whole number 9.

A short view of the three expressions that have been here obtained for the value of the root of the cubick equation $x^3 - qx = t$.

189. It appears therefore that the root of the cubick equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, will be equal to either of the three following expressions; to wit,

First, to $2\sqrt[3]{g} \times$ the infinite series $1 + \frac{Cz}{gg} - \frac{Ez^4}{g^4} + \frac{Gz^6}{g^6} - \frac{Iz^8}{g^8} + \frac{Lz^{10}}{g^{10}} - \frac{Nz^{12}}{g^{12}} + \frac{Pz^{14}}{g^{14}} - \frac{Rz^{16}}{g^{16}} + \frac{Tz^{18}}{g^{18}} - \&c \text{ ad infinitum}$; which is an expression wholly transcendental;

And, 2dly, to the expression $\sqrt[3]{g+z} + \sqrt[3]{g-z} + 4\sqrt[3]{g} \times$ the infinite series $\frac{Cz}{gg} + \frac{Gz^6}{g^6} + \frac{Lz^{10}}{g^{10}} + \frac{Pz^{14}}{g^{14}} + \frac{Tz^{18}}{g^{18}} + \&c \text{ ad infinitum}$; which is an expression partly algebraïck and partly transcendental, and in which the transcendental part is added to the algebraïck part;

And, 3dly, to the expression $4\sqrt[3]{g} \times$ the infinite series $1 - \frac{Ez^4}{g^4} - \frac{Iz^8}{g^8} - \frac{Nz^{12}}{g^{12}} - \frac{Rz^{16}}{g^{16}} - \&c \text{ ad infinitum} - \sqrt[3]{g+z} - \sqrt[3]{g-z}$; which is also an expression partly transcendental and partly algebraïck, but of which the greater part is transcendental, and the algebraïck part is subtracted from the transcendental part.

And we may observe that the algebraïck part of the two latter expressions, to wit, the quantity $\sqrt[3]{g+z} + \sqrt[3]{g-z}$, is similar to the algebraïck expression $\sqrt[3]{e+s} + \sqrt[3]{e-s}$, given by Cardan's second rule for the root of the equation $y^3 - qy = r$, when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$. See on this subject Monsieur Clairaut's *Elémens d'Algèbre*, pages 282, 283, &c, to page 297.

Of the resolution of the equation $qy - y^3 = r$ by means of either of the three foregoing expressions of the value of the root of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$.

190. The greatest possible magnitude of the absolute term r in the equation $qy - y^3 = r$ is $\frac{2q\sqrt{q}}{3\sqrt{3}}$, which is the value of the compound quantity $qy - y^3$ when y is $= \frac{\sqrt{q}}{\sqrt{3}}$. And, when r is equal to this quantity $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qy - y^3 = r$ will have but one root, to wit, $\frac{\sqrt{q}}{\sqrt{3}}$; and, when r is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, the equation $qy - y^3 = r$ will have two roots, of which the lesser will be less than $\frac{\sqrt{q}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{q}}{\sqrt{3}}$, but less than $\sqrt{q}$. See my Dissertation on the Use of the Negative Sign in Algebra, art. 114, page 92. Therefore, whenever we have an equation of this form, $qy - y^3 = r$, proposed to us to be resolved, the absolute term r of such equation will be always equal to the absolute term t of some equation of the opposite form $x^3 - qx = t$, in which the absolute term t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, and which therefore cannot be resolved by means of the expressions given by Cardan's second rule; and consequently we may suppose the compound quantity $qy - y^3$, which forms the first side of the equation $qy - y^3 = r$, to be equal to the compound quantity $x^3 - qx$, which forms the first side of the equation $x^3 - qx = t$. Now, upon this supposition that $qy - y^3$ is equal to $x^3 - qx$, we may reduce the equation $qy - y^3 = r$ to a quadratick equation, and derive the values of its two roots from the value of x (the only root of the equation $x^3 - qx = t$) by proceeding in the manner following.

Since $qy - y^3$ is ($= r = t$) $= x^3 - qx$, add y^3 to both sides; and we shall have $qy = x^3 - qx + y^3$. Now let qx be added to both sides; and we shall have $qx + qy = x^3 + y^3$. Therefore $\frac{qx + qy}{x + y}$ will be $= \frac{x^3 + y^3}{x + y}$, and consequently q will be $= xx - xy + yy$. Add xy to both sides; and we shall have $q + xy = xx + yy$, and (subtracting yy from both sides) $q + xy - yy = xx$. Further, since q is less than xx , and consequently than the other side of the equation, let it be subtracted from both sides; and we shall have $xy - yy = xx - q$. And, because $\frac{xx}{4}$, or the square of $\frac{x}{2}$, is greater than $x - y \times y$, or than $xx - xy$, and consequently than the other side of the equation, both sides may be subtracted from $\frac{xx}{4}$. Let them be so subtracted; and we shall have $\frac{xx}{4} - xy + yy (= \frac{xx}{4} - xx + q = \frac{xx}{4} - \frac{4xx}{4} + \frac{4q}{4}) = \frac{4q - 3xx}{4}$. Therefore the square-root of $\frac{xx}{4} - xy + yy$ will be $= \frac{\sqrt{4q - 3xx}}{2}$. But the quantity $\frac{xx}{4} - xy + yy$ has two square-

roots, a greater and a lesser, to wit, $\frac{x}{2} - y$ and $y - \frac{x}{2}$, according as $\frac{x}{2}$ is greater, or less, than y . Therefore for the determination of the greater value of y we shall have the equation $y - \frac{x}{2} = \frac{\sqrt{4q - 3xx}}{2}$, and consequently $y = \frac{x + \sqrt{4q - 3xx}}{2}$; and for the determination of the lesser value of y we shall have the equation $\frac{x}{2} - y = \frac{\sqrt{4q - 3xx}}{2}$, and consequently $\frac{x}{2} = y + \frac{\sqrt{4q - 3xx}}{2}$, and $y = \frac{x - \sqrt{4q - 3xx}}{2}$. And consequently the greater root of the cubick equation $qy - y^3 = r$ will be $\frac{x + \sqrt{4q - 3xx}}{2}$, and the lesser root of the same equation will be $\frac{x - \sqrt{4q - 3xx}}{2}$. Therefore, if t , or r , is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or if $\frac{tt}{4}$, or $\frac{rr}{4}$, is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, we may obtain the values of the two roots of the cubick equation $qy - y^3 = r$ by first computing the value of the root x of the opposite equation $x^3 - qx = t$ by means of either of the three expressions set down in the preceeding article 189, and then computing the two quadrattick expressions $\frac{x + \sqrt{4q - 3xx}}{2}$ and $\frac{x - \sqrt{4q - 3xx}}{2}$.

Q. E. I.

Of the Trisection of a circular arc by means of either of the three foregoing expressions for the value of the root of the cubick equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or when $\frac{tt}{4}$ is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$.

191. If a be the radius of a circle, and consequently $2a$ be its diameter, and k be the chord of any arc in it, and y be the chord of the third part of the lesser of the two arcs whereof k is the common chord, the relation between the chords k and y will be expressed by this equation $3aay - y^3 = aak$, of which the said chord y will be the lesser root; as is demonstrated in many books of mathematics, and, amongst the rest in my Dissertation on the Use of the Negative Sign in Algebra, published in the year 1758, pages 183, 184, art. 220, 221. Now let q be substituted in this equation instead of $3aa$, and let r be substituted in it instead of aak ; and the equation $3aay - y^3 = aak$ will thereby be converted into the equation $qy - y^3 = r$, in which $\frac{\sqrt{q}}{\sqrt{3}}$ will represent a , or the radius of the circle, and $\frac{2\sqrt{q}}{\sqrt{3}}$ will represent $2a$, or the diameter of the circle, and $(\frac{r}{aa})$, or r divided

vided by $\frac{q}{3}$, or $r \times \frac{3}{q}$, or) $\frac{3r}{q}$ will represent k , or the given chord of the greater arch, which is to be trisected, or of the third part of which we are to find the chord y ; and the chord y will be the lesser root of the said equation $qy - y^3 = r$. We must therefore endeavour to find the lesser root of this equation $qy - y^3 = r$. Now it appears from the foregoing art. 190, that, if the absolute term t of the equation $x^3 - qx = t$ be equal to the absolute term r of the equation $qy - y^3 = r$, the lesser root of the equation $qy - y^3 = r$ will be equal to the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$. Therefore the value of the chord of the arc which is equal to one third part of the lesser of the two arcs of which the given chord k , or $\frac{3r}{q}$, is the common chord, in the circle of which a , or $\frac{\sqrt{q}}{\sqrt{3}}$, is the radius, and $2a$, or $\frac{2\sqrt{q}}{\sqrt{3}}$, is the diameter, will be equal to the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$. Therefore, if r , or t , is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or if aak is less than $2a^3$, but greater than $\sqrt{2} \times a^3$, or if k is less than $(\frac{2a^3}{aa})$, or) $2a$, but greater than $(\sqrt{2} \times \frac{a^3}{aa})$, or) $\sqrt{2} \times a$, that is, if the given chord k is less than the diameter $2a$, but greater than $\sqrt{2} \times a$, or the chord of a quadrantal arc, or of an arc of 90 degrees, or, if the given arc that is to be trisected, is less than the semicircumference, but greater than the arch of a quadrant, we may find the value of the chord y by first finding the root x of the opposite equation $x^3 - qx = t$, by means of one of the three expreffions set down in art. 189, and then computing the quadratick expreffion $\frac{x - \sqrt{4q - 3xx}}{2}$.

Q. E. I.

Of the analogy, or harmony, between circular arcs and logarithms, or between the measures of angles and the measures of ratios.

192. This application of the expreffions obtained in the foregoing articles to the trisection of a circular arc is an instance of the analogy, or *harmony* (as Mr. Cotes calls it) that subsists between logarithms and circular arcs, or between the measures of ratios and the measures of angles. For from the expreffion $\sqrt[3]{e + s} + \sqrt[3]{e - s}$ given by Cardan's second rule for the root of the equation $y^3 - qy = r$ in the first case of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$ (the value of which expreffion is to be obtained by extracting the cube-roots of the given quantities $e + s$ and $e - s$, that is, by trisecting the ratios of $e + s$ to 1 and of $e - s$ to 1) we have derived the three expreffions set down in art. 189, by either of which we are enabled to find the value of

of the root x of the equation $x^3 - qx = t$, when t is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, from which value we may afterwards, by computing the quadratick expression $\frac{x - \sqrt{4q - 3xx}}{2}$, obtain the value of the lesser root of the equation $qy - y^3 = r$, or the value of the chord of an arc that is equal to the third part of the lesser of the two arcs of which the given quantity $\frac{3r}{q}$, or k , is the common chord, in a circle of which the given quantity $\frac{\sqrt{q}}{\sqrt{3}}$, or a , is the radius; or, in other words, may trisect the said lesser of the two arcs of which $\frac{3r}{q}$, or k , is the common chord; provided that the said arc is greater than the arch of a quadrant. And consequently problems that require the trisection of a circular arc, or of an angle, may, by means of the method here explained, be solved by the trisection of a ratio, or by the help of a table of logarithms.

Of the resolution of the cubick equation $x^3 - qx = t$ by means of expressions derived from Cardan's rules, when t is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{t^3}{4}$ is less than $\frac{q^3}{54}$.

193. When the absolute term t of the equation $x^3 - qx = t$, or the absolute term r of the opposite equation $qy - y^3 = r$, is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or $\frac{t^3}{4}$, or $\frac{rr}{4}$, is less than $\frac{q^3}{54}$, the lesser root of the equation $qy - y^3 = r$ may be found by means of the expressions investigated in the preceeding discourse contained in pages 379, 380, 381, &c. 440 of this volume of tracts; which expressions are derived from the expression given by Cardan's first rule for the resolution of the cubick equation $y^3 + qy = r$, or $qy + y^3 = r$: and, when the value of the said lesser root of the equation $qy - y^3 = r$ has been so obtained, the value of the root x of the equation $x^3 - qx = t$ may be derived from it by computing the quadratick expression $\frac{y + \sqrt{4q - 3yy}}{2}$. For, since $x^3 - qx$ is $(= t = r) = qy - y^3$, we shall (by adding y^3 to both sides) have $x^3 - qx + y^3 = qy$, and (by adding qx to both sides) $x^3 + y^3 = qx + qy = q \times [x + y]$. Therefore $\frac{x^3 + y^3}{x + y}$ will be $= \frac{q \times x + y}{x + y} = q$; that is, $xx - xy + yy$ will be $= q$; and consequently, subtracting yy (which is less than q , and therefore than the other side of the equation) from both sides, we shall have $xx - xy = q - yy$, or $xx - yx = q - yy$; and (adding $\frac{yy}{4}$ to both sides) $xx - yx + \frac{yy}{4} = q - yy + \frac{yy}{4} =$

$$\frac{4q - 4yy + yy}{4}$$

$\frac{4q - 4yy + yy}{4} = \frac{4q - 3yy}{4}$. Therefore $x - \frac{y}{2}$ will be $= \frac{\sqrt{4q - 3yy}}{2}$, and x will be $= \frac{y}{2} + \frac{\sqrt{4q - 3yy}}{2}$, or $\frac{y + \sqrt{4q - 3yy}}{2}$. Q. E. D.

Conclusion concerning both branches of the second or irreducible case of the cubick equation $y^3 - qy = r$.

194. It appears therefore that in both the branches of the equation $x^3 - qx = t$, or of the second case of the cubick equation $y^3 - qy = r$, that is, both when t , or r , is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, but greater than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, and when it is less than $\sqrt{2} \times \frac{q\sqrt{q}}{3\sqrt{3}}$, or both when $\frac{tt}{4}$, or $\frac{rr}{4}$, is less than $\frac{q^3}{27}$, but greater than $\frac{q^3}{54}$, and when it is less than $\frac{q^3}{54}$, the value of the root x , or y , of the equation $x^3 - qx = t$, or $y^3 - qy = r$, may always be found by means either of the expressions derived in the present discourse from Cardan's second rule, given for the resolution of the equation $y^3 - qy = r$ in its first case, or of the expressions derived in the last preceeding discourse in this volume of tracts from Cardan's first rule, given for the resolution of the equation $qy + y^3 = r$, or $y^3 + qy = r$. And thus that which is called the *irreducible* case of the equation $y^3 - qy = r$ is shewn to be always capable of being resolved by means of expressions derived from one or other of Cardan's rules by the help of Sir Isaac Newton's binomial and residual theorems, which therefore seem to form a sort of bridge of communication between the problems belonging to the trisection of a ratio, and the problems belonging to the trisection of an angle, or between the problems depending on, or resolvable by means of, the asymptotick areas of an hyperbola, or a table of logarithms, and the problems depending on, or resolvable by means of, circular arches, or a table of sines and tangents.

A S C H O L I U M.

195. The first person who seems to have discovered the possibility of thus extending Cardan's rules to the second, or *irreducible*, case of the cubick equation $y^3 - qy = r$, was the famous German mathematician and philosopher, of the latter part of the last century, Mr. Leibnitz of Hanover, as is shewn in the scholium at the end of the last preceeding discourse in this volume of tracts, pages 438, 439, 440. But he has done it by the means of what are called *impossible* quantities, or quantities involving the square-roots of negative quantities, which, in the expressions to which his calculations lead him, are of such magnitudes, and so connected by the signs $+$ and $-$, as ultimately to destroy, or expunge, each other; whereas in this and the preceeding tract I have carefully avoided all mention of these unintelligible mysteries.

$$\frac{x - 2y + 3z}{4} = \frac{4x - 2y}{4} \quad \text{Therefore } x - \frac{y}{2} \text{ will be } = \frac{4x - 2y}{4} \text{ and } y \text{ will be}$$

$$= \frac{y}{2} + \frac{\sqrt{4x - 2y}}{2} \text{ or } \frac{y}{2} + \frac{\sqrt{4x - 2y}}{2}$$

Consider the case of the cubic equation $x^3 - 2x^2 + 3x - 4 = 0$.
 The first term of the equation is x^3 , the second $-2x^2$, the third $+3x$, and the fourth -4 .
 The first term is x^3 , the second $-2x^2$, the third $+3x$, and the fourth -4 .
 The first term is x^3 , the second $-2x^2$, the third $+3x$, and the fourth -4 .

It appears therefore that in both the branches of the equation $x^3 - 2x^2 + 3x - 4 = 0$, or of the second case of the cubic equation $x^3 - 2x^2 + 3x - 4 = 0$, that is, both when x or y is less than $\frac{2x}{3}$, but greater than $\frac{x}{3}$, and when it is less than $\frac{x}{3}$, or both when $\frac{x}{3}$ or $\frac{2x}{3}$ is less than $\frac{x}{3}$, but greater than $\frac{x}{3}$, and when it is less than $\frac{x}{3}$, the value of the root x or y of the equation $x^3 - 2x^2 + 3x - 4 = 0$, may always be found by means either of the expressions

derived in the present discourse from Cardan's second rule, given for the resolution of the equation $x^3 - 2x^2 + 3x - 4 = 0$ in its full case, or of the expressions derived in the last preceding discourse in this volume of notes from Cardan's first rule, given for the resolution of the equation $x^3 - 2x^2 + 3x - 4 = 0$. And thus that which is called the *irreducible case* of the equation $x^3 - 2x^2 + 3x - 4 = 0$ is shown to be always capable of being resolved by means of expressions derived from one or other of Cardan's rules by the help of Sir Isaac Newton's binomial and radical theorems, which therefore seem to form a sort of bridge of communication between the problems belonging to the resolution of a ratio, and the problems belonging to the resolution of an angle, or between the problems depending on, or resolvable by means of, the asymptotic areas of an hyperbola, or a circle of logarithms, and the problems depending on, or resolvable by means of, circular arches, or a table of sines and tangents.

The first person who seems to have discovered the possibility of thus extending Cardan's rules to the second, or irreducible, case of the cubic equation $x^3 - 2x^2 + 3x - 4 = 0$, was the famous German mathematician and philosopher of the latter part of the last century, Mr. Lambert of Hanovert, as is known in the last part of the end of the last preceding discourse in this volume of notes, page 438, 439, 440. But he has done it by the means of what are called *imaginary quantities*, or quantities involving the square-root of negative quantities, which in the expressions to which his calculations lead him, are of such nature, as not to be connected by the signs $+$ and $-$ as ultimately to define, or express, each other: whereas in this and the preceding that I have carefully avoided all mention of these unintelligible quantities.

C O N J E C T U R E

CONCERNING THE METHOD BY WHICH

C A R D A N ' s R U L E S

FOR RESOLVING

*The Cubick Equation $x^3 + qx = r$ in all Cases (or in all
Magnitudes of the known Quantities q and r)*

*And the Cubick Equation $x^3 - qx = r$ in the first Case of it
(or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$)*

*were probably discovered by Scipio Ferreus of Bononia, and
Nicholas Tartalea, or whoever else were the first Inventors
of them.*

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHÈQUER.

N. B. This Tract was first published in the Philosophical Transactions for the year 1780.

CARDAN'S RULES CONCERNING THE METHOD BY WHICH CUBIC EQUATIONS

FOR RESOLVING

The Cubic Equation $x^3 + px = r$ in all Cases (or in all
 Magnitudes of the known Quantities p and r)

And the Cubic Equation $x^3 - px = r$ in the first Case of it
 (or when r is greater than $\frac{27}{4}p^2$, or $\frac{r}{p}$ is greater than $\frac{27}{4}$)

were probably discovered by Scipio Ferro of Bononia, and
 Nicholas Tartalea, or, whoever else were the first Inventors
 of them.

BY FRANCIS MASTERS, Esq. F.R.S.

CURRIER BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

M. D. CCLXXIII. This Treatise was first published in the Philosophical Transactions for the year 1780.

A
C O N J E C T U R E
C O N C E R N I N G
C A R D A N ' s R U L E S.

Art. 1. **T**HERE is nothing more amusing, or more grateful to an inquisitive mind, in the study of the sciences of Geometry or Algebra (for, if we banish from it the ridiculous mysteries arising from the supposition of *negative* quantities, or quantities *less than nothing*, the latter may deserve the name of a *science* as well as the former), than to contemplate the methods by which the several ingenious and surprising truths that are delivered in the books that treat of them were first discovered. This we are sometimes enabled to do by the authors themselves to whom we are indebted for these discoveries, who have candidly informed their readers of the several steps, and sometimes of the accidents, by which they have been led to them: but it also often happens, that the authors of these discoveries have neglected to give their readers this satisfaction, and have contented themselves with either barely delivering the propositions they have found out, without any demonstrations, or with giving formal and positive demonstrations of them, which command indeed the assent of the understanding to their truth, but afford no clue whereby to discover the train of reasoning by which they were first found out; and consequently contribute but little to enable the reader to make similar discoveries himself on the like subjects. This seems to be the case with those ingenious rules for the resolution of certain cubick equations, which are usually known by the name of CARDAN's Rules. We are told to make certain substitutions of some quantities for others in these equations $x^3 + qx = r$ and $x^3 - qx = r$ (which are the objects of those rules), and certain suppositions concerning the quantities so substituted; by doing which, we find that those equations will be transformed into other equations which will involve the sixth power of the unknown quantity contained in them, but which (though of double the dimensions of the original equations $x^3 + qx = r$ and $x^3 - qx = r$, from which they were derived) will be more easy to resolve than those equations, because they will contain only the sixth power and the cube of the unknown quantity which is their root, and consequently will be of the same form as quadratick equations; so that by resolving them as quadratick equations we may obtain the value of the cube of the unknown quantity which is their root; and afterwards, by extracting the cube-root of the said value, we

may obtain the value of the said root, or unknown quantity, itself; and then, at last, by the relation of this last root to x , or the root of the original equation (which relation is derived from the suppositions that have been made in the course of the preceeding transformations), we may determine the value of x . And, if we please to examine the several steps of this process with sufficient attention, we may perceive, as we go along, that all these substitutions are legitimate and practicable, or are founded upon possible suppositions; though I cannot but observe, that the writers on algebra, for the most part, have not been so kind as to shew us that they are so. But still the question recurs, "How came SCIPIO FERREUS, of Bononia (who, as CARDAN tells us, was the first inventor of the former of these rules, and NICHOLAS TARTALEA, who was the first inventor of the latter of them), or the other persons, whoever they were, that invented them, to think of making these lucky substitutions which thus transform the original cubick equations into equations of the sixth power, which contain only the sixth and third powers of the unknown quantities which are their roots, and consequently are of the form of quadratick equations?" To answer this question as well as I can *by conjecture* (for I know of no historical account of this matter in any book of algebra), and in a manner that appears to me to be *probable*, is the design of the following pages.

Art. 2. The most probable conjecture concerning the invention of these rules, called CARDAN'S Rules, by SCIPIO FERREUS, of Bononia, and by NICHOLAS TARTALEA, or whoever else were the inventors of them, seems to be this: that the said inventors tried a great variety of methods of reducing the three cubick equations of the third class, to wit, $x^3 + qx = r$ and $x^3 - qx = r$, and $qx - x^3 = r$ (to some one of which all other cubick equations may, by proper substitutions, be reduced), to a lower degree, or to a more simple form, by substituting various quantities in the stead of x , in hopes that some of the terms arising by such substitutions might be equal to others of them, and, having contrary signs prefixed to them, might destroy them, and thereby render the new equation more simple and manageable than the old one. And, amongst other trials, it seems natural to imagine, that they would substitute the sum or difference of two other quantities instead of x , as being the most simple and obvious substitutions that could be made. And, by making these substitutions, the above-mentioned rules would of course come to be discovered, as well as the aforesaid limitation of them in the resolution of the equation $x^3 - qx = r$ (which restrains the rule to those cases only in which r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$), and their utter inutility in all the cases of the equation $qx - x^3 = r$. This will appear by examining each of these equations separately in the following manner.

Of the equation $x^3 + qx = r$.

Art. 3. In the equation $x^3 + qx = r$ the investigator of these rules would naturally be inclined to substitute the difference of two quantities (which we will here

here call y and z , and of which we will suppose y to be the greater) instead of x , rather than their sum, or would suppose x to be equal to $y - z$, rather than to $y + z$; because, if he was to suppose x to be equal to the sum of the two quantities y and z , and was to substitute that sum, or the binomial quantity $y + z$, instead of x in the equation $x^3 + qx = r$, it is evident that (as the signs of x^3 and qx are, both of them, affirmative) the terms of the new equation, arising from such substitution, would all of them be likewise affirmative; and consequently none of them, though they should happen to be exactly equal to each other, could exterminate each other, and thereby render the new equation more simple than the old one, which was the only view with which the substitution would have been made. He would therefore suppose x to be equal to $y - z$; and by substituting this quantity instead of x in the original equation $x^3 + qx = r$, he would transform that equation into the following one, to wit,

$$y^3 - 3yz + 3yz - z^3 + qy - qz = r,$$

$$\text{or } y^3 - 3yz \times y - z - z^3 + q \times y - z = r.$$

Now in this equation it is evident that the terms $3yz \times y - z$ and $q \times y - z$ have contrary signs; and therefore, if their co-efficients $3yz$ and q can be supposed to be equal to each other, those terms will mutually destroy each other, and the equation will be reduced to the following short one, $y^3 - z^3 = r$. And, if in this equation we substitute, instead of z , its value $\frac{q}{3y}$, derived from the same supposition of the equality of q and $3yz$, the equation will be $y^3 - \frac{q^3}{27y^3} = r$; and, by multiplying both sides by y^3 , it will be $y^6 - \frac{q^3}{27} = ry^3$; and, by adding $\frac{q^3}{27}$ to both sides, it will be $y^6 = ry^3 + \frac{q^3}{27}$; and, by subtracting ry^3 from both sides, it will be $y^6 - ry^3 = \frac{q^3}{27}$; which equation, though it rises to the sixth power of the unknown quantity y , is evidently of the form of a quadratick equation, and may therefore be resolved, so far as to find the value of the cube of y , in the same manner as a quadratick equation; after which it will be possible to find the value of y itself by the mere extraction of the cube root; and then at last, from the relation of y to x (derived from the foregoing suppositions that $y - z$ was equal to x , and that $3yz$ was equal to q , and consequently z equal to $\frac{q}{3y}$), we shall be able to determine the value of x .

Art. 4. It would therefore remain for the investigator of this method to enquire, whether or no the supposition "that $3yz$ was equal to q ," was a possible supposition; that is, whether it was possible (whatever might be the magnitudes of q and r) for two quantities, y and z , to exist, whose nature should be such that their difference $y - z$ should be equal to the unknown quantity x in the equation $x^3 + qx = r$, and that three times their product should at the same time be equal to q , or their simple product to the third part of q . And this supposition he would soon find to be always possible, whatever may be the magnitudes of q and r ; because, if the lesser quantity z is supposed to increase from *o ad infinitum*, and the greater quantity y is likewise supposed to increase with equal swiftness,

ness, or to receive equal increments in the same times, and thereby to preserve their difference $y - z$ always of the same magnitude, or equal to x , it is evident that the product or rectangle yz will increase continually at the same time from *o ad infinitum*, and consequently will pass successively through all degrees of magnitude, and therefore must at one point of time during its increase become equal to $\frac{q}{3}$.

And having thus found this supposition of the equality of yz and $\frac{q}{3}$, or of $3yz$ and q , to be always possible, whatever might be the magnitudes of q and r , our investigator would justly consider his solution of the equation $x^3 + qx = r$ (which was founded on that supposition) as legitimate and complete. And thus we see in what manner it seems probable, that CARDAN's rule for resolving the cubick equation $x^3 + qx = r$ may have been discovered.

Of the equation $x^3 - qx = r$.

Art. 5. In this second equation $x^3 - qx = r$, in which the second term qx is subtracted from the first, or marked with the sign $-$, it seems to have been natural for the person who invented these rules to substitute the *sum* as well as the *difference* of two other quantities, y and z , instead of x , in the terms x^3 and qx , in hopes of such an extermination of equal terms, and consequential reduction of the equation to one of a simpler and more manageable form, as was found to be so useful in the case of the former equation $x^3 + qx = r$. We will therefore try both these substitutions; and, as that of the difference $y - z$ has in the former case proved so successful, we will begin by that.

Art. 6. Now, by substituting the difference $y - z$ instead of x in the equation $x^3 - qx = r$, we shall transform it into the following equation, to wit, $y^3 - 3yyz + 3yzz - z^3 - q \times [y - z] = r$, or $y^3 - 3yz \times [y - z] - z^3 - q \times y + z = r$; in which the terms $3yz \times y - z$ and $q \times y - z$ have both of them the same sign $-$ prefixed to them, and consequently can never exterminate each other, whether $3yz$ be equal or unequal to q . This substitution therefore is in this case of no use.

Art. 7. We will now therefore try the substitution of the *sum* of y and z , instead of their difference, in the equation $x^3 - qx = r$.

Now, if x be supposed to be equal to $y + z$, and $y + z$ be substituted instead of it in the equation $x^3 - qx = r$, that equation will be thereby transformed into the following one, to wit,

$$y^3 + 3yyz + 3yzz + z^3 - q \times \overline{y + z} = r,$$

$$\text{or } y^3 + 3yz \times \overline{y + z} + z^3 - q \times \overline{y + z} = r.$$

Now in this equation, the terms $3yz \times \overline{y + z}$ and $q \times \overline{y + z}$ have contrary signs. Consequently, if they can be supposed to be equal to each other, they will destroy each other, and the equation will be thereby reduced to the following

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ing short one, $y^3 + z^3 = r$; that is, if $3yz$ and q can be supposed to be equal to each other, or if yz can be supposed to be equal to $\frac{q}{3}$, the equation will be reduced to the short equation $y^3 + z^3 = r$. And, if in this short equation we substitute instead of z its value $\frac{q}{3y}$ (derived from the same supposition of the equality of $3yz$ and q), the equation thence resulting will be $y^3 + \frac{q^3}{27y^3} = r$; and by multiplying both sides by y^3 , it will be $y^6 + \frac{q^3}{27} = ry^3$; and, by subtracting y^6 from both sides, it will be $ry^3 - y^6 = \frac{q^3}{27}$; which, though it rises to the sixth power of y , is evidently of the form of a quadratick equation, and consequently may be resolved in the same manner as a quadratick equation, so far as to find the value of y^3 , or the cube of the root y ; after which it will be possible to find the value of y itself by the mere extraction of the cube root; and, lastly, from the relation of y to x (contained in the two suppositions, that $y + z$ is equal to x , and that $3yz$ is equal to q , and consequently that z is equal $\frac{q}{3y}$) we may determine the value of x .

Art. 8. The only thing, therefore, that would remain for the investigator of these rules to do, in order to know whether the foregoing method of resolving the equation $x^3 - qx = r$ was practicable or not, would be to enquire, whether it was possible in all cases, that is, in all magnitudes of the known quantities q and r , for $3yz$ to be equal to q , or for yz (or the product, or rectangle, of the two quantities y and z , whose sum is equal to x) to be equal to $\frac{q}{3}$; and, if it was not possible in all cases, but only in some, to determine in what cases it *was* possible, or what must be the relation between q and r to make it possible.

Art. 9. Now, in order to determine this question, it would be proper and natural to observe, that the quantity yz , or the product of the multiplication of the two quantities y and z , whose sum is supposed to be equal to x , can never be greater than the square of half that sum, that is, than the square of $\frac{x}{2}$, or than $\frac{xx}{4}$, by El. 2, 5, but may be of any magnitude that does not exceed that square. Therefore, if $\frac{q}{3}$ is greater than $\frac{xx}{4}$, it will be impossible for yz to be equal to it; but, if $\frac{q}{3}$ is either equal to, or less than, $\frac{xx}{4}$, it will be possible for yx to be equal to it; and, if $\frac{q}{3}$ is exactly equal to $\frac{xx}{4}$, z will be exactly equal to y , and each of them equal to one half of x . We must therefore enquire what is the magnitude of x when $\frac{q}{3}$ is equal to $\frac{xx}{4}$. Now, when $\frac{xx}{4} = \frac{q}{3}$, xx will be $= \frac{4q}{3}$, and $x = \frac{2\sqrt{q}}{\sqrt{3}}$; therefore, when x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be impossible for

yz

yz to be equal to $\frac{q}{3}$; but when x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, it will be possible for yz to be equal to $\frac{q}{3}$.

But when x is $= \frac{2\sqrt{q}}{\sqrt{3}}$, x^3 will be $= \frac{8q\sqrt{q}}{3\sqrt{3}}$, and qx will be $= \frac{2q\sqrt{q}}{\sqrt{3}}$ or $\frac{6q\sqrt{q}}{3\sqrt{3}}$, and consequently $x^3 - qx$ will be $= \frac{8q\sqrt{q}}{3\sqrt{3}} - \frac{6q\sqrt{q}}{3\sqrt{3}} = \frac{2q\sqrt{q}}{3\sqrt{3}}$.

Therefore, if it be true (as we shall presently see that it is) that, while x increases from being equal to $\sqrt{q}$ (which is evidently its least possible magnitude) to any other magnitude, the compound quantity $x^3 - qx$, or the excess of x^3 above qx , will also continually increase from 0 (to which it is equal when x is $= \sqrt{q}$, or xx is $= q$) to some correspondent magnitude without ever decreasing; it will follow that, when x is less than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $x^3 - qx$ will be less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and when x is greater than $\frac{2\sqrt{q}}{\sqrt{3}}$, the compound quantity $x^3 - qx$ will be greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$; and, *è converso*, if the compound quantity $x^3 - qx$ is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be less than $\frac{2\sqrt{q}}{\sqrt{3}}$; and, if the compound quantity $x^3 - qx$ is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, x will be greater than $\frac{2\sqrt{q}}{\sqrt{3}}$. Consequently, if the compound quantity $x^3 - qx$, or, its equal, the absolute term r in the equation $x^3 - qx = r$, is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, it will be impossible for yz to be equal to $\frac{q}{3}$; but if $x^3 - qx$, or r , is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be possible for yz to be equal to $\frac{q}{3}$. Therefore, if $x^3 - qx$, or r , is less than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, the foregoing method of resolving the cubick equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx = r$, or r , is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be practicable.

Art. 10. It now only remains to be proved, that while x increases, from being equal to $\sqrt{q}$, *ad infinitum*, the compound quantity $x^3 - qx$ will likewise increase from 0 *ad infinitum*, without ever decreasing. Now this may be demonstrated as follows.

Art. 11. It is evident that while x increases from being equal to $\sqrt{q}$ *ad infinitum*, both the quantities x^3 and qx will increase *ad infinitum* likewise. But it does not therefore follow, that the excess of x^3 above qx will continually increase at the same time. This will depend upon the relation of the contemporary increments of x^3 and qx : if the increment of x^3 in any given time is equal to the contemporary increment of qx , the compound quantity $x^3 - qx$ will neither increase nor decrease, but continue always of the same magnitude during the said time, notwithstanding the increase of x ; if the former increment is less than the latter,

latter, the said compound quantity will decrease; and if it is greater, it will increase. We must therefore enquire whether the increment of x^3 in any given time is greater or less than the contemporary increment of qx .

Art. 12. Now, if $\dot{x}$ be put for the increment which x receives in any given time, the increment of x^3 in the same time will be the excess of $x + \dot{x}$ above x^3 , that is, the excess of $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ above x^3 ; and the increment of qx in the same time will be the excess of $q \times x + \dot{x}$, or $qx + q\dot{x}$, above qx ; that is, the increment of x^3 will be $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, and that of qx will be $q\dot{x}$. Now in the equation $x^3 - qx = r$ it is evident that xx must be greater than q ; for otherwise x^3 would not be greater than qx , as it is supposed to be. Consequently, $xx \times \dot{x}$ must be greater than $q\dot{x}$; and, *à fortiori*, $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ (which is more than triple of $x^2\dot{x}$) must be greater than $q\dot{x}$; that is, the increment of x^3 will be greater than the contemporary increment of qx . Therefore the excess of x^3 above qx , or the compound quantity $x^3 - qx$, will increase continually, without decreasing, while x increases from $\sqrt[3]{q}$ *ad infinitum*. Q. E. D.

Art. 13. It follows, therefore, upon the whole of these enquiries, that, if the compound quantity $x^3 - qx$, or, its equal, the absolute term r , is less than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is less than $\frac{q^3}{27}$, it will be impossible for yx to be equal to $\frac{q}{3}$, and consequently the foregoing method of resolving the equation $x^3 - qx = r$ will be impracticable; but, if $x^3 - qx$, or r , is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, it will be possible for yx to be equal to $\frac{q}{3}$, and consequently, the foregoing method of resolving the equation $x^3 - qx = r$ will be practicable. And thus we see in what manner it is probable that CARDAN's rule for resolving the cubick equation $x^3 - qx = r$ in the first case of it, or when r is greater than $\frac{2q\sqrt[3]{q}}{3\sqrt[3]{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, together with the restriction of it to that first case, may have been discovered.

Of the Equation $qx - x^3 = r$.

Art. 14. In the third equation $qx - x^3 = r$ the terms x^3 and qx have different signs, as well as in the second equation $x^3 - qx = r$; and therefore it seems to have been natural for the inventor of CARDAN's rules to try both the substitutions of $y - z$ and $y + z$ instead of x in this equation, as well as in that second equation, in hopes of an extermination of equal terms that are marked with contrary signs, and a consequent reduction of the equation to another which, though of double the dimensions of the equation $qx - x^3 = r$, should have been of a simpler form, and more easy to be resolved. But it will be found, upon trial, that neither of these substitutions will answer the end proposed.

Art. 15. For, in the first place, let us suppose x to be $= y - z$. Then we shall have $x^3 = y^3 - 3yyz + 3yyz - z^3 = y^3 - 3yz \times y - z - z^3$, and $qx = q \times y - z$ and consequently $qx - x^3 = q \times y - z - y^3 + 3yz \times y - z + z^3$. Therefore, $q \times y - z - y^3 + 3yz \times y - z + z^3$ will be $= r$. Now in this equation it is evident, the terms $q \times y - z$ and $3yz \times y - z$ have the same signs, and therefore can never destroy each other. Therefore, no such method of resolving this equation $qx - x^3 = r$ as was found above for resolving the two former equations $x^3 + qx = r$ and $x^3 - qx = r$, can be obtained by substituting the difference $y - z$ in it instead of x .

Art. 16. We will now try the substitution of $y + z$ instead of x in the terms of this equation.

Now, if x be supposed to be $= y + z$, we shall have $x^3 = y^3 + 3yyz + 3yyz + z^3 = y^3 + 3yz \times y + z + z^3$, and $qx = q \times y + z$, and consequently $qx - x^3 = q \times y + z - y^3 - 3yz \times y + z - z^3$. Therefore, $q \times y + z - y^3 - 3yz \times y + z - z^3$ will be $= r$.

In this equation it is true indeed that the terms $q \times y + z$ and $3yz \times y - z$ have different signs. But they cannot be equal to each other: for, since the three terms y^3 and $3yz \times y - z$ and z^3 are all marked with the sign $-$, or are to be subtracted from the first term $q \times y + z$, and the remainder is $= r$, it is evident that $q \times y + z$ must be greater than the sum of all the three terms y^3 , $3yz \times y - z$, and z^3 , taken together, and therefore, *à fortiori*, greater than $3yz \times y + z$ alone. Therefore, no such extermination of equal terms marked with contrary signs as took place in the transformed equations derived from the two former equations $x^3 + qx = r$ and $x^3 - qx = r$, can take place in this transformed equation derived from the equation $qx - x^3 = r$ by substituting $y + z$ in its terms instead of x ; and consequently no such method of resolving the equation $qx - x^3 = r$ as has been found for the resolution of the equations $x^3 + qx = r$ and $x^3 - qx = r$, can be obtained by means of that substitution.

Art. 17. These are the methods of investigation by which I conceive it to be probable that CARDAN's rules for the resolution of the cubick equations $x^3 + qx = r$ and $x^3 - qx = r$, together with the limitation of the rule relating to the latter of those equations, and their inapplicability to the third equation $qx - x^3 = r$, may have been discovered by the first inventors of them.

AN APPENDIX

TO THE

TRACT

CONTAINED IN THE FOREGOING PART OF THIS SECOND VOLUME OF
MATHEMATICAL TRACTS, IN PAGES 153, 154, 155, &c, TO 169;

INTITLED,

“A DEMONSTRATION OF SIR ISAAC NEWTON’S BINOMIAL THEOREM
IN THE CASE OF INTEGRAL POWERS, OR POWERS OF WHICH
THE INDEXES ARE WHOLE NUMBERS:”

*Containing an Investigation of the Law by which the co-
efficients of the third and fourth and other following terms
of the series which is equal to any integral power of a
binomial quantity, are derived from the co-efficient of the
second term of the said series, grounded on a probable in-
duction from particular examples.*

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF HIS MAJESTY’S COURT OF EXCHEQUER.

Art. 1. **I**T is shewn in art. 5 of the foregoing tract, pages 155, 156, that, in all integral powers whatsoever of the binomial quantity $a + b$, the literal parts of the terms of the series which is equal to $(a + b)^m$ (in which the letter m denotes any whole number whatsoever), will always be $a^m, a^{m-1}b, a^{m-2}b^2, a^{m-3}b^3, \&c$, of which every term is generated from the next before it by the multiplication of the fraction $\frac{b}{a}$. And it is also shewn in art. 6 of the said tract, page 156, that the numeral co-efficient of the first term of the series that is equal to $(a + b)^m$ must always be 1, or that the first term of the said series will always be a^m , and that the numeral co-efficient of the second term of the said series will always be m , or the index of the power to which $a + b$ is raised, or that the second term of the said series will always be $m \times a^{m-1}b$, to whatever whole number the letter m may be supposed to be equal.

Art. 2. And it is further shewn in the said tract that the numeral co-efficients of the third, and fourth, and fifth, and other following terms of the series which is

equal to $\overline{a + b}^m$ may always be derived from m , the co-efficient of the second term of the said series, by the continual multiplication of the following fractions, to wit, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, till we come to the fraction $\frac{m-m}{m+1}$, which is $= 0$, or till the said series of fractions is terminated or exhausted; which fractions, $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, are therefore called *the generating fractions* of the co-efficients of the third and other following terms of the series which is equal to $\overline{a + b}^m$.

Art. 3. And the method by which it is shewn in the said tract that the said fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, are in all cases, or when m is supposed to represent any whole number whatsoever, the generating fractions of the co-efficients of the terms of the series that is equal to $\overline{a + b}^m$, or the fractions by which the co-efficients of the third and fourth, and other following terms of the said series, are derived from m , the co-efficient of the second term $m \times a^{m-1} b$, consists of the three following parts; to wit, first, of a demonstration that, if it be true that these are the generating fractions of the co-efficients of the third and other following terms of the series that is equal to $\overline{a + b}^m$ when m is equal to *any one* whole number whatsoever, it will also be true that they will be the generating fractions of the co-efficients of the third and other following terms of the series that is equal to $\overline{a + b}^m$ when m is equal to any other whole number greater than the former; and, secondly, of a proof, by actual trials of the co-efficients of the terms of the several serieses that are equal to $\overline{a + b}^2$, $\overline{a + b}^3$, $\overline{a + b}^4$, and $\overline{a + b}^5$, that in these four serieses, or when m is equal either to the number 2, or the number 3, or the number 4, or the number 5, the said fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, and $\frac{m-5}{6}$, are the generating fractions by which the co-efficients of the third and other following terms of the said serieses (that are equal to $\overline{a + b}^2$, $\overline{a + b}^3$, $\overline{a + b}^4$, and $\overline{a + b}^5$) are derived from the co-efficients of the second terms of the said serieses respectively; and, thirdly, of a conclusion evidently following from the former two propositions, to wit, that, to whatever whole number the index m be supposed to be equal, it will always be true that the said fractions $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, will be the generating fractions by the continual multiplication of which the co-efficients of the third and other following terms of the series that is equal to $\overline{a + b}^m$ will be derived from m , the co-efficient of the second term. These reasonings I take to be just and clear, and such as must give every reader full satisfaction as to the truth of the proposition, or conclusion, obtained by means of them.

Art. 4.

Art. 4. But it may be asked, "How came it to be suspected that the fractions $\frac{m-1}{2}, \frac{m-2}{3}, \frac{m-3}{4}, \frac{m-4}{5}, \frac{m-5}{6},$ &c, were the generating fractions by which the co-efficients of the third and other following terms of the series that is equal to $(a+b)^m$ are derived from m , the co-efficient of the second term, in any one value of the index m , since it is by no means apparent from the mere inspection of the terms of the serieses that are equal to $(a+b)^m$ when m is equal to the small numbers 2, 3, 4, and 5?"

This is a very natural and reasonable question, and well deserves to be considered; more especially if we recollect that Dr. Wallis informs us that he had sought for these generating fractions without being able to discover them. And till a person had first suspected, and then found upon trial, that these are the generating fractions of the co-efficients of the terms of the series that is equal to $(a+b)^m$ in some of the lower values of m , he could never think of shewing, in the method above described, that the same generating fractions would enable us to find the co-efficients of the terms of the like serieses in all other integral values of m .

Art. 5. Now, in answer to this question it may be observed, that these fractions will occur to our notice as the generating fractions of the co-efficients of the third and other following terms of the serieses that are equal to $(a+b)^m$ in some of the lower values of m , if we divide the several successive terms of these serieses by the terms next before them, in order to discover the generating fractions by which they are derived one from another, and then reduce the generating fractions so obtained to their lowest denominations. Thus, for example, the sixth power of the binomial quantity $a+b$ is $= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; of which terms if we divide the second term $6a^5b$ by the first term a^6 , the quotient will be $\frac{6a^5b}{a^6}$, or $6 \times \frac{b}{a}$; and, if we divide in like manner the third term by the second, and the fourth term by the third, and the fifth term by the fourth, and the sixth term by the fifth, and the seventh, or last, term by the sixth, the quotients will be $\frac{15a^4b^2}{6a^5b}$, or $\frac{15}{6} \times \frac{b}{a}$, and $\frac{20a^3b^3}{15a^4b^2}$, or $\frac{20}{15} \times \frac{b}{a}$, and $\frac{15a^2b^4}{20a^3b^3}$, or $\frac{15}{20} \times \frac{b}{a}$, and $\frac{6ab^5}{15a^2b^4}$, or $\frac{6}{15} \times \frac{b}{a}$, and $\frac{b^6}{6ab^5}$, or $\frac{1}{6} \times \frac{b}{a}$. And consequently, if we multiply the first term a^6 by the quotient $6 \times \frac{b}{a}$, we shall thereby produce the second term $6a^5b$; and, if we multiply the second term $6a^5b$ by $\frac{15}{6} \times \frac{b}{a}$, we shall thereby produce the third term $15a^4b^2$; and, if we multiply the third term $15a^4b^2$ by $\frac{20}{15} \times \frac{b}{a}$, we shall thereby produce the fourth term $20a^3b^3$; and, if we multiply the fourth term $20a^3b^3$ by $\frac{15}{20} \times \frac{b}{a}$, we shall thereby produce the fifth term $15a^2b^4$; and,

if we multiply the fifth term $15a^2b^4$ by $\frac{6}{15} \times \frac{b}{a}$, we shall thereby produce the sixth term $6ab^5$; and, if we multiply the sixth term $6ab^5$ by $\frac{1}{6} \times \frac{b}{a}$, we shall thereby produce the seventh, or last, term b^6 . Therefore the generating fractions, by the continual multiplication of which the second, and other following terms of the series $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$ (which is equal to $(a+b)^6$) are derived from the first term a^6 , are $\frac{6}{1} \times \frac{b}{a}$, $\frac{15}{6} \times \frac{b}{a}$, $\frac{20}{15} \times \frac{b}{a}$, $\frac{15}{20} \times \frac{b}{a}$, $\frac{6}{15} \times \frac{b}{a}$, and $\frac{1}{6} \times \frac{b}{a}$; and consequently the generating fractions, by the continual multiplication of which the numeral co-efficients of the second and other following terms of the said series $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$ (independently of the literal parts of the said terms) are derived from 1, the numeral co-efficient of the first term a^6 , will be $\frac{6}{1}$, $\frac{15}{6}$, $\frac{20}{15}$, $\frac{15}{20}$, $\frac{6}{15}$, and $\frac{1}{6}$. Now let these co-efficients be reduced to their lowest terms; and they will then be $\frac{6}{1}$, $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{1}{6}$, of which the numerator of the first term $\frac{6}{1}$ is the index 6 of the power of the binomial quantity to which the said series of terms is equal, and the numerators of the following fractions $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, $\frac{1}{6}$ are derived from the said index 6 by the continual subtraction of 1, and the denominators of the said fractions are the natural numbers 1, 2, 3, 4, 5, 6, which begin with an unit, and increase by the continual addition of 1. This observation on the increase and decrease of the denominators and numerators of the fractions $\frac{5}{2}$, $\frac{4}{3}$, $\frac{3}{4}$, $\frac{2}{5}$, and $\frac{1}{6}$, and their derivation from the index 6, or $\frac{6}{1}$, in the case of the series which is equal to $(a+b)^6$, is sufficient to have induced the person who should have made it, to conjecture, that possibly, when the index m was equal to any other number (such as 5, or 4, or 7, or 8), the generating fractions whereby the numeral co-efficients of the third and fourth and other following terms of the series that was equal to $(a+b)^m$, were derived from the numeral co-efficient m of the second term, and from each other, might likewise, when properly reduced, be found to consist of numerators and denominators that did in like manner decrease from the index m by the continual subtraction of an unit, and increase from 1 by the continual addition of an unit; or, in other words, might be equal to $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c. And this conjecture might have produced a trial whether this rule took place in some particular examples, and more especially in the serieses that were equal to $(a+b)^2$, $(a+b)^3$, $(a+b)^4$, $(a+b)^5$, $(a+b)^6$, $(a+b)^7$, and $(a+b)^8$, and perhaps a few more of the lower integral powers of $a+b$; after which trials, and the success that would have attended them, it would have become so highly probable that the same rule would take

take place in the serieses that were equal to any other integral powers of $a + b$, that it would have been almost impossible to doubt of it. And then it would have been natural to endeavour to find some general demonstration of the truth of the rule in all integral powers of the binomial quantity $a + b$ whatsoever, which might have led to such a demonstration as that which is given in the tract above-mentioned, which is contained in pages 153, 154, 155, &c 169, of this volume.

N. B. This method of discovering (by a conjecture grounded on a trial or two, in some particular examples), that the generating fractions by which the co-efficients of the third and fourth and other following terms of the series that is equal to $(a + b)^m$ (or any integral power of the binomial quantity $a + b$) are derived from m (the index of the power to which the said binomial quantity is raised) or from the co-efficient of the second term of the said series (which is equal to the said index) are $\frac{m-1}{2}$, $\frac{m-2}{3}$, $\frac{m-3}{4}$, $\frac{m-4}{5}$, $\frac{m-5}{6}$, &c, is suggested by professor Saunderson, in the second volume of his Algebra, in the chapter on the binomial theorem, where the reader will find a good explanation and illustration of the said celebrated theorem, by a variety of examples, both in the case of integral powers and in the case of roots, and other fractional powers, and even in the case of negative powers, and of powers that are both fractional and negative; but no demonstration of it in any case, not even in that of integral and affirmative powers.

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take place in the cases that were equal to any other integral power of $x + A$. And then it would have been almost impossible to doubt of it. And then it would have been natural to endeavour to find some general demonstration of the truth of the theorem in all integral powers of the binomial quantity $x + A$ whatever, which might have led to such a demonstration as that which is given in the above mentioned, which is contained in pages 133, 134, 135, 136, 137, 138, 139, of this volume.

N. B. This method of discovering (by a conjecture grounded on a trial or two, in some particular examples), that the generating fractions by which the coefficients of the third and fourth and other following terms of the series that is equal to $x + A$ (or any integral power of the binomial quantity $x + A$) are derived from the indices of the power to which the said binomial quantity is raised, or from the coefficient of the second term of the said series (which is equal to the said A) are $\frac{m-1}{2}, \frac{m-2}{3}, \frac{m-3}{4}, \frac{m-4}{5}, \frac{m-5}{6}, \frac{m-6}{7}, \frac{m-7}{8}, \frac{m-8}{9}, \frac{m-9}{10}, \frac{m-10}{11}, \frac{m-11}{12}, \frac{m-12}{13}, \frac{m-13}{14}, \frac{m-14}{15}, \frac{m-15}{16}, \frac{m-16}{17}, \frac{m-17}{18}, \frac{m-18}{19}, \frac{m-19}{20}, \frac{m-20}{21}, \frac{m-21}{22}, \frac{m-22}{23}, \frac{m-23}{24}, \frac{m-24}{25}, \frac{m-25}{26}, \frac{m-26}{27}, \frac{m-27}{28}, \frac{m-28}{29}, \frac{m-29}{30}, \frac{m-30}{31}, \frac{m-31}{32}, \frac{m-32}{33}, \frac{m-33}{34}, \frac{m-34}{35}, \frac{m-35}{36}, \frac{m-36}{37}, \frac{m-37}{38}, \frac{m-38}{39}, \frac{m-39}{40}, \frac{m-40}{41}, \frac{m-41}{42}, \frac{m-42}{43}, \frac{m-43}{44}, \frac{m-44}{45}, \frac{m-45}{46}, \frac{m-46}{47}, \frac{m-47}{48}, \frac{m-48}{49}, \frac{m-49}{50}, \frac{m-50}{51}, \frac{m-51}{52}, \frac{m-52}{53}, \frac{m-53}{54}, \frac{m-54}{55}, \frac{m-55}{56}, \frac{m-56}{57}, \frac{m-57}{58}, \frac{m-58}{59}, \frac{m-59}{60}, \frac{m-60}{61}, \frac{m-61}{62}, \frac{m-62}{63}, \frac{m-63}{64}, \frac{m-64}{65}, \frac{m-65}{66}, \frac{m-66}{67}, \frac{m-67}{68}, \frac{m-68}{69}, \frac{m-69}{70}, \frac{m-70}{71}, \frac{m-71}{72}, \frac{m-72}{73}, \frac{m-73}{74}, \frac{m-74}{75}, \frac{m-75}{76}, \frac{m-76}{77}, \frac{m-77}{78}, \frac{m-78}{79}, \frac{m-79}{80}, \frac{m-80}{81}, \frac{m-81}{82}, \frac{m-82}{83}, \frac{m-83}{84}, \frac{m-84}{85}, \frac{m-85}{86}, \frac{m-86}{87}, \frac{m-87}{88}, \frac{m-88}{89}, \frac{m-89}{90}, \frac{m-90}{91}, \frac{m-91}{92}, \frac{m-92}{93}, \frac{m-93}{94}, \frac{m-94}{95}, \frac{m-95}{96}, \frac{m-96}{97}, \frac{m-97}{98}, \frac{m-98}{99}, \frac{m-99}{100}$ is suggested by professor Sturgeson in the second volume of his Algebra, in the chapter on the binomial theorem, where the reader will find a good explanation and illustration of the said celebrated theorem, by a variety of examples, both in the case of integral powers and in the case of roots, and other fractional powers, and even in the case of negative powers, and powers that are both fractional and negative; but no demonstration will be attempted, not even in the case of integral and negative powers.



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Nature And Construction Of Logarithms, Mentioned In Dr. Hutton's
Historical Introduction To His New Edition Of Sherwin's Mathematical
Tables Together With Some Tracts On The Binomial Theorem And Other
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